Chapter 3

Some Characterizations in $QTAG$-Modules

3.1 Introduction

Singh [48] proved that a $QTAG$-module $M$ is a direct sum of uniserial modules if and only if $M$ is the union of an ascending chain of bounded submodules. This indicates that $M$ is a direct sum of uniserial modules if and only if $\text{Soc}(M) = \bigoplus_{k \in \omega} S_k$ and $H(x) = k$ for every $x \in S_k$. This motivates us to define summable modules. A reduced $QTAG$-module $M$ of length $\sigma$ is said to be summable if $\text{Soc}(M) = \bigoplus_{\rho < \sigma} S_{\rho}$ and the nonzero elements of $S_{\rho}$ are contained in $(H_{\rho}(M) \setminus H_{\rho+1}(M))$.

Here we investigate the direct sums of countably generated $QTAG$-modules. Since there exist summable modules $M$ of length greater than $\omega$ such that $M/H_{\omega}(M)$ is not a direct sum of uniserial modules and therefore they are not the direct sums of countably generated modules. This made us to investigate the other conditions as summability is not enough.

In section 2, we prove some significant results related to the modules which are the direct sums of countably generated modules. Section 3 deals with some applications of these results.
3.2 Main Results

We start with the following:

**Definition 3.2.1.** A submodule $N$ of a QTAG-module $M$ is height finite, if the heights of the elements of $N$ take only finitely many values.

**Remark 3.2.1.** We may observe that if $M$ is a summable QTAG-module of countable length then $Soc(M)$ is the union of an ascending chain of height finite submodules. Since $Soc(M) = \bigoplus_{\rho<\sigma} S_\rho$, $S_\rho \subseteq (H_\rho(M) \setminus H_{\rho+1}(M)) \cup \{0\}$ if $\rho_1, \rho_2, \ldots, \rho_n, \ldots$ are the ordinals less than $\sigma$, then $Soc(M) = \bigcup_{k=1}^\infty T_k$, $T_k = S_{\rho_1} \oplus \ldots \oplus S_{\rho_k}$ and $T_k$ are height finite.

We prove the following lemma:

**Lemma 3.2.1.** Let $N$ be a submodule of a QTAG-module $M$ such that

1. $H_\alpha(M/N) = \frac{H_\alpha(M) + N}{N}$, for every $\alpha < \beta$;
2. $M/N$ is countably generated of length $\leq \beta$.

If $K/N$ is finitely generated submodule of $M/N$, then $K$ also satisfies these two conditions.

**Proof.** Assume that $H_\alpha(M/K) = \frac{H_\alpha(M) + K}{K}$, for all $\alpha < \gamma < \beta$. Consider $x + K \in H_\gamma(M/K)$. If $\gamma = \alpha + 1$, then there exists $y + K \in H_\alpha(M/K) = \frac{H_\alpha(M) + K}{K}$ such that $d\left(\frac{(y + K)R}{(x + K)R}\right) = 1$, therefore $y \in H_\alpha(M)$ and $x + K \in \frac{H_\alpha(M) + K}{K}$. Otherwise, if $\gamma$ is a limit ordinal, then for each $\alpha < \gamma$, $x + K = z + K$, where $z \in H_\alpha(M)$. Thus $x - z = u + v$ where $u \in K$, $v \in N$. Since $u$ assumes only finitely many values, there is a fixed $u_0 \in K$ and a set $T \subseteq \{\alpha \mid \alpha < \gamma\}$ such that $\sup T = \gamma$ and $x - z = u_0 + v$ where $v \in N$. Now $x - u_0 + K = z + N \in \frac{H_\alpha(M) + N}{N}$. Since $\sup T = \gamma$, $x - u_0 + N = z + N \in \frac{H_\alpha(M) + N}{N}$ for all $\alpha \in T$ and $x - u_0 + N \in H_\gamma(M/N) = \frac{H_\gamma(M) + N}{N}$. Now we may write $x - u_0 + N = z + N$ with $z \in H_\gamma(M)$ and $x + K = z + K \in \frac{H_\alpha(M) + K}{K}$.

To prove the second condition it is sufficient to prove $H_\beta(M/K) = 0$ i.e., length of $M/K$ is $\beta$. For $x + K \in H_\beta(M/K)$ we find $u \in K$ such that $x - u + N \in H_\beta(M/N) = 0$. Therefore $x - u \in N$ or $x + K = 0$. \(\square\)
Theorem 3.2.1. If $N$ is a submodule of a $QTAG$-module $M$ such that

(i) $H_\alpha(M/N) = \frac{H_\alpha(M) + N}{N}$ for every $\alpha < \beta$;

(ii) $H_\alpha(M) \cap N = H_\alpha(N)$ for every $\alpha < \beta$;

(iii) $M/N$ is countably generated of length $\leq \beta$;

then $N$ is a summand of $M$.

Proof. Since $M/N$ is countably generated, we have to show that if $K/N$ is a finitely generated submodule of $M/N$, every height increasing homomorphism $f : K \to N$ ($f|_N = I_N$) can be extended to a height increasing homomorphism $\tilde{f} : K + xR \to N$ whenever $y \in K$ such that $d \left( \frac{xR}{yR} \right) = 1$. Consider $x \notin K$. Now $H_{M/K}(x + K) = \alpha$, for some $\alpha < \beta$. Therefore $x + K = z + K$ with $z \in H_\alpha(M)$. Since $u \in K$ where $d \left( \frac{zR}{uR} \right) = 1$ and $xR + K = zR + K$, $x \in H_\alpha(M)$. Now $H_M(x + v) \leq H_{M/K}(x + K) = H_M(x) = \alpha$ for all $v \in K$. Again $\alpha + 1 = \beta$, we have $w \in H_\alpha(N)$ such that $w' = f(y) \in H_{\alpha+1}(M) \cap N = H_{\alpha+1}(N)$ where $d \left( \frac{wR}{w' R} \right) = 1$. Now we may define $\tilde{f}$ on $K + xR$ such that $\tilde{f}(xr + u) = wr + f(u)$, $r \in R$, $u \in K$. Here $\tilde{f}$ is a well defined height increasing homomorphism, which is an extension of $f$. \qed

Corollary 3.2.1. Let $N$ be a submodule of a $QTAG$-module $M$ such that

(i) $\frac{M/N}{H_\beta(M/N)}$ is countably generated;

(ii) $\frac{N + H_\beta(M)}{H_\beta(M)}$ is a summand of $M/H_\beta(M)$;

(iii) $N \cap H_\beta(M) = H_\beta(N)$ and

(iv) $H_\beta(M) = H_\beta(N) \oplus T$;

then $M = N \oplus K$ with $K \supseteq T$.

Proof. We may put

$$\frac{M}{H_\beta(M)} = \frac{N + H_\beta(M)}{H_\beta(M)} \oplus \frac{Q}{H_\beta(M)}.$$
Thus, \((N + H_\alpha(M)) \cap Q = H_\alpha(M) \cap Q\) for all \(\alpha \leq \beta\). Now suppose \(H_\alpha(M) \cap N = H_\alpha(N)\) for \(\alpha < \gamma \leq \beta\). If \(\gamma\) is a limit ordinal, we have \(H_\gamma(M) \cap N = H_\gamma(N)\), otherwise \(\gamma = \alpha + 1\).

For \(x \in H_\gamma(M) \cap N\), there exist \(y \in N\), \(z \in Q\) such that \(d\left(\frac{(y + z)R}{xR}\right) = 1\) and \(y + z \in H_\alpha(M)\). Then \(z \in (N + H_\alpha(M)) \cap Q = H_\alpha(M) \cap Q\) and \(y = y + z - z \in H_\alpha(M) \cap N = H_\alpha(N)\). If \(d\left(\frac{yR}{y'R}\right) = 1\), then \(x - y' \in N \cap Q = N \cap H_\beta(M) = H_\beta(N)\) and \(x = (x - y') + y' \in H_\gamma(N)\). In order to show that

\[
H_\alpha(M/N) = \frac{H_\alpha(M) + N}{N}, \text{ for all } \alpha \leq \beta,
\]

consider

\[
\frac{M}{N + H_\alpha(M)} = \frac{N + H_\alpha(M) + Q}{N + H_\alpha(M)}
\]

\[
\Rightarrow \frac{Q}{Q \cap (N + H_\alpha(M))} = \frac{Q}{Q \cap H_\alpha(M)}
\]

\[
\Rightarrow \frac{H_\alpha(M) + Q}{H_\alpha(M)} \subseteq \frac{M}{H_\alpha(M)} \text{, for all } \alpha \leq \beta.
\]

Now we apply Theorem 3.2.1 to \(\frac{N \oplus T}{T} = \frac{N + H_\beta(M)}{T}\) and get

\[
H_\alpha\left(\frac{M}{N \oplus T}/T\right) \cong H_\alpha\left(\frac{M}{N \oplus T}\right) = H_\alpha\left(\frac{M}{(N + H_\beta(M))}\right)
\]

\[
\cong H_\alpha\left(\frac{M/N}{H_\beta(M/N)}\right) = \frac{H_\alpha(M/N)}{H_\beta(M/N)}
\]

\[
\cong \frac{H_\alpha(M) + N}{H_\beta(M) + N} \cong \frac{(H_\alpha(M) + N)/T}{(H_\beta(M) + N)/T}
\]

\[
= \frac{(H_\alpha(M/T) + N \oplus T)/T}{(N \oplus T)/T}.
\]

These isomorphisms give identity on \(\frac{M/T}{(N \oplus T)/T}\) when combined together and we have

\[
\frac{(H_\alpha(M) + N)/T}{(H_\beta(M) + N)/T} = \frac{(H_\alpha(M/T) + N \oplus T)/T}{(N \oplus T)/T}, \text{ for all } \alpha < \beta.
\]
Now,

\[ H_\alpha(M/T) \cap ((N \oplus T)/T) = (H_\alpha(M)/T) \cap ((N \oplus T)/T) \]
\[ = ((H_\alpha(M) \cap N) \oplus T)/T = (H_\alpha(N) \oplus T)/T \]
\[ \subseteq H_\alpha((N \oplus T)/T), \text{ for all } \alpha \leq \beta. \]

Also,

\[ \frac{(M/T)}{(N \oplus T)/T} \cong \frac{M}{N + H_\beta(M)} \cong \frac{(M/H_\beta(M))}{(N + H_\beta(M))/(H_\beta(M))} \]
\[ = \frac{M/N}{H_\beta(M/N)} \]

is countably generated module of length at most \( \beta \). Again by Theorem 3.2.1 we have

\[ \frac{M}{T} = \left( \frac{N \oplus T}{T} \right) \oplus \left( \frac{K}{T} \right) \]

and therefore \( M = N \oplus K \).

In order to prove some important results, we need the following lemmas:

**Lemma 3.2.2.** For a QTAG-module \( M \), a submodule \( N \subseteq M \) and a limit ordinal \( \lambda \) if

\[ \text{Soc}(H_\alpha(M/N)) = \frac{\text{Soc}(H_\alpha(M)) + N}{N}, \text{ for every } \alpha < \lambda, \text{ then} \]

\[ H_\alpha(M/N) = \frac{H_\alpha(M) + N}{N}, \text{ for all } \alpha < \lambda. \]

**Proof.** Assume that \( \beta < \lambda \) and \( H_\alpha(M/N) = \frac{H_\alpha(M) + N}{N} \), for all \( \alpha < \beta \). If \( \beta = \alpha + 1 \) and \( x + N \in H_\beta(M/N) \), then \( d\left(\frac{(y + N)}{(x + N)}\right) = 1 \) for some \( y \in H_\alpha(M/N) = \frac{H_\alpha(M) + N}{N} \). Without loss of generality, we may assume that \( y \in H_\alpha(M) \) and \( x + N = y' + N \in \frac{H_\beta(M) + N}{N} \), where \( d\left(\frac{(y'N)}{(x'N)}\right) = 1 \). Otherwise, if \( \beta \) is a limit ordinal and we assume \( H^k(H_\beta(M/N)) \subseteq \frac{H_\beta(M) + N}{N} \) and \( x + N \in H^{k+1}(H_\beta(M/N)) \). If \( d\left(\frac{(x + N)}{(y + N)}\right) = k \), then \( y + N \in \text{Soc}(H_{\beta+k}(M/N)) = \frac{\text{Soc}(H_{\beta+k}(M)) + N}{N} \) as \( \beta + k < \lambda \). Therefore \( y + N = z + N \) such that \( z' \in H^{k+1}(H_\beta(M)) \), \( d\left(\frac{(z'N)}{(x'N)}\right) = k \) and \( (x - z') + N \in H^k(H_\beta(M/N)) \subseteq \frac{H_\beta(M) + N}{N} \). Therefore \( x - z' + N = u + N \) with \( u \in H_\beta(M) \) and \( x + N = (z' + u) + N \in \frac{H_\beta(M) + N}{N} \). \( \square \)
Lemma 3.2.3. Let $N \subseteq K$ be submodules of a QTAG-module $M$ such that $K/N$ is $h$-divisible. If $\text{Soc} \left( H_{\alpha} \left( \frac{M}{N} \right) \right) = \frac{\text{Soc}(H_{\alpha}(M)) + N}{N}$, then

$$\text{Soc} \left( H_{\alpha} \left( \frac{M}{K} \right) \right) = \frac{\text{Soc}(H_{\alpha}(M)) + K}{K}.$$ 

Proof. Since $K/N$ is $h$-divisible, it is a direct summand of $M/N$ and $\frac{M}{N} = \frac{K}{N} \oplus \frac{Q}{N}$, for some $Q \supseteq N$. For $x + K \in \text{Soc}(H_{\alpha}(M/K))$, $x = y + z$, $y \in Q$, $z \in N$ and we may define a homomorphism $f : M/K \rightarrow Q/N$ such that $f(x + K) = y + N$. Therefore $y + N \in \text{Soc}(H_{\alpha}(Q/N)) \subseteq \text{Soc}(H_{\alpha}(M/N)) = (\text{Soc}(H_{\alpha}(M)) + N)/N$ and there exists a $u \in \text{Soc}(H_{\alpha}(M))$ such that $y + N = u + N$. Therefore

$$x + K = y + K = u + K \in \frac{\text{Soc}(H_{\alpha}(M)) + K}{K}.$$ 

which completes the proof. \hfill \Box

Lemma 3.2.4. For $k < \omega$, let $K$ be a $H_{\alpha+k}(M)$-high submodule of $M$. Then $H^i(M) \subseteq H^i(K) + H_{\alpha}(M)$ for every $i \leq k + 1$.

Proof. Since $\text{Soc}(M) = \text{Soc}(K) \oplus \text{Soc}(H_{\alpha+k}(M)) \subseteq \text{Soc}(K) + H_{\alpha}(M)$, we suppose that $H^i(M) \subseteq H^i(K) + H_{\alpha}(M)$ for every $i \leq k$. Consider $x' \in H^{i+1}(M)$. Then there exists $x \in H^i(M)$ such that $d \left( \frac{x' R}{x R} \right) = i$ and $x = y + z'$ where $y \in \text{Soc}(K)$ and $d \left( \frac{z R}{z' R} \right) = k$ with $z \in H_{\alpha}(M)$. Now $y = x - z' \in H_i(M) \cap K = H_i(K)$, as $K$ is $h$-pure in $M$. Therefore $d \left( \frac{y' R}{y R} \right) = i$ for some $y' \in K$. Now if $d \left( \frac{z'' R}{z' R} \right) = i$, then $x - y' - z'' \in H^i(M) \subseteq H^i(K) + H_{\alpha}(M)$ which implies that $x \in H^{i+1}(K) + H_{\alpha}(M)$. \hfill \Box

The above discussion leads to the following:

Remark 3.2.2. For a submodule $N$ of a QTAG-module $M$, if $\text{Soc}(H_{\alpha}(M/N)) = \frac{\text{Soc}(H_{\alpha}(M)) + N}{N}$ for all $\alpha \leq \beta$, then $H_{\alpha}(M) \cap N = H_{\alpha}(N)$ for all $\alpha \leq \beta$. For $\beta \leq \omega$, the converse holds.

Lemma 3.2.5. If $N$ is a direct summand of a $H_{\alpha}(M)$-high submodule of $M$, then $\text{Soc}(H_{\gamma}(M/N)) = \frac{\text{Soc}(H_{\gamma}(M)) + N}{N}$, for all $\gamma \leq \alpha$. 

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Proof. Being a direct summand of a $h$-pure submodule of $M$, $N$ is also $h$-pure in $M$. Remark 3.2.2 suggests that for all $\gamma < \omega$, $\text{Soc}(H_\gamma(M/N)) = \frac{\text{Soc}(H_\alpha(M)) + N}{N}$. Without loss of generality, we may assume that $\alpha \geq \omega$. Let $\alpha = \beta + k$, $k < \omega$, $\beta$ a limit ordinal and $Q \oplus N$, a $H_\alpha(M)$-high submodule of $M$. Suppose the result holds for all ordinals $\delta < \alpha$. If $\omega \leq \delta < \alpha$ and if $T$ is a $H_\delta(N)$-high submodule of $N$, then $T$ is a direct summand of $H_\delta(M)$-high submodule of $M$. Therefore if $K$ is $H_\delta(Q)$-high in $Q$, then $T \oplus K$ is $H_\delta(M)$-high in $M$. Inductively, $\text{Soc}(H_\delta(M/T)) = \frac{\text{Soc}(H_\delta(M)) + T}{T}$. As $\frac{N}{T}$ is $h$-divisible by Corollary 3.2.1, $\frac{\text{Soc}(H_\delta(M)) + N}{N} = \text{Soc}(H_\delta(M/N))$ and the result holds for all $\gamma < \alpha$.

In order to prove that $\text{Soc}(H_\alpha(M/N)) = \frac{\text{Soc}(H_\alpha(M)) + N}{N}$, we have to show that $H^{k+1}(H_\beta(M/N)) = \frac{H^{k+1}(H_\beta(M)) + N}{N}$. Consider $x + N \in H^{k+1}(H_\beta(M))$. Since $N$ is $h$-pure, $x \in H^{k+1}(M)$ and by Lemma 3.2.4, $x = x_1 + x_2 + x_3$ where $x_1 \in H^{k+1}(N)$, $x_2 \in H^{k+1}(Q)$ and $x_3 \in H_\beta(M)$. As $\beta$ is a limit ordinal, for every ordinal $\gamma < \beta$, by Lemma 3.2.2, we may write $x + N = z + N$, where $z \in H_\gamma(M)$. Thus for each $\gamma < \beta$, we have $x = x_1 + x_2 + x_3 = z + z_1$ where $z_1 \in N$.

Now, $x_1 - z_1 + x_2 \in H_\gamma(M) \cap (N \oplus Q) = H_\gamma(N) \oplus H_\gamma(Q)$ and $x_2 \in \bigcap_{\gamma < \beta} H_\gamma(Q) = H_\beta(M)$ implying that $x + N = (x_2 + x_3) + N \in \frac{H^{k+1}(H_\beta(M)) + N}{N}$. \qed

We need some more lemmas to prove our main results:

**Lemma 3.2.6.** Let $K = \bigoplus_{i \in I} K_i$ be a submodule of a QTAG-module $M$ such that each $K_i$ is countably generated. If $N$ is a submodule of $M$, $N \cap K = \bigoplus_{i \in I'} K_i$, $I' \subset I$ and if $T/N$ is a countably generated submodule of $\frac{M}{N}$, then there is a submodule $Q \subseteq M$ such that $T \subseteq Q$, $\frac{Q}{N}$ is countably generated and $Q \cap K = \bigoplus_{i \in J} K_i$ for some subset $J$ of $I$. If $\frac{M}{K}$ and $\frac{N}{N \cap K}$ are $h$-divisible, then $M$ can be chosen so that $\frac{Q}{Q \cap K}$ is also $h$-divisible.

**Proof.** Since $\frac{T \cap K}{N \cap K}$ is countably generated, there is a countable subset $I''$ of $I$ such that $\bigoplus_{i \in I''} K_i$ contains a complete set of representatives of $\frac{T \cap K}{N \cap K}$. Set $Q = T + (\bigoplus_{i \in I''} K_i) = T + (\bigoplus_{i \in J} K_i)$, where $J = I' \cup I''$. 

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Now, suppose \( \frac{M}{K} \) and \( \frac{N}{N \cap K} \) are \( h \)-divisible. Consider a submodule \( L \supseteq N \) such that \( \frac{L}{N} \) is countably generated i.e., \( \frac{L}{N} = \sum (b_i + N)R \). Consider the elements \( x_1, x_2, \ldots, x_n, \ldots \) such that \( b_i - y_i \in K \) where \( d \left( \frac{x_iR}{y_iR} \right) = 1. \)

If \( P = L + \sum x_iR \), then \( \frac{P}{N} \) is countably generated and \( L \subseteq H_1(P) + K \). In this way we can choose two ascending sequences of submodules \( T \subseteq Q_1 \subseteq Q_2 \subseteq \ldots \subseteq Q_k \subseteq \ldots \) and \( P_1 \subseteq P_2 \subseteq \ldots \subseteq P_k \subseteq \ldots \), where \( \frac{M_k}{N} \) and \( \frac{P_k}{N} \) are countably generated for every \( k \), \( Q_k \cap K = \bigoplus_{i \in I_k} K_k \), \( Q_k \subseteq P_k \subseteq Q_{k+1} \) and \( Q_k \subseteq H_1(P_k) + K \) for each \( k \). If we put \( Q = \bigcup_{k=1}^\infty Q_k \), \( P = \bigcup_{k=1}^\infty P_k \) and \( J = \bigcup_{k=1}^\infty I_k \), then \( P_k \subseteq H_1(P_{k+1}) + K \) ensures that \( \frac{Q}{K \cap Q} \) is \( h \)-divisible.

**Lemma 3.2.7.** Let \( \{K_k\}_{k=1}^\infty \) be a sequence of submodules of a \( QTAG \)-module \( M \) and \( K_k = \bigoplus_{i \in I_k} K_k^{(i)} \), where each \( K_k^{(i)} \) is countably generated. Let \( A \) be a set of positive integers such that \( M/K_k \) is \( h \)-divisible, for every \( k \in A \). If \( N \) is a submodule of \( M \) such that \( N \cap K_k = \bigoplus_{i \in I_k} K_k^{(i)} \) for each \( k \) and \( \frac{N}{N \cap K_k} \) is \( h \)-divisible, whenever \( k \in A \) and if \( \frac{T}{N} \) is a countably generated submodule of \( \frac{M}{N} \), then there is a submodule \( Q \) of \( M \) such that \( T \subseteq Q \), \( \frac{Q}{N} \) is countably generated, \( Q \cap K_k = \bigoplus_{i \in I_k} K_k^{(i)} \) and \( \frac{Q}{Q \cap K_k} \) is \( h \)-divisible, whenever \( k \in A \).

**Proof.** Let \( n_1, n_2, \ldots \) be a sequence of positive integers such that for every pair of positive integers \( k \) and \( n \) there is an integer \( j > k \) such that \( n_j = n \). By Lemma 3.2.6, we may construct a chain of submodules of \( M \) containing \( T, Q_1 \subseteq Q_2 \subseteq \ldots \subseteq Q_k \subseteq \ldots \) such that for each \( k \), \( \frac{Q_k}{N} \) is countably generated. \( Q_k \cap K_{n_k} = \bigoplus_{i \in I_{n_k}} K_{n_k}^{(i)} \) and \( \frac{Q_k}{Q_k \cap K_{n_k}} \) is \( h \)-divisible, whenever \( n_k \in A \). Set \( Q = \bigcup_{k=1}^\infty Q_k \) and for each \( n \), let \( J_n \) be the union of all \( I_{n_k} \) for which \( n_k = n \). Since for each \( n \), \( Q \) is the union of those \( Q_k \) such that \( n_k = n \), \( Q \) is the required submodule. \( \square \)

**Lemma 3.2.8.** Let \( N \) be a submodule of \( M \) such that \( \text{Soc} \left( H_\alpha \left( \frac{M}{N} \right) \right) = \frac{\text{Soc}(H_\alpha(M)) + N}{N} \) for all \( \alpha < \lambda \). If \( \text{Soc}(M) = \text{Soc}(N) \oplus T \), where \( T \) is a summable subsocle such that \( T \cap H_\lambda(M) = 0 \), then \( \frac{M}{N} \) is a summable module of length at most \( \lambda \).
Proof. Let \( T = \bigoplus_{\alpha<\lambda} T_\alpha \), where \( S_\alpha \subseteq H_\alpha(M) \), \( S_\alpha \cap H_{\alpha+1}(M) = 0 \), for every \( \alpha \). Then

\[
\text{Soc}(M/N) = \frac{\text{Soc}(M) + N}{N} = \frac{T \oplus N}{N} = \bigoplus_{\alpha<\lambda} \left( \frac{T_\alpha \oplus N}{N} \right).
\]

We have to show that for \( x + N(\neq 0) \in \frac{T_\alpha + N}{N} \), \( H_{M/N}(x + N) = \alpha \). Since \( \text{Soc}(M) = \text{Soc}(N) \oplus T \), \( \text{Soc}(H_\alpha(M)) = (\text{Soc}(N) \cap H_\alpha(M)) \oplus (T \cap H_\alpha(M)) = (\text{Soc}(N) \cap H_\alpha(M)) \oplus (\bigoplus_{\beta \geq \alpha} S_\beta) \). Therefore

\[
\text{Soc}(H_\alpha(M/N)) = \frac{\text{Soc}(H_\alpha(M)) + N}{N} = \left( \frac{T \cap H_\alpha(M) + N}{N} \right) / \bigoplus_{\beta \geq \alpha} \left( \frac{T_\beta \oplus N}{N} \right)
\]

\[
= (\frac{T \cap H_\alpha(M) + N}{N}) / \bigoplus_{\beta \geq \alpha} \left( \frac{T_\beta \oplus N}{N} \right)
\]

\[
= (\frac{T_\alpha \oplus N}{N}) \oplus \bigoplus_{\beta \geq \alpha+1} \left( \frac{T_\beta \oplus N}{N} \right)
\]

\[
= (\frac{T_\alpha \oplus N}{N}) \oplus \text{Soc}(H_{\alpha+1}(M/N)).
\]

and we are done. \( \square \)

Now we are able to prove one of our main results.

**Theorem 3.2.2.** Let \( M \) be a QTAG-module of length \( \sigma \), where \( \sigma \) is a countable limit ordinal. Then \( M \) is a direct sum of countably generated modules if \( M \) is summable and for each \( \rho < \sigma \), \( M \) contains a \( H_\rho(M) \)-high submodule which is a direct sum of countably generated modules.

**Proof.** For an ordinal \( \rho \) such that \( \omega \leq \rho < \sigma \), let \( K_\rho \) be a \( H_\rho(M) \)-high submodule of \( M \) which is a direct sum of countably generated submodules. Put \( K_\rho = \bigoplus_{i \in I_\rho} K_\rho^i \) such that each \( K_\rho^i \) is countably generated and \( \text{Soc}(M) = \bigoplus_{i \in I} T_i \), where each \( T_i \) is countably generated. Now the elements of the minimal generating set of \( M \), \( \{x_\alpha\} \) can be well ordered \( \alpha < \delta \). Since \( \sigma \) is countable, \( K_\rho \)'s can be enumerated and by Lemma 3.2.7, we may define submodules \( \{N_\alpha\} \) satisfying the following conditions:

(i) \( N_0 = 0, N_\alpha \subseteq N_\beta \) for every \( \alpha < \beta \) and \( N_\alpha = \bigcup_{\beta < \alpha} N_\beta \), if \( \alpha \) is a limit ordinal;

(ii) \( x_\alpha \in N_{\alpha+1}; \)
(iii) $N_\alpha \cap K_\rho = \bigoplus_{i \in I_\rho^\alpha} K_\rho^i$, for each $\alpha$;
(iv) $N_{\alpha + 1}/N_\alpha$ is countably generated;
(v) $N_\alpha/(N_\alpha \cap K_\rho)$ is $h$-divisible, for each $\alpha$;
(vi) $\text{Soc}(N_\alpha) = \bigoplus_{i \in I_\rho} T_i$.

By Lemma 3.2.5,

$$\text{Soc}(H_\rho(M/N_\alpha)) = \frac{\text{Soc}(H_\rho(M)) + N_\alpha}{N_\alpha},$$

for all $\rho < \sigma$.

Therefore by Remark 3.2.2, each $N_\alpha$ is isotype and again by Lemma 3.2.2,

$$H_\rho(M/N_\alpha) = \frac{H_\rho(M) + N_\alpha}{N_\alpha},$$

for all $\rho < \sigma$.

Now we infer that

$$H_\rho(N_{\alpha + 1}/N_\alpha) = \frac{H_\rho(N_{\alpha + 1}) + N_\alpha}{N_\alpha},$$

for all $\rho < \sigma$.

By Lemma 3.2.8, $\frac{N_{\alpha + 1}}{N_\alpha}$ has length at most $\sigma$ and

$$H_\rho(N_{\alpha + 1}) \cap N_\alpha = (H_\rho(M) \cap N_{\alpha + 1}) \cap N_\alpha = H_\rho(M) \cap N_\alpha = H_\rho(N_\alpha),$$

for all $\rho \leq \sigma$.

Now using Theorem 3.2.1, we get $N_{\alpha + 1} = N_\alpha + Q_\alpha$, for each $\alpha < \delta$. Consequently, $M = \bigoplus_{\alpha} Q_\alpha$, $\alpha < \delta$ is a direct sum of countably generated modules.

To prove some other interesting results, we need some more lemmas.

**Lemma 3.2.9.** If $\rho$ is a countable ordinal and $x \in H_\rho(M)$, then there is a countably generated submodule $N$ of $M$ such that $x \in H_\rho(N)$.

**Proof.** The statement holds if $\rho = 0$. We shall prove the result inductively by assuming that it holds for all ordinals $\sigma < \rho$. If $\rho = \sigma + 1$, then there exists an element $y \in H_\rho(M)$ such that $d\left(\frac{yR}{xR}\right) = 1$. By assumption, there is a countably generated submodule $N$ such that $y \in H_\sigma(N)$. Now $x \in H_{\sigma + 1}(N) = H_\rho(N)$ as $d\left(\frac{yR}{xR}\right) = 1$. Otherwise, if $\rho$ is a limit ordinal, then $x \in H_\rho(M)$ for all $\sigma < \rho$. Inductively, there exists a countable submodule $N_\sigma$ for each $\sigma < \rho$, such that $x \in H_\sigma(N_\sigma)$. If $N = \Sigma N_\sigma$, then $N$ is countably generated and $x \in \bigcap_{\sigma < \rho} H_\sigma(N_\rho) \subseteq \bigcap_{\sigma < \rho} H_\sigma(N) = H_\rho(N)$. \qed
Lemma 3.2.10. Let $\sigma$ be a countable ordinal and $N$ a submodule of a $QTAG$-module $M$ such that $H_\sigma(M) \cap N = H_\sigma(N)$. If $T/N$ is a countably generated submodule of $M/N$, then there exists a submodule $K \subseteq M$ such that $T \subseteq K$, $K/N$ is countably generated and $H_\sigma(M) \cap K = H_\sigma(K)$.

Proof. As
\[
\frac{T \cap H_\sigma(M)}{H_\sigma(N)} \cong \frac{T \cap H_\sigma(N)}{N} \subseteq \frac{T}{N},
\]
we may select elements $x_1, x_2, \ldots, x_k, \ldots$ such that $\frac{T \cap H_\sigma(M)}{H_\sigma(N)} = \Sigma x_i R$. By Lemma 3.2.9, for each $k$, there is a countably generated submodule $N_k$ such that $x_k \in H_\sigma(N_k)$. Put $K_1 = T + \Sigma N_k$. Now $\frac{K_1}{N}$ is countably generated and $T \cap H_\sigma(M) \subseteq H_\sigma(K_1)$. On repeating this process, we may construct a chain of submodules $K_1 \subseteq K_2 \subseteq \ldots$ such that $\frac{K_k}{N}$ is countably generated and $K_k \cap H_\sigma(M) \subseteq H_\sigma(K_{k+1})$. Now $K = \bigcup_{k=1}^{\infty} K_k$ is the required module. □

Lemma 3.2.11. Let $\sigma$ be a countable ordinal and $T$ a submodule of $M$ such that $T = \bigoplus_{i \in I} T_i, \quad \frac{M}{T} = \bigoplus_{j \in J} Q_j$ and each $T_i$ and $Q_j$ is countably generated. Suppose $N$ is a submodule of $M$ such that $N \cap T = \bigoplus_{i \in I'} T_i, \quad \frac{N+T}{T} = \bigoplus_{j \in J'} Q_j$ and $H_\sigma(M) \cap N = H_\sigma(N)$. If $\frac{Q}{N}$ is a countably generated submodule of $\frac{M}{N}$, then there exists a submodule $P$ of $M$ such that $Q \subseteq P$, $\frac{P}{N}$ is countably generated, $P \cap T = \bigoplus_{i \in I''} T_i, \quad \frac{P+T}{T} = \bigoplus_{j \in J''} Q_j$ and $H_\sigma(M) \cap P = H_\sigma(P)$.

Proof. Since $Q/N$ is a countably generated module, so $\frac{Q+T}{N+T}$ is also countably generated. Therefore there exists a subset $J' \subseteq J$ such that $\bigoplus_{j \in J'} Q_j$ contains a set of representatives of $\left( \frac{Q+T}{T} \right) / \left( \frac{N+T}{T} \right)$. If $L$ is generated by a complete set of representatives of $\bigoplus_{j \in J'} Q_j$, then $L$ is a countably generated submodule of $M$ such that $\frac{L+T}{T} = \bigoplus_{j \in J'} Q_j$. If $P_1 = Q + L$ and $J_1 = J' \cup \bar{J}$, then $\frac{P_1}{N}$ is countably generated and $\frac{P_1+T}{T} = \bigoplus_{j \in J_1} Q_j$. Now by Lemma 3.2.10 and Lemma 3.2.6, there exist ascending chains of submodules $P_1 \subseteq P_2 \subseteq \ldots \subseteq P_k \subseteq \ldots; L_1 \subseteq L_2 \subseteq \ldots \subseteq L_k \subseteq \ldots$ and $K_1 \subseteq K_2 \subseteq \ldots \subseteq K_k \subseteq \ldots$ such that $N \subseteq P_k \subseteq L_k \subseteq K_k \subseteq P_{k+1}$, $P_k/N$ is countably generated and $K_k \cap H_\sigma(M) \subseteq H_\sigma(K_{k+1})$. Therefore $K_k \cap H_\sigma(M) \subseteq H_\sigma(P_{k+1})$. □
generated for each $k$, $\frac{P_k + T}{T} = \bigoplus_{j \in J_k} Q_j$, $L_k \cap T = \bigoplus_{i \in I_k} T_i$ and $H_\sigma(M) \cap K_k = H_\sigma(K_k)$.

Now $P = \bigcup_{k=1}^\infty P_k$, $I'' = \bigcup_{k=1}^\infty I_k$ and $J'' = \bigcup_{k=1}^\infty J_k$ and we are done. \qed

Now we are able to prove the following result.

**Theorem 3.2.3.** If $\sigma$ is a countable ordinal and if $M$ is a $h$-reduced QTAG-module such that $M/H_\sigma(M)$ and $H_\sigma(M)$ are direct sums of countably generated modules, then $M$ is a direct sum of countably generated modules.

**Proof.** Let $\frac{M}{H_\sigma(M)} = \bigoplus_{j \in J} Q_j$ and $H_\sigma(M) = \bigoplus_{i \in I} P_i$, where each $Q_j$ and $P_i$ is countably generated. We may define a well ordering relation on the set of generators of $M$, $\{x_\alpha\}$, $\alpha < \beta$. By Lemma 3.2.11, we may construct a well ordered family of the submodules $\{N_\alpha\}$ of $M$ satisfying the following conditions:

(i) $N_0 = 0$, $N_\alpha \subseteq N_\beta$ if $\alpha < \beta$ and $N_\alpha = \bigcup_{\beta < \alpha} N_\beta$, if $\alpha$ is a limit ordinal;

(ii) $x_\alpha \in N_{\alpha+1}$;

(iii) $N_\alpha \cap H_\sigma(M) = \bigoplus_{i \in I_\alpha} T_i$;

(iv) $\frac{N_\alpha + H_\sigma(M)}{H_\sigma(M)} = \bigoplus_{j \in J_\alpha} Q_j$;

(v) $H_\sigma(M) \cap N_\alpha = H_\sigma(N_\alpha)$;

(vi) $N_{\alpha+1}/N_\alpha$ is countably generated.

Now $H_\sigma(N_{\alpha+1}) \cap N_\alpha = H_\sigma(N_\alpha)$ and $H_\sigma(N_\alpha) = \bigoplus_{i \in I_\alpha} T_i$, is a direct summand of $H_\sigma(N_{\alpha+1}) = \bigoplus_{i \in I_{\alpha+1}} T_i$. If we define the canonical map $f : \frac{N_{\alpha+1} + H_\sigma(M)}{H_\sigma(M)} \to \frac{N_{\alpha+1}}{H_\sigma(N_{\alpha+1})}$, then $f$ is an isomorphism and $f \left( \frac{N_\alpha + H_\sigma(M)}{H_\sigma(M)} \right) = \frac{N_\alpha + H_\sigma(N_{\alpha+1})}{H_\sigma(N_{\alpha+1})}$. As $\frac{N_{\alpha+1} + H_\sigma(M)}{H_\sigma(N_{\alpha+1})}$ is a direct summand of $\frac{N_{\alpha+1}}{H_\sigma(N_{\alpha+1})}$ and all the conditions of Corollary 3.2.1 are satisfied, therefore for each $\alpha < \beta$ we have $N_{\alpha+1} = N_\alpha \oplus N'_\alpha$. Thus $M = \bigoplus_{\alpha < \beta} N'_\alpha$ is a direct sum of countably generated modules. \qed

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Since $h$-pure, isotype and high submodules are very significant in the study of $QTAG$-modules. We prove some results related to these submodules.

**Proposition 3.2.1.** For a countable ordinal $\sigma$ and a $QTAG$-module $M$, if $M/H_\sigma(M)$ is a direct sum of countably generated modules, then every $H_\sigma(M)$-high submodule of $M$ is a direct sum of countably generated submodules.

**Proof.** Suppose the statement holds for all the ordinal less than $\sigma$ and $\sigma \geq \omega$. Now we put $\sigma = \rho + k$, where $\rho$ is a limit ordinal and $k < \omega$. Let $N$ be an $H_\sigma(M)$-high submodule of $M$. Then by Lemma 3.2.5 and Lemma 3.2.2,

$$M/N = H_\sigma(M/N) = \frac{H_\sigma(M) + N}{N} \text{ for all } \alpha < \rho.$$ 

Therefore,

$$\frac{N}{H_\alpha(N)} \cong \frac{H_\alpha(M) + N}{H_\alpha(M)} = \frac{M}{H_\alpha(M)} \cong \frac{(M/H_\alpha(M))}{H_\alpha(M/H_\alpha(M))}$$

is a direct sum of countably generated submodules for all $\alpha < \rho$. Now, by induction, every $H_\alpha(M)$-high submodule of $N/H_\rho(N)$ is a direct sum of countably generated submodules. Since $\frac{N + H_\rho(M)}{H_\rho(M)} \cong \frac{N}{H_\rho(N)}$ and $\frac{N + H_\rho(M)}{H_\rho(M)}$ is isotype in the summable module $M/H_\rho(M)$, $N/H_\rho(N)$ is also summable.

Now by Theorem 3.2.2, $N/H_\rho(N)$ is a direct sum of countably generated modules, therefore by Theorem 3.2.3, $N$ is a direct sum of countably generated modules. \qed

**Theorem 3.2.4.** Let $N$ be an isotype submodule of a $QTAG$-module $M$ of countable length and $M$, a direct sum of countably generated $h$-reduced $QTAG$-modules. Then $N$ is also a direct sum of countably generated modules.

**Proof.** Let $\sigma$ be the length of $N$. As $\frac{N + H_\sigma(M)}{H_\sigma(M)}$ is isotype in $\frac{M}{H_\sigma(M)}$, without loss of generality, we assume that the length of $M$ is also $\sigma$. Suppose the statement holds for all lengths $\rho < \sigma$. Now $\frac{N + H_\rho(M)}{H_\rho(M)}$ is isotype in $\frac{M}{H_\rho(M)}$ for all $\rho < \sigma$. Inductively,

$$\frac{N}{H_\rho(N)} \cong \frac{N + H_\rho(M)}{H_\rho(M)}$$

is a direct sum of countably generated modules for all $\rho < \alpha$. If $\sigma = \rho + 1$, then $\frac{N}{H_\rho(N)}$ and $H_\rho(N)$ are the direct sums of countably generated modules and the result follows from Theorem 3.2.3.
Otherwise, if \( \sigma \) is a limit ordinal, \( \frac{N}{H_\rho(N)} \) is a direct sum of countably generated submodules for all \( \rho < \sigma \) and \( N \) is summable as it is isotype in the summable module \( M \) of countable length. Therefore, \( N \) is a direct sum of countably generated modules. \( \square \)

### 3.3 Some Applications

In this section we establish some interesting results using the results of section two. We start with the following lemma:

**Lemma 3.3.1.** Let \( N, K \) be \( H_\rho(M) \)-high submodules of \( M \) where \( \sigma = \rho + n, \ n < \omega \). If \( x + H_\rho(N) \to y + H_\rho(K) \) if and only if \( x - y \in H_\rho(M) \), then this mapping is a height preserving isomorphism from \( \text{Soc}(N/H_\rho(N)) \) to \( \text{Soc}(K/H_\rho(K)) \).

**Proof.** Let \( x \in N \) and \( x' \in H_\rho(N) \) such that \( d\left(\frac{xR}{x'R}\right) = 1 \). Suppose \( x \notin K \), then \( (K + xR) \cap H_\sigma(M) \neq 0 \) and therefore there exists \( y' \in K \) and \( z \in H_\rho(M) \) such that \( y' + x'' = z' \neq 0 \). Here \( d\left(\frac{xR}{x''R}\right) = i, d\left(\frac{zR}{z'R}\right) = n \). Since \( K \cap H_\sigma(M) = 0 \), \( i \leq n \) and there is a \( y'' \in K \) such that \( d\left(\frac{y''R}{y'R}\right) = i \). By Lemma 3.2.4, \( y'' + x + z'' \in H^i(M) \subseteq H^i(K) + H_\rho(M) \), where \( d\left(\frac{z''R}{z'R}\right) = i \). Now \( x = (u - y'') + z'' + v \) for some \( u \in H^i(K) \) and \( v \in H_\rho(M) \). Put \( y = u - y'' \). Now the map \( x + H_\rho(N) \to y + H_\rho(K) \) is a height preserving isomorphism. \( \square \)

**Theorem 3.3.1.** If a \( H_\sigma(M) \)-high submodule of a QTAG-module \( M \) is a direct sum of countably generated modules, then every \( H_\sigma(M) \)-high submodule of \( M \) is a direct sum of countably generated submodules.

**Proof.** We may assume that \( \sigma \) is countable and \( \sigma \geq \omega \). Now \( \sigma = \rho + n \), where \( \rho \) is a limit ordinal and \( n < \omega \). Let \( N, K \) be two \( H_\sigma(M) \)-high submodules of \( M \) and \( N \) a direct sum of countably generated submodules. Now \( N/H_\rho(N) \) is also a direct sum of countably generated modules and by Lemma 3.3.1, \( K/H_\rho(K) \) is summable. As \( \rho \) is a limit ordinal, by Proposition 3.2.1, we have \( N/H_\alpha(N) \cong M/H_\alpha(M) \cong K/H_\alpha(K) \) for all \( \alpha < \rho \). Therefore \( K/H_\rho(K) \) is a direct sum of countably generated modules.
Now $H_\rho(K)$ is bounded, therefore by Theorem 3.2.3, $K$ is a direct sum of countably generated modules.

**Proposition 3.3.1.** For a $QTAG$-module $M$, if $M/H_\rho(M)$ is summable for all $\rho \leq \sigma$, where $\sigma$ is countable then $M/H_\sigma(M)$ is a direct sum of countably generated modules.

*Proof.* We assume that the result holds for all ordinals less than $\sigma$. If $\sigma = \alpha + 1$, then by induction we may say that $M/H_\alpha(M)$ is a direct sum of countably generated modules. Since
\[
\frac{M/H_\alpha(M)}{H_\alpha(M/H_\sigma(M))} \cong \frac{M}{H_\alpha(M)}
\]
and $H_\alpha(M/H_\alpha(M))$ are direct sums of countably generated modules, the result follow from Theorem 3.2.3. Otherwise, if $\sigma$ is a limit ordinal, then by assumption
\[
\frac{M/H_\sigma(M)}{H_\rho(M/H_\sigma(M))} \cong \frac{M}{H_\rho(M)}
\]
is a direct sum of countably generated modules for each $\rho < \sigma$. Since $\frac{M}{H_\sigma(M)}$ is summable, $\frac{M}{H_\sigma(M)}$ is a direct sum of countably generated modules.

An immediate consequence of this proposition may be stated as follows:

**Corollary 3.3.1.** Let $\sigma$ be a countable limit ordinal and let $M$ be a $QTAG$-module of length $\sigma$. Then the following conditions are equivalent:

(i) $M$ is a direct sum of countably generated modules;

(ii) $M$ is summable and $\frac{M}{H_\rho(M)}$ is a direct sum of countably generated modules for all $\rho < \sigma$;

(iii) $\frac{M}{H_\rho(M)}$ is summable for all $\rho \leq \sigma$;

(iv) $M$ is summable and for each $\rho < \sigma$, the $H_\rho(M)$-high submodules of $M$ are direct sums of countably generated modules.

$Ulm$ factors play a very important role in the study of $QTAG$-modules. In order to establish our next result involving $Ulm$ factors we need the following lemma:
Lemma 3.3.2. Let $\sigma$ be a countable ordinal. Then for a QTAG-module $M$, $M/H_\rho(M)$ is a direct sum of countably generated modules for all $\rho < \omega \sigma$ if and only if $\frac{M}{M^\alpha}$ is a direct sum of countably generated modules for all $\alpha < \sigma$. Here $M^\alpha$ is the $\alpha^{th}$ Ulm submodule of $M$.

Proof. If $\rho < \omega \sigma$, then $\rho = \omega \alpha + n$, $n < \omega$, $\alpha < \sigma$. Now we have

$$\frac{M}{M^\alpha} = \frac{M}{H_{\omega \alpha}(M)} \cong \frac{(M/H_\rho(M))}{H_{\omega \alpha}(M)/H_\rho(M)} = \frac{M/H_\rho(M)}{H_{\omega \alpha}(M/H_\rho(M))}.$$ 

Therefore, $H_{\omega \alpha} \left( \frac{M}{H_\rho(M)} \right)$ is bounded and if $\frac{M}{M^\alpha}$ is a direct sum of countably generated modules, then by Theorem 3.2.3, $\frac{M}{H_\rho(M)}$ is a direct sum of countably generated modules.

Theorem 3.3.2. Let $M$ be a QTAG-module of countable length. Then $M$ is a direct sum of countably generated modules if and only if

(i) all Ulm factors of $M$ are direct sums of uniserial modules and

(ii) $M/M^\rho$ is summable, for all limit ordinals $\rho$.

Proof. The second condition implies that $M$ is summable. Now, $M$ will be a direct sum of countably generated modules if $M/H_\tau(M)$ is a direct sum of countably generated modules for all $\tau$ less than $\sigma$, the length of $M$. Assume that $\rho < \sigma$ and $\frac{M}{M^\tau}$ is a direct sum of countably generated modules for all $\tau < \rho$. In the light of Lemma 3.3.2, we have to show that $\frac{M}{M^\rho}$ is a direct sum of countably generated modules for all $\rho < \sigma$. If $\rho = \alpha + 1$, then

$$\frac{(M/M^\rho)}{H_{\omega \alpha}(M/M^\rho)} = \frac{(M/M^\rho)}{M^\alpha/M^\rho} \cong \frac{M}{M^\alpha}$$

is a direct sum of countably generated modules by induction. Now

$$H_{\omega \alpha} \left( \frac{M}{M^\rho} \right) = \frac{M^\alpha}{M^\rho} = \frac{M^\alpha}{M^{\alpha+1}}$$

is a direct sum of uniserial modules by (i). Therefore by applying Theorem 3.2.3, $\frac{M}{M^\rho}$ is a direct sum of countably generated modules. Otherwise, if $\rho$ is a limit ordinal, then $\frac{M}{M^\rho} = \frac{M}{H_{\omega \rho}(M)}$ is summable by (ii). Inductively, $\frac{M}{M^\alpha}$ is a direct sum of countably generated modules.
generated modules for all $\alpha < \rho$. By Lemma 3.3.2, $M/H_{\tau}(M)$ is a direct sum of countably generated modules for all $\tau < \omega \rho$. Thus $\frac{M}{H_{\omega \rho}(M)} = \frac{M}{M^{\rho}}$ is a direct sum of countably generated modules.

As an immediate consequence of this result, we can say that a summable $QTAG$-module $M$ of length $\omega$ is a direct sum of countably generated modules if and only if all of its $Ulm$ factors are direct sums of uniserial modules.