CHAPTER-4
LATTICE OF CLOSURE OPERATORS

We denote the set of all closure operators on a fixed set \( X \) by \( L(X) \). It is a complete lattice with the partial order \( \leq \) defined on it as in 3.1.6. In this chapter we study some properties of this lattice.

4.1 INFRA AND ULTRA CLOSURE OPERATORS
4.1.1 DEFINITIONS

The closure operator \( D \) on \( X \) defined as \( D(A) = A \) for every \( A \) in \( P(X) \) is called the discrete closure operator.

The closure operator \( I \) on \( X \) defined by

\[
I(A) = \varnothing \text{ if } A = \varnothing \\
= X \text{ otherwise}
\]

is called the indiscrete closure operator.

4.1.2 REMARKS

Note that \( D \) and \( I \) are the closure operators associated with the discrete and the indiscrete topologies on \( X \) respectively.

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Moreover $D$ is the unique closure operator whose associated topology is discrete.

Also $I$ and $D$ are the smallest and the largest elements of $L(X)$ respectively.

4.1.3 DEFINITIONS

A closure operator on $X$, other than $I$, is called an infra closure operator if the only closure operator on $X$, strictly smaller than it, is $I$.

A closure operator on $X$, other than $D$, is called an ultra closure operator if the only closure operator on $X$, strictly larger than it, is $D$.

4.1.4 REMARK

Note that the infra closure operators and the ultra closure operators are precisely the atoms and the dual atoms respectively of the lattice $L(X)$.

4.1.5 NOTATION

For $a, b$ in $X$, $a \neq b$, let

$$V_{a, b}(A) = \varnothing \quad \text{if } A = \varnothing$$

$$= X \setminus \{b\} \quad \text{if } A = \{a\}$$

$$= X \quad \text{otherwise}$$
$V_{a,b}$ can be verified to be a closure operator on $X$.

Now we can characterize the infra closure operators.

4.1.6 THEOREM

A closure operator on $X$ is an infra closure operator if and only if it is of the form $V_{a,b}$ for some $a,b$ in $X$, $a \neq b$.

PROOF

If $V$ is a closure operator on $X$ strictly smaller than $V_{a,b}$, then $V(\{a\})$ will be strictly larger than $X\backslash\{b\}$ and hence equal to $X$. Also $V(A) = X$ for every $A$ in $P(X)$ other than $\emptyset$ and $\{a\}$. Hence $V = I$. Thus all closure operators of the form $V_{a,b}$ are infra closure operators.

Now let $V$ be any closure operator on $X$ other than $I$. Then there exists a non-empty subset $A$ of $X$ such that $V(A) \neq I(A) = X$. Now choose an element $a$ of $A$ and an element $b$ of $X\backslash V(A)$. Then $b$ is not an element of $V(\{a\})$ since $b$ is not an element of $V(A)$. Also $V_{a,b}(M) = X$ for every nonempty subset $M$ of $X$ other than $\{a\}$. Then $V_{a,b} \leq V$. Thus all infra closure operators are of the form $V_{a,b}$ for some $a,b$ in $X$, $a \neq b$. 

4.1.7 NOTE

A topology T on X which is not discrete, is called an ultra topology if the discrete topology is the only topology strictly larger than T. In [9] O. FRÖLICH proved that the ultra topologies on X are precisely the topologies of the form $\mathcal{P}(X \setminus \{a\}) \cup \mathcal{U}$ where $a \in X$ and $\mathcal{U}$ is an ultra filter on X which does not contain $\{a\}$.

The closure operator $V$ associated with an ultratopology $\mathcal{P}(X \setminus \{a\}) \cup \mathcal{U}$ is given by

$$V(A) = \begin{cases} A & \text{if } A = \emptyset, \ a \in A \text{ or } X \setminus A \in \mathcal{U} \\ A \cup \{a\} & \text{otherwise} \end{cases}$$

4.1.8 THEOREM

A closure operator on X is an ultra closure operator if and only if it is the closure operator associated with some ultra topology on X.

PROOF

Let $\mathcal{P}(X \setminus \{a\}) \cup \mathcal{U}$ be an ultra topology on X and $V$ the associated closure operator. Let $V'$ be a closure operator on X strictly larger than $V$. Then there exists a subset $A$ of X such that $V'(A) \subseteq V(A)$,
but $V'(A) \neq V(A)$. Then $V(A) = A \cup \{a\}$ and $V'(A) = A$, which means that $X \setminus A$ is open in $(X, V')$ and not open in $(X, V)$. Also every open set in $(X, V)$ is open in $(X, V')$. Thus the associated topology of $V'$ is strictly larger than the ultra topology and hence is discrete. Then $V' = D$ by Remark 4.1.2. Hence the closure operator associated with an ultra topology is an ultra closure operator.

To prove that every ultra closure operator is the closure operator associated with an ultra topology. Let $V$ be a closure operator on $X$ other than $D$. It suffices to prove that there exists a closure operator associated with an ultratopology larger than $V$. Since $V \neq D$, there exists an element $a$ of $X$ such that $\{a\}$ is not open in $(X, V)$. Now, consider

$$\mathcal{F} = \{A \subseteq X : a \in A \text{ and } a \notin V(X \setminus A)\}$$

$\mathcal{F}$ can be verified to be a filter on $X$. Here $\{a\}$ is not an element of $\mathcal{F}$, for $a \in V(X \setminus \{a\})$ by the choice of $a$. Then $\mathcal{F} \cup \{X \setminus \{a\}\}$ is a family with finite intersection property. For, otherwise there will be an $F$ in $\mathcal{F}$ such that $F \cap (X \setminus \{a\}) = \emptyset$. Then $F \subseteq \{a\}$. Then $\{a\} \in \mathcal{F}$, for $F \in \mathcal{F}$, a contradiction. By Zorn's lemma,
we have an ultrafilter \( \mathcal{U} \) on \( X \) containing \( \bigcup \{X \setminus \{a\}\} \). Clearly \( \mathcal{U} \) does not contain \( \{a\} \). Now consider the ultratopology \( P(X \setminus \{a\}) \cup \mathcal{U} \). Let \( V' \) be the closure operator associated with it. Then \( V \leq V' \). For, otherwise there exists a nonempty subset \( M \) of \( X \) such that \( V'(M) \nsubseteq V(M) \). But then there exists an element \( a \) of \( X \) such that \( a \in V'(M) \) but \( a \notin V(M) \). Since \( a \notin V(M) \), \( X \setminus M \in \mathcal{U} \). Then \( V'(M) = M \), a contradiction. Hence the result.

### 4.1.9 REMARKS

In the course of the proof of Theorems 4.1.6 and 4.1.8, we also proved that every element of \( L(X) \) other than \( I \) is larger than or equal to an atom and every element of \( L(X) \) other than \( D \) is smaller than or equal to a dual atom.

### 4.1.10 DEFINITIONS

Let \( x \) be an element of \( X \). Then the set

\[
\mathcal{U}(x) = \left\{ A \subseteq X : x \in A \right\}
\]

is an ultrafilter on \( X \). Such ultrafilters are called principal ultrafilters. An ultratopology \( P(X \setminus \{a\}) \cup \mathcal{U} \) is called a principal ultratopology or a nonprincipal ultratopology according as \( \mathcal{U} \) is principal or not. The closure operator associated
with an ultratopology is called principal ultra closure operator or nonprincipal ultra closure operator according as the ultratopology is principal or not.

4.1.11 THEOREM

Infra closure operators are less than or equal to any nonprincipal ultra closure operator.

PROOF

Let \( V_{x,y} \) be an infra closure operator and \( V \) a non principal ultra closure operator. Since \( V_{x,y}(A) = X \) for all \( A \) in \( P(X) \) other than \( \varnothing \) and \( \{x\} \), we need show only that

\[
V(\{x\}) \subset V_{x,y}(\{x\}) = X \setminus \{y\}
\]

But since all nonprincipal ultratopologies are \( T_1 \) (see [35]), \( V(\{x\}) = \{x\} \) for all \( x \) in \( X \). Hence the result.

4.1.12 NOTATION

We denote the principal ultra closure operator associated with the principal ultra topology \( P(X \setminus \{a\}) \cup \bigcup(b) \) by \( T_{a,b} \).
4.1.13 THEOREM

An infra closure operator $V_{x,y}$ is less than or equal to a principal ultra closure operator $T_{a,b}$ if and only if $x \neq b$ or $y \neq a$. Also $V_{a,b}$ and $T_{b,a}$ are incomparable.

PROOF

At first we prove that $V_{a,b}$ and $T_{b,a}$ are incomparable. We have $V_{a,b}(\{a\}) = X \setminus \{b\}$ and

$T_{b,a}(\{a\}) = \{a,b\}$. Then

$T_{b,a}(\{a\}) \notin V_{a,b}(\{a\})$

Thus $V_{a,b} \nleq T_{b,a}$. Also $V_{a,b} \nmeq T_{b,a}$ for $V_{a,b}$ is an atom and $T_{b,a}$ is a dual atom of the lattice $L(X)$. Hence $V_{a,b}$ and $T_{b,a}$ are incomparable.

Now when $A$ is a non empty subset of $X$, other than $\{x\}$,

$T_{a,b}(A) \subseteq X = V_{x,y}(A)$

Also when $x \neq b$,

$T_{a,b}(\{x\}) = \{x\} \subseteq X \setminus \{y\} = V_{x,y}(\{x\})$
and when $y \neq a$

$$T_{a,b}(\{x\}) \subset \{x,a\} \subset X \setminus \{y\} = v_{x,y}(\{x\})$$

Thus

$$v_{x,y} \leq T_{a,b} \text{ if } x \neq b \text{ or } y \neq a$$

4.2 COMPLEMENTATION IN THE LATTICE $L(X)$

In this section we study the complementation in the lattice $L(X)$.

4.2.1 THEOREM

The lattice $L(X)$ is complemented when $X$ is finite.

PROOF

To every closure operator we can associate a reflexive relation $\rho V$ as explained in 3.3.2. Then $\rho$ is a one-one map from $L(X)$ onto the lattice of reflexive relations with the partial order of set inclusion such that $V_1 \leq V_2$ if and only if $\rho V_2 \subset \rho V_1$ for $V_1$ and $V_2$ in $L(X)$ by Theorem 3.3.11. Thus $\rho$ is a dual isomorphism and also the lattice of reflexive relations with the partial order of set inclusion is complemented. Hence $L(X)$ is complemented.
4.2.2 THEOREM

No nonprincipal ultra closure operator has a complement.

PROOF

On the contrary let \( V \) be a nonprincipal ultra closure operator with a complement \( V' \) in the lattice \( L(X) \). Since \( V' \) is not indiscrete, there exists an infra closure operator \( V_x, y \leq V' \) by 4.1.9. But \( V_{x, y} \leq V \) by Theorem 4.1.11. This contradicts the fact that \( V \) and \( V' \) are complements in the lattice \( L(X) \) and hence the result.

4.2.3 REMARK

When \( X \) is infinite we can prove that there exists a non principal ultra closure operator on \( X \) since then, assuming axiom of choice, we can prove that there exists a non principal ultrafilter. Thus by the previous theorem \( L(X) \) is not complemented when \( X \) is infinite.

But some elements of \( L(X) \) do have complements as proved below in
4.2.4 THEOREM

The infra closure operators $V_{x,y}$ and the principal ultra closure operators $T_{y,x}$ are lattice complements of each other in $L(X)$ for any $x,y$ in $X$, $x \neq y$.

PROOF

Here $V_{x,y}$ is an atom and $T_{y,x}$ is a dual atom of the lattice $L(X)$ and they are incomparable by Theorem 4.1.13. Hence the result follows.

4.2.5 THEOREM

Let $V$ and $V'$ be closure operators on $X$ which are not discrete such that an ultra closure operator is greater than or equal to $V$ if and only if it is greater than or equal to $V'$. Then $V = V'$.

PROOF

Suppose $V \neq V'$. Then without loss of generality we assume that there exists a nonempty subset $A$ of $X$ such that $V(A) \subsetneq V'(A)$. Thus there exists an element $a \in V(A)$ such that $a \notin V'(A)$. Now let

$$\mathcal{F} = \left\{ F \subset X : a \in F \text{ and } a \notin V(X \setminus F) \right\}$$
\[ \mathcal{Y}' = \{ F \subseteq X : a \in F \text{ and } a \not\in V'(X \setminus F) \} \]

We can easily verify that \( \mathcal{Y} \) and \( \mathcal{Y}' \) are filters on \( X \) such that \( A \in \mathcal{Y}' \) and \( A \not\in \mathcal{Y} \).

Now consider \( \mathcal{B} = \mathcal{Y} \cup \{ A \} \). It is a family with finite intersection property for otherwise there exists \( F \in \mathcal{Y} \) such that \( F \cap A = \emptyset \) and then \( X \setminus A \in \mathcal{Y} \) as \( F \subseteq X \setminus A \) which is a contradiction since \( a \not\in X \setminus A \) as \( A \in \mathcal{Y}' \).

Then \( \mathcal{B} \) is contained in some ultra filter \( \mathcal{U} \) on \( X \) which does not contain \( \{ a \} \) for \( a \not\in A \) and \( A \in \mathcal{U} \).

Let \( S \) be the ultra closure operator on \( X \) associated with the ultra topology \( \mathcal{P}(X \setminus \{ a \}) \cup \mathcal{U} \). But \( S(M) \subseteq M \cup \{ a \} \) for every \( M \) in \( \mathcal{P}(X) \) and whenever \( a \not\in V(M) \), \( X \setminus M \in \mathcal{Y} \subseteq \mathcal{U} \) and hence \( S(M) = M \). Thus \( S(M) \subseteq V(M) \) for every \( M \) in \( \mathcal{P}(X) \). That is \( V \subseteq S \).

But \( V' \not\subseteq S \) for \( S(A) = A \cup \{ a \} \) as \( A \not\subseteq \emptyset \), \( a \notin A \) and \( X \setminus A \in \mathcal{U} \) and \( a \notin V'(A) \). This contradicts the hypothesis and hence the result.

4.2.6 REMARKS

Let \( V \) be any closure operator on \( X \) which is not
discrete. Then $V$ is the greatest lower bound of the set of all ultra closure operators greater than or equal to $V$, by the previous theorem. Thus $L(X)$ is dually atomic. Eventhough $L(X)$ is atomic when $X$ is finite, it is not atomic when $X$ is infinite, for then there will be many nonprincipal ultra closure operators, all of them greater than every infra closure operator.

4.2.7 THEOREM

If $V'$ is the complement of $V$ in the lattice $L(X)$, then every ultra closure operator is greater than or equal to one of $V$ and $V'$.

PROOF

On the contrary assume that there exists an ultra closure operator $U$ associated with the ultra topology $P(X \setminus \{a\} \cup \mathcal{U})$ such that $U \not\geq V$ and $U \not\leq V'$. Since $U \not\geq V$, $U(A) \not\subseteq V(A)$ for some nonempty subset $A$ of $X$. But $U(A) \subseteq A \cup \{a\}$. Therefore $a \notin V(A)$. Also $A \in \mathcal{U}$ for otherwise $X \setminus A \in \mathcal{U}$ and hence $U(A) = A \cup V(A)$, a contradiction. Similarly there exists a nonempty subset $B$ of $X$ such that $a \not\in V'(B)$ and $B \in \mathcal{U}$. Let $b \in A \cap B \neq \emptyset$ since $A \in \mathcal{U}$, $B \in \mathcal{U}$. Then $a \not\in V(A \cap B)$ and $a \not\in V'(A \cap B)$.
Thus $V_{b,a} \leq V$ and $V_{b,a} \leq V'$. This contradicts the fact that $V'$ is the lattice complement of $V$ in $L(X)$ and hence the result.

4.2.8 THEOREM

In the lattice $L(X)$ of closure operators on $X$, no element has more than one complement.

PROOF

Clearly $I$ and $D$ have unique complements in $L(X)$. Let a closure operator $V$ on $X$ other than $I$ and $D$ have lattice complements $V_1$ and $V_2$ in $L(X)$. Then both the set of ultra closure operators greater than or equal to $V_1$ and the set of ultra closure operators greater than or equal to $V_2$ are the same as the set of all ultra closure operators which are not greater than or equal to $V$, by Theorem 4.2.7. But then $V_1 = V_2$ by Theorem 4.2.5. Hence the result.

4.2.9 REMARK

The lattice of topologies on a set $X$ is not even modular when $|X| \geq 3$. Also it is not self dual when $|X| > 3$ (See [33]).
But the lattice of closure operators on a finite set $X$ is dually isomorphic to the lattice of reflexive relations on $X$ as shown in 4.2.1. The latter lattice is isomorphic with the lattice of all subsets of the set $\{(x,y) : x \in X, y \in X, x \neq y\}$ and hence distributive and self-dual. Thus the lattice of closure operators on a finite set $X$ is also distributive and self-dual. But when $X$ is infinite, the lattice of closure operators is not self-dual. Since then the number of dual atoms will be $2^{\left|X\right|}$, as in the case of the lattice of topologies, but the number of atoms is equal to $\left|X\right|$.

An important problem is to determine whether the lattice of closure operators on an infinite set is distributive or even modular or not. We could not yet solve it.

4.3 SOME SUB-LATTICES OF $L(X)$

In this section we discuss some sublattices of $L(X)$.

4.3.1 NOTE

The lattice of topologies has two important sub-
lattices, namely, the lattice of principal topologies and the lattice of $T_1$ topologies generated by the set of all principal ultratopologies and the set of all non-principal ultratopologies respectively. A topology $T$ on $X$ is principal if and only if $T$ is the intersection of all principal ultratopologies finer than $T$ if and only if arbitrary intersection of open sets of $(X,T)$ is open in it if and only if $\bigcup A_\alpha = \bigcup \overline{A_\alpha}$ for every collection $\{A_\alpha\}$ of subsets of $X$ where $\overline{A}$ denotes the closure $\overline{A} \subseteq X$, in $(X,T)$. Also $\kappa$ is a dual isomorphism from the lattice of pre-orders on $X$ onto the lattice of principal topologies on $X$ (See [1], [27]).

In the case of $L(X)$, we have

4.3.2 THEOREM

The set of all $T_1$ closure operators on $X$ form a sublattice of $L(X)$ generated by the set of all non-principal ultra closure operators.

PROOF

Nonprincipal ultra closure operators are $T_1$ since nonprincipal ultratopologies are $T_1$ (see [33]).
Let $V$ be any $T_1$ closure operator. It is the infimum of all ultra closure operators greater than or equal to it. Also since $V$ is $T_1$, no principal ultra closure operator can be greater than or equal to $V$. Hence $V$ is the infimum of all non principal ultra closure operator greater than or equal to it. Then the result follows by noting that the $T_1$ closure operators on $X$ form a subinterval of $L(X)$ containing all closure operators greater than or equal to the closure operator $C_0$ associated with the cofinite topology on $X$.

4.3.3 NOTE

The natural generalization of the concept of a principal topology is that of a quasi-discrete closure operator in the sense of [8]. A closure operator $V$ on $X$ is called quasi-discrete if $V(\bigcup_{\alpha} A_{\alpha}) = \bigcup_{\alpha} V(A_{\alpha})$ for every collection $\{A_{\alpha}\}$ of subsets of $X$. Also it is proved in [8], that $\triangledown$ is a dual isomorphism from the lattice of reflexive relations onto the lattice of quasi-discrete closure operators. When $X$ is finite, every closure operator on $X$ is quasi-discrete. But when $X$ is infinite, the quasi-discrete closure operators on $X$ do not form a sublattice of $L(X)$ as shown below.
Let $X = A \cup B$, where $A$ and $B$ are disjoint subsets of $X$ having the same cardinality. Define $V_1 : P(X) \rightarrow P(X)$ such that

$$V_1(M) = \begin{cases} \emptyset & \text{if } M = \emptyset \\ M \cup A & \text{if } M \cap A = \emptyset \text{ and } M \cap B \neq \emptyset \\ M \cup B & \text{if } M \cap A \neq \emptyset \text{ and } M \cap B = \emptyset \\ X & \text{if } M \cap A \neq \emptyset \text{ and } M \cap B \neq \emptyset \end{cases}$$

Then $V_1$ can be verified to be quasi-discrete closure operator on $X$. Also define $V_2 : P(X) \rightarrow P(X)$ such that

$$V_2(M) = \begin{cases} \emptyset & \text{if } M = \emptyset \\ M \cup A & \text{if } M \cap A \neq \emptyset \text{ and } M \cap B = \emptyset \\ M \cup B & \text{if } M \cap A = \emptyset \text{ and } M \cap B \neq \emptyset \\ X & \text{if } M \cap A \neq \emptyset \text{ and } M \cap B \neq \emptyset \end{cases}$$

Then $V_2$ can be verified to be quasi-discrete closure operator on $X$.

The lattice join $V_1 \vee V_2$ of $V_1$ and $V_2$ is the closure operator $C_o$ on $X$ defined by

$$C_o(A) = \begin{cases} A & \text{if } A \text{ is finite} \\ X & \text{otherwise} \end{cases}$$
But $C_0$ is not quasi-discrete. Hence the quasi-discrete closure operators on $X$ do not form a sublattice of $L(X)$.

4.3.4 THEOREM

The function $\mu$ is a dual isomorphism from the lattice of reflexive relations on $X$ onto the sublattice of $L(X)$ containing all closure operators on $X$ less than or equal to the closure operator $C_0$ associated with the cofinite closure operator.

PROOF

If $R$ is any reflexive relation, $\mu R$ is a closure operator on $X$ less than or equal to $C_0$. Also, it can easily be seen that every closure operator on $X$ less than or equal to $C_0$ is of the form $\mu R$ for some reflexive relation $R$ on $X$. By Theorem 3.3.5, $\mu$ is one-one and $R_1 \subseteq R_2$ if and only if $\mu R_2 \leq \mu R_1$. Hence the result.

4.3.5 NOTE

In view of the fact that the largest element of $L(X)$ and the dual atoms of it are topological, we would like to ask the following question. What are the topologies on $X$ such that all closure operators on $X$ greater than the closure operator associated with it are topological?
A more general problem is to determine the intervals in the lattice $\mathcal{L}(X)$ containing only topological closure operators. Some partial solutions to the first problem are given below.

4.3.6 THEOREM

Let $A$ be a proper nonempty subset of $X$ and

$$T = \{ N \subseteq X : \text{either } X \setminus A \subseteq M \text{ or } M \subseteq X \setminus A \}$$

Then every closure operator greater than the closure operator associated with the topology $T$ are topological.

PROOF

Let $V$ be the closure operator on $X$ associated with $T$. Then

$$V(B) = \begin{cases} B & \text{if } B \subseteq A \text{ or } A \subseteq B \\ B \cup A & \text{otherwise} \end{cases}$$

Let $V'$ be any closure operator such that $V < V'$. Let $N$ be a nonempty proper subset of $X$ such that

$$N \neq V'(N)$$

Since $N \neq V'(N) \subseteq V(A)$, $V(N) = N \cup A$

Then $V'(N) = N \cup A'$ for some $A' \subseteq A$
Thus \( V'(V'(N)) = V'(N \cup A') = V'(N) \cup V'(A') \)

\[ = V'(N) \cup A' \quad (\text{Since } A' \subset V'(A) \subset V(A') = A') \]

\[ = V'(N) \]

Thus \( V' \) is topological. Hence the result.

4.3.6 DEFINITION (See [1] )

A topological space \((X, T)\) is called a TF space if given a finite subset \( F \) of \( X \) and \( x \) in \( X \setminus F \), either there exists an open set containing \( x \) which is disjoint with \( F \) or there exists an open set containing \( F \) which does not contain \( x \).

4.3.7 REMARK

In [1], it is proved that a topological space \((X, T)\) is TF if and only if \( \rho(T) \) is a partial order of length at most 1. It can be seen that a partially ordered set \((X, \leq)\) is of length at most 1 if and only if it does not contain three distinct elements \( x, y \) and \( z \) such that \( x < y < z \).

4.3.8 THEOREM

If every closure operator on \( X \) greater than the
closure operator associated with a topology $T$ on $X$, is
topological, then $(X, T)$ is $T_F$.

**PROOF**

Suppose not. Let $R = \rho(T)$. Then $(X, R)$ is a
partially ordered set containing three distinct elements
$x, y$ and $z$ such that $xRy$ and $yRz$. Let $R' = R \setminus \{(x, z)\}$.
Now $V \preceq \vee R$ since $R = \rho(T)$ is the same as $\rho V$ associated
with the closure operator $V$ associated with $T$. Also
$\vee R \preceq \vee R'$ by Theorem 3.3.5. Thus $V \preceq \vee R'$. Also $\vee R'$
is not topological since

\[
\vee R'(\{x\}) = \{x, y\} \quad \text{and} \\
\vee R'(\{x, y\}) = \vee R'(\{x\}) \cup \vee R'(\{y\}) \\
= \{x, y\} \cup \{y, z\} \\
= \{x, y, z\} \neq \vee R'(\{x\})
\]

Hence the result.

4.3.9 **REMARK**

The original problem mentioned in 4.3.5 remains
to be solved.

4.4 **GROUP OF AUTOMORPHISMS OF $L(X)$**

In this last section we would like to discuss a
problem related to the group of automorphisms of the lattice \( L(X) \). This is to determine the points of \( L(X) \) which are left fixed by every automorphism of the lattice \( L(X) \).

4.4.1 NOTE

Let \( C_0 \) be a function from \( P(X) \) into \( P(X) \) defined by

\[
C_0(A) = \begin{cases} A & \text{if } A \text{ is finite} \\ X & \text{otherwise} \end{cases}
\]

Then \( C_0 \) is a closure operator on \( X \). It is actually the closure operator associated with the cofinite topology and is called the cofinite closure operator. All infra closure operators are smaller than \( C_0 \) since \( C_0(\{x\}) = \{x\} \) for every \( x \) in \( X \). Also if \( V \) is a closure operator larger than every infra closure operator, \( V(\{x\}) = \{x\} \) for every \( x \) in \( X \). Then \( V(F) = F \) for every finite subset \( F \) of \( X \). Thus \( C_0 \leq V \). Hence \( C_0 \) is the least upper bound of all infra closure operators.

4.2.2 THEOREM

The elements \( I, D \) and \( C_0 \) are left fixed by every automorphism of \( L(X) \).
PROOF.

The fact that I and D are left fixed by every automorphism of L(X) follows since they are respectively the smallest and the largest elements.

\( C_0 \) is left fixed by every automorphism since it is the least upper bound of all infra closure operators on \( X \), infra closure operators are mapped into infra closure operators by any automorphism of L(X) being atoms and automorphisms preserve least upper bound of arbitrary subsets.

The following theorem gives a sufficiency condition.

4.4.3 THEOREM

If a closure operator \( V \) on \( X \) is left fixed by every automorphism of L(X), then \((X,V)\) is completely homogeneous.

PROOF

Let \( f \) be any permutation of \( X \). If \( V \) is an element of L(X), consider the function \( T_f V \) from P(X) into P(X) defined by

\[
T_f V(A) = f^{-1}(V(f(A)))
\]
\( T_f V \) can be verified to be a closure operator on \( X \),
also the function \( T_f \) from \( L(X) \) into \( L(X) \) defined by
\( T_f(V) = T_f V \) can be verified to an automorphism of
\( L(X) \).

If \( V \) is a closure operator on \( X \) left fixed
by every automorphism of \( L(X) \), then clearly \( T_f V = V \)
for every permutation \( f \) of \( X \).

Then \( V(A) = f^{-1}(V(f(A))) \) for every \( A \subseteq X \).

Then \( f(V(A)) = V(f(A)) \)

Then \( f \) is a closure isomorphism of \((X,V)\) for every
permutation \( f \) of \( X \). Thus \((X,V)\) is completely homogeneous.

4.4.4 REMARK

According to Theorem 3.2.3 the completely homo­
geneous closure operators are precisely the closure
operators associated with completely homogeneous topologies.
Also, by Theorem 1.2.13, a topology is completely homo­
geneous if and only if it is either discrete, indiscrete
or of the form
\[ T_\alpha = \{ \varnothing \} \cup \{ A \subseteq X : \text{card} (X \setminus A) < \alpha \} \]

for some infinite cardinal number \( \alpha \leq |X| \). Then in view of Theorem 4.4.3, every closure operator left fixed by every automorphism of \( L(X) \) are closure operators associated with topologies among these. Also by Theorem 4.4.2, three of these are shown to be left fixed by every automorphism of \( L(X) \). Now the problem reduces to determine whether the remaining completely homogeneous closure operators are left fixed by every automorphism of \( L(X) \) or not. This problem remains yet to be solved.