Chapter 3
λ-SEQUENCE SPACES OF NON-ABSOLUTE TYPE
In the present chapter, we introduce the notions of $\lambda$-convergence and boundedness. Further, we define the $\lambda$-sequence spaces of non-absolute type. Moreover, we establish some inclusion relations via the idea of matrix transformations. The most important materials of this chapter can be found in [85, 86, 89] and [90].

### 3.1 Notions of $\lambda$-convergence and boundedness

Throughout this chapter, let $\lambda = (\lambda_k)_{k=0}^\infty$ be a strictly increasing sequence of positive reals tending to infinity, that is

$$0 < \lambda_0 < \lambda_1 < \cdots \text{ and } \lambda_k \to \infty \text{ as } k \to \infty. \quad (3.1.1)$$

We say that a sequence $x = (x_k) \in w$ is $\lambda$-convergent to the number $l \in \mathbb{C}$, called as the $\lambda$-limit of $x$, if $\Lambda_n(x) \to l$ as $n \to \infty$, where

$$\Lambda_n(x) = \frac{1}{\lambda_n} \sum_{k=0}^n (\lambda_k - \lambda_{k-1})x_k; \quad (n \in \mathbb{N}). \quad (3.1.2)$$

In particular, we say that $x$ is a $\lambda$-null sequence if $\Lambda_n(x) \to 0$ as $n \to \infty$, i.e., if $x$ is $\lambda$-convergent to naught. Further, we say that $x$ is $\lambda$-bounded if $\sup_n |\Lambda_n(x)| < \infty$.

Here and in the sequel, we shall use the convention that any term with a negative subscript is equal to zero, e.g. $\lambda_{-1} = 0$ and $x_{-1} = 0$.

Now, it is known by [79, p.319] that if $\lim_{n \to \infty} x_n = a$ in the ordinary sense of convergence, then

$$\lim_{n \to \infty} \left( \frac{1}{\lambda_n} \sum_{k=0}^n (\lambda_k - \lambda_{k-1})|x_k - a| \right) = 0.$$

This implies that

$$\lim_{n \to \infty} |\Lambda_n(x) - a| = \lim_{n \to \infty} \left| \frac{1}{\lambda_n} \sum_{k=0}^n (\lambda_k - \lambda_{k-1})(x_k - a) \right| = 0$$

which yields that $\lim_{n \to \infty} \Lambda_n(x) = a$ and hence $x$ is $\lambda$-convergent to $a$. We therefore deduce that the ordinary convergence implies the $\lambda$-convergence to the same limit. This leads us to the following basic result:

**Lemma 3.1.1** Every convergent sequence is $\lambda$-convergent to the same ordinary limit.

We shall later show that the converse implication need not be true. Before that, the following result is immediate by Lemma 3.1.1.

**Lemma 3.1.2** If a $\lambda$-convergent sequence converges in the ordinary sense, then it must converge to the same $\lambda$-limit.
Now, let \( n \geq 1 \). Then, by using (3.1.2), we derive that
\[
x_n - \Lambda_n(x) = \frac{1}{\lambda_n} \sum_{i=0}^{n} (\lambda_i - \lambda_{i-1})(x_n - x_i)
\]
\[
= \frac{1}{\lambda_n} \sum_{i=0}^{n-1} (\lambda_i - \lambda_{i-1})(x_n - x_i)
\]
\[
= \frac{1}{\lambda_n} \sum_{i=0}^{n-1} (\lambda_i - \lambda_{i-1}) \sum_{k=i+1}^{n} (x_k - x_{k-1})
\]
\[
= \frac{1}{\lambda_n} \sum_{k=1}^{n} (x_k - x_{k-1}) \sum_{i=0}^{k-1} (\lambda_i - \lambda_{i-1})
\]
\[
= \frac{1}{\lambda_n} \sum_{k=1}^{n} \lambda_{k-1}(x_k - x_{k-1}).
\]

Therefore, we have for every \( x = (x_k) \in w \) that
\[
x_n - \Lambda_n(x) = S_n(x); \quad (n \in \mathbb{N}),
\]
where the sequence \( S(x) = (S_n(x))_{n=0}^{\infty} \) is defined by
\[
S_0(x) = 0 \text{ and } S_n(x) = \frac{1}{\lambda_n} \sum_{k=1}^{n} \lambda_{k-1}(x_k - x_{k-1}) \text{ for } n \geq 1.
\]

Thus, the following result is obtained from Lemma 3.1.2 by using (3.1.3).

**Lemma 3.1.3**  A \( \lambda \)-convergent sequence \( x \) converges in the ordinary sense if and only if \( S(x) \in c_0 \).

Similarly, the following results can easily be proved.

**Lemma 3.1.4**  Every bounded sequence is \( \lambda \)-bounded.

**Lemma 3.1.5**  A \( \lambda \)-bounded sequence \( x \) is bounded in the ordinary sense if and only if \( S(x) \in \ell_\infty \).

Finally, we define the infinite matrix \( \Lambda = (\lambda_{nk})_{n,k=0}^{\infty} \) by
\[
\lambda_{nk} = \begin{cases} 
\frac{\lambda_k - \lambda_{k-1}}{\lambda_n}; & (0 \leq k \leq n), \\
0; & (k > n)
\end{cases}
\]

for all \( n, k \in \mathbb{N} \). Then, the \( \Lambda \)-transform of \( x \in w \) is the sequence \( \Lambda(x) = (\Lambda_n(x))_{n=0}^{\infty} \), where \( \Lambda_n(x) \) is given by (3.1.2) for every \( n \in \mathbb{N} \). Thus, the sequence \( x \) is \( \lambda \)-convergent if and only if \( x \) is \( \Lambda \)-summable. Further, if \( x \) is \( \lambda \)-convergent then the \( \lambda \)-limit of \( x \) is nothing but the \( \Lambda \)-limit of \( x \). Moreover, it is obvious that our matrix \( \Lambda \) is a triangle. Also, it follows by Lemma 3.1.1 that the method \( \Lambda \) is regular.
Remak 3.1.6 We may note by (2.1.2) that the matrix Λ is the special case \( r = \lambda, s = e \) and \( t = \Delta \lambda \) of the matrix \( \tilde{A}(r, s, t) \) of generalized means, where \( \Delta \lambda = (\lambda_k - \lambda_{k-1})_{k=0}^{\infty} \).

Remak 3.1.7 By taking \( q_k = \lambda_k - \lambda_{k-1} \) for all \( k \), the matrix Λ is the special case \( Q_n \to \infty (n \to \infty) \) of the matrix \( (\tilde{N}, q) \) of weighted means, where \( Q_n = \sum_{k=0}^{n} q_k = \lambda_n \) for all \( n \) (see [48, 69]). On the other hand, the matrix Λ is reduced, in the special case \( \lambda_k = k + 1 (k \in \mathbb{N}) \), to the Cesàro matrix \( C_1 \) of arithmetic means [95, 105].

### 3.2 The sequence spaces \( \ell^\lambda_\infty \), \( c^\lambda \) and \( c^\lambda_0 \)

We define the sequence spaces \( \ell^\lambda_\infty \), \( c^\lambda \) and \( c^\lambda_0 \) as the sets of all \( \lambda \)-bounded, \( \lambda \)-convergent and \( \lambda \)-null sequences, respectively; that is

- \( \ell^\lambda_\infty = \left\{ x = (x_k) \in w : \sup_n \left| \frac{1}{\lambda_n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1}) x_k \right| < \infty \right\} \)
- \( c^\lambda = \left\{ x = (x_k) \in w : \lim_{n \to \infty} \left( \frac{1}{\lambda_n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1}) x_k \right) \text{ exists} \right\} \)
- \( c^\lambda_0 = \left\{ x = (x_k) \in w : \lim_{n \to \infty} \left( \frac{1}{\lambda_n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1}) x_k \right) = 0 \right\} \)

With the notation of (1.2.4), we may redefine the spaces \( \ell^\lambda_\infty \), \( c^\lambda \) and \( c^\lambda_0 \) as the matrix domains of the triangle Λ in the spaces \( \ell_\infty \), \( c \) and \( c_0 \), respectively; that is

\[ \ell^\lambda_\infty = (\ell_\infty)_\Lambda, \quad c^\lambda = c_\Lambda \quad \text{and} \quad c^\lambda_0 = (c_0)_\Lambda. \quad (3.2.1) \]

Then, it is obvious by Remark 3.1.6 that the spaces \( \ell^\lambda_\infty \), \( c^\lambda \) and \( c^\lambda_0 \) are special cases of the spaces \( \ell_\infty(r, s, t) \), \( c(r, s, t) \) and \( c_0(r, s, t) \) of generalized means, respectively. Therefore, it follows by Corollary 2.2.4 (a) that \( \ell^\lambda_\infty \), \( c^\lambda \) and \( c^\lambda_0 \) are BK spaces with the same norm given by

\[ \| x \|_{\ell^\lambda_\infty} = \| \Lambda(x) \|_{\ell_\infty} = \sup_{n} | \Lambda_n(x) |. \quad (3.2.2) \]

Remak 3.2.1 It can easily be seen that the absolute property does not hold on the spaces \( \ell^\lambda_\infty \), \( c^\lambda \) and \( c^\lambda_0 \), that is \( \| x \|_{\ell^\lambda_\infty} \neq \| \| x \|_{\ell^\lambda_\infty} \) for at least one sequence \( x \) in each of these spaces, where \( | x | = (| x_k |)_{k=0}^{\infty} \). This means that \( \ell^\lambda_\infty \), \( c^\lambda \) and \( c^\lambda_0 \) are BK spaces of non-absolute type.

Remak 3.2.2 We may note by Remark 3.1.7 that the spaces \( \ell^\lambda_\infty \), \( c^\lambda \) and \( c^\lambda_0 \) can be obtained as special cases of the sequence spaces of weighted means studied in [48, 69], that is \( \ell^\lambda_\infty = (\tilde{N}, \Delta \lambda)_\infty \), \( c^\lambda = (\tilde{N}, \Delta \lambda) \) and \( c^\lambda_0 = (\tilde{N}, \Delta \lambda)_0 \). On the other hand, the spaces \( \ell^\lambda_\infty \), \( c^\lambda \) and \( c^\lambda_0 \) are reduced in the special case \( \lambda_k = k + 1 (k \in \mathbb{N}) \) to the Cesàro sequence spaces \( X_\infty \), \( \tilde{c} \) and \( \tilde{c}_0 \) of non-absolute type (see [95, 105]).

For brevity, we shall omit those results* concerning the bases, duals and matrix mappings on the spaces \( c^\lambda_0 \), \( c^\lambda \) and \( \ell^\lambda_\infty \) which can immediately be obtained from the

*See [85] for the detailed proofs of these results.
results of Chapter 2 by means of Remark 3.1.6. Thus, in the following, we shall confine ourselves to establish some inclusion relations via the idea of matrix transformations.

**Theorem 3.2.3** The inclusions $c_0^λ \subset c^λ \subset ℓ_∞^λ$ strictly hold.

**Proof.** It is clear that the inclusions $c_0^λ \subset c^λ \subset ℓ_∞^λ$ hold. Further, since the inclusion $c_0 \subset c$ is strict, it follows by Lemma 3.1.1 that the inclusion $c_0^λ \subset c^λ$ is also strict. Moreover, consider the sequence $x = (x_k)$ defined by $x_k = (-1)^k(λ_k + λ_{k-1})/(λ_k - λ_{k-1})$ for all $k \in N$. Then, we have for every $n \in N$ that

$$\Lambda_n(x) = \frac{1}{λ_n}\sum_{k=0}^{n}(-1)^k(λ_k + λ_{k-1}) = (-1)^n.$$ 

This shows that $Λ(x) \in ℓ_∞ \setminus c$. Thus, the sequence $x$ is in $ℓ_∞^λ$ but not in $c^λ$. Hence, the inclusion $c^λ \subset ℓ_∞^λ$ strictly holds. This completes the proof. □

Further, the following result is immediate by the regularity of the matrix $Λ$ and by Lemma 3.1.3.

**Lemma 3.2.4** The inclusions $c_0 \subset c_0^λ$ and $c \subset c^λ$ hold. Furthermore, the equalities hold if and only if $S(x) \in c_0$ for every sequence $x$ in the spaces $c_0^λ$ and $c^λ$, respectively.

**Proof.** The first part is obvious by Lemma 3.1.1. Thus, we turn to the second part. For this, suppose firstly that the equality $c_0^λ = c_0$ holds. Then, we have for every $x \in c_0^λ$ that $x \in c_0$ and hence $S(x) \in c_0$ by Lemma 3.1.3.

Conversely, let $x \in c_0^λ$ be given. Then, we have by the hypothesis that $S(x) \in c_0$. Thus, it follows, by Lemma 3.1.3 and then Lemma 3.1.2, that $x \in c_0$. This shows that the inclusion $c_0^λ \subset c_0$ holds. Hence, by combining the inclusions $c_0^λ \subset c_0$ and $c_0 \subset c_0^λ$, we get the equality $c_0^λ = c_0$.

Similarly, one can show that the equality $c^λ = c$ holds if and only if $S(x) \in c_0$ for every $x \in c^λ$. This concludes the proof. □

Moreover, the following result can be proved similarly by means of Lemmas 3.1.4 and 3.1.5.

**Lemma 3.2.5** The inclusion $ℓ_∞ \subset ℓ_∞^λ$ holds. Furthermore, the equality $ℓ_∞^λ = ℓ_∞$ holds if and only if $S(x) \in ℓ_∞$ for every $x \in ℓ_∞^λ$.

Now, it is obvious by Lemma 3.2.4 that $c_0 \subset c_0^λ \cap c$. Also, it follows by Lemma 3.1.2 that $c_0^λ \cap c \subset c_0$. Thus, we have the following result:

**Theorem 3.2.6** The equality $c_0^λ \cap c = c_0$ holds.

On the contrary of Theorem 3.2.6, the equality $c^λ \cap ℓ_∞ = c$ need not be held. For example, let $λ_k = k + 1$ and $x_k = (-1)^k$ for all $k$. Then $x \in c^λ \cap ℓ_∞$ while $x \notin c$.

Now, let $n \geq 1$. Then, by bearing in mind the relations (3.1.2), (3.1.3) and (3.1.4), we derive that

$$S_n(x) = \frac{1}{λ_n}\sum_{k=1}^{n}λ_{k-1}(x_k - x_{k-1})$$

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\[
\lambda_n \left[ \sum_{k=1}^{n} \lambda_{k-1} x_k - \sum_{k=1}^{n} \lambda_{k-1} x_{k-1} \right] \\
= \frac{1}{\lambda_n} \left[ \sum_{k=0}^{n} \lambda_{k-1} x_k - \sum_{k=0}^{n-1} \lambda_{k} x_k \right] \\
= \frac{1}{\lambda_n} \left[ \lambda_{n-1} x_n - \sum_{k=0}^{n-1} (\lambda_k - \lambda_{k-1}) x_k \right] \\
= \frac{\lambda_{n-1}}{\lambda_n} \left[ x_n - \Lambda_{n-1}(x) \right] \\
= \frac{\lambda_{n-1}}{\lambda_n} \left[ S_n(x) + \Lambda_n(x) - \Lambda_{n-1}(x) \right].
\]

Therefore, we have for every \( x \in \mathcal{W} \) that
\[
S_n(x) = \frac{\lambda_{n-1}}{\lambda_n - \lambda_{n-1}} \left[ \Lambda_n(x) - \Lambda_{n-1}(x) \right]; \quad (n \in \mathbb{N}). \tag{3.2.3}
\]

On the other hand, by taking into account the definition of the sequence \( \lambda \) given by (3.1.1), we have \( \lambda_{k+1}/\lambda_k > 1 \) for all \( k \in \mathbb{N} \). Thus, there are only two distinct cases of the sequence \( \lambda \); either \( \lim\inf_{k \to \infty} \lambda_{k+1}/\lambda_k > 1 \) or \( \lim\inf_{k \to \infty} \lambda_{k+1}/\lambda_k = 1 \). Obviously, the first case holds if and only if \( \lim\inf_{k \to \infty} (\lambda_{k+1} - \lambda_k)/\lambda_{k+1} > 0 \) which is equivalent to say that the sequence \( (\lambda_k/(\lambda_k - \lambda_{k-1}))_{k=0}^{\infty} \) is a bounded sequence. Similarly, the second case holds if and only if the above sequence is unbounded. Therefore, we have the following lemma:

**Lemma 3.2.7** For any sequence \( \lambda = (\lambda_k)_{k=0}^{\infty} \) satisfying (3.1.1), we have
(a) \( \left( \frac{\lambda_k}{\lambda_k - \lambda_{k-1}} \right)_{k=0}^{\infty} \notin \ell_{\infty} \) if and only if \( \lim\inf_{k \to \infty} \frac{\lambda_{k+1}}{\lambda_k} = 1 \).
(b) \( \left( \frac{\lambda_k}{\lambda_k - \lambda_{k-1}} \right)_{k=0}^{\infty} \in \ell_{\infty} \) if and only if \( \lim\inf_{k \to \infty} \frac{\lambda_{k+1}}{\lambda_k} > 1 \).

**Remark 3.2.8** Clearly, Lemma 3.2.7 still holds if the sequence \( (\lambda_k/(\lambda_k - \lambda_{k-1}))_{k=0}^{\infty} \) is replaced by \( (\lambda_k/\lambda_{k+1} - \lambda_k))_{k=0}^{\infty} \).

Now, we are going to prove the following result which gives the necessary and sufficient condition for the matrix \( \Lambda \) to be stronger than convergence and boundedness both, i.e., for the inclusions \( c_0 \subset c_0^\Lambda \), \( c \subset c^\Lambda \) and \( \ell_{\infty} \subset \ell_{\infty}^\Lambda \) to be strict.

**Theorem 3.2.9** The inclusions \( c_0 \subset c_0^\Lambda \), \( c \subset c^\Lambda \) and \( \ell_{\infty} \subset \ell_{\infty}^\Lambda \) strictly hold if and only if \( \lim\inf_{n \to \infty} \lambda_{n+1}/\lambda_n = 1 \).

**Proof.** Suppose that the inclusion \( \ell_{\infty} \subset \ell_{\infty}^\Lambda \) is strict. Then, Lemma 3.2.5 implies the existence of a sequence \( x \in \ell_{\infty}^\Lambda \) such that \( S(x) = (S_n(x))_{n=0}^{\infty} \notin \ell_{\infty} \). Since \( x \in \ell_{\infty}^\Lambda \), we have \( \Lambda(x) = (\Lambda_n(x))_{n=0}^{\infty} \in \ell_{\infty} \) and hence \( (\Lambda_n(x) - \Lambda_{n-1}(x))_{n=0}^{\infty} \in \ell_{\infty} \). Therefore, we deduce from (3.2.3) that \( (\lambda_{n-1}/(\lambda_n - \lambda_{n-1}))_{n=0}^{\infty} \notin \ell_{\infty} \) and hence \( \lambda_n/(\lambda_n - \lambda_{n-1}))_{n=0}^{\infty} \notin \ell_{\infty} \). This leads us with Lemma 3.2.7 to the consequence that \( \lim\inf_{n \to \infty} \lambda_{n+1}/\lambda_n = 1 \).

Similarly, by using Lemma 3.2.4 instead of Lemma 3.2.5, it can be shown that if the inclusions \( c_0 \subset c_0^\Lambda \) and \( c \subset c^\Lambda \) are strict, then \( \lim\inf_{n \to \infty} \lambda_{n+1}/\lambda_n = 1 \). This proves the necessity of the condition.
To prove the sufficiency, suppose that \( \liminf_{n \to \infty} \lambda_{n+1}/\lambda_n = 1 \). Then, we have by Lemma 3.2.7 (a) that \( (\lambda_n/(\lambda_n - \lambda_{n-1}))_{n=0}^\infty \notin \ell_\infty \). Let us now define the sequence \( x = (x_k) \) by \( x_k = (-1)^k \lambda_k/(\lambda_k - \lambda_{k-1}) \) for all \( k \). Then, we have for every \( n \in \mathbb{N} \) that

\[
|\Lambda_n(x)| = \frac{1}{\lambda_n} \left| \sum_{k=0}^{n} (-1)^k \lambda_k \right| \leq \frac{1}{\lambda_n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1}) = 1
\]

which shows that \( \Lambda(x) \in \ell_\infty \). Thus, the sequence \( x \) is in \( \ell^\lambda_\infty \) but not in \( \ell_\infty \). Therefore, by combining this with the fact that the inclusion \( \ell_\infty \subset \ell^\lambda_\infty \) always holds by Lemma 3.2.5, we conclude that this inclusion is strict.

Similarly, if \( \liminf_{k \to \infty} \lambda_{k+1}/\lambda_k = 1 \) then we deduce from Lemma 3.2.7 (a) that \( \liminf_{k \to \infty} (\lambda_k - \lambda_{k-1})/\lambda_k = 0 \). Thus, there is a subsequence \( (\lambda_{k_r})_{r=0}^{\infty} \) of the sequence \( \lambda = (\lambda_k)_{k=0}^{\infty} \) such that

\[
\lim_{r \to \infty} \left( \frac{\lambda_{k_r} - \lambda_{k_r-1}}{\lambda_{k_r}} \right) = 0. \tag{3.2.4}
\]

Obviously, our subsequence can be chosen such that \( k_{r+1} - k_r \geq 2 \) for all \( r \in \mathbb{N} \). Now, let us define the sequence \( y = (y_k)_{k=0}^{\infty} \) by

\[
y_k = \begin{cases} 
1; & (k = k_r), \\
-\left( \frac{\lambda_{k-1} - \lambda_{k-2}}{\lambda_k - \lambda_{k-1}} \right); & (k = k_r + 1), \quad (r \in \mathbb{N}) \\
0; & \text{(otherwise)}
\end{cases}
\]

for all \( k \in \mathbb{N} \). Then \( y \notin c \). On the other hand, we have for every \( n \in \mathbb{N} \) that

\[
\Lambda_n(y) = \begin{cases} 
\frac{\lambda_n - \lambda_{n-1}}{\lambda_n}; & (n = k_r), \\
0; & (n \neq k_r).
\end{cases} \tag{3.2.5}
\]

This and (3.2.4) together imply that \( \Lambda(y) \in c_0 \) and hence \( y \in c^\lambda_0 \). Therefore, the sequence \( y \) is in both spaces \( c^\lambda_0 \) and \( c^\lambda \) but not in any of the spaces \( c_0 \) or \( c \). Hence, by combining this with Lemma 3.2.4, we deduce that the inclusions \( c_0 \subset c^\lambda_0 \) and \( c \subset c^\lambda \) are strict. This concludes the proof.

Now, as a consequence of Theorem 3.2.9, we have the following result which gives the necessary and sufficient condition for the matrix \( \Lambda \) to be equivalent to convergence and boundedness both.

**Corollary 3.2.10** The equalities \( c^\lambda_0 = c_0 \), \( c^\lambda = c \) and \( \ell^\lambda_\infty = \ell_\infty \) hold if and only if \( \liminf_{n \to \infty} \lambda_{n+1}/\lambda_n > 1 \).

**Proof.** The necessity is immediate by Theorem 3.2.9. For, if the equalities hold then the inclusions in Theorem 3.2.9 cannot be strict and hence \( \liminf_{n \to \infty} \lambda_{n+1}/\lambda_n \neq 1 \) which implies that \( \liminf_{n \to \infty} \lambda_{n+1}/\lambda_n > 1 \).

*In the special case \( \lim_{k \to \infty} \lambda_{k+1}/\lambda_k = 1 \), we may take \( k_r = 2r \) \((r \in \mathbb{N})\) and drop the third case in (3.2.5). Also, we can take \( k_r = 2r + 1 \) with \( y_0 = 0 \).*
Conversely, suppose that \( \liminf_{n \to \infty} \lambda_{n+1}/\lambda_n > 1 \). Then, it follows by part (b) of Lemma 3.2.7 that \( (\lambda_n/(\lambda_n - \lambda_{n-1}))_{n=0}^{\infty} \in \ell_\infty \) and hence \( (\lambda_{n-1}/(\lambda_n - \lambda_{n-1}))_{n=0}^{\infty} \in \ell_\infty \).

Now, let \( x \in c_0^\lambda \) be given. Then, we have \( \Lambda(x) = (\Lambda_n(x))_{n=0}^{\infty} \in c \) and hence \( (\Lambda_n(x) - \Lambda_n-1(x))_{n=0}^{\infty} \in c_0 \). Thus, we obtain by (3.2.3) that \( (S_n(x))_{n=0}^{\infty} \in c_0 \). This shows that \( S(x) \in c_0 \) for every \( x \in c_0^\lambda \) and hence for every \( x \in c_0^\lambda \). Consequently, we deduce by Lemma 3.2.4 that the equalities \( c_0^\lambda = c_0 \) and \( c^\lambda = c \) hold.

Similarly, by using Lemma 3.2.5 instead of Lemma 3.2.4, one can show that the equality \( \ell_\infty^\lambda = \ell_\infty \) holds if \( \liminf_{n \to \infty} \lambda_{n+1}/\lambda_n > 1 \). This completes the proof. \( \square \)

**Lemma 3.2.11** The following statements are true:

(a) Although the spaces \( c_0^\lambda \) and \( c \) overlap, the space \( c_0^\lambda \) does not include the space \( c \).

(b) Although the spaces \( c^\lambda \) and \( \ell_\infty \) overlap, the space \( c^\lambda \) does not include the space \( \ell_\infty \).

**Proof.** Part (a) is immediate by Theorem 3.2.6. To prove (b), it is obvious by Lemma 3.2.4 that \( c \subset c^\lambda \cap \ell_\infty \), that is, the spaces \( c^\lambda \) and \( \ell_\infty \) overlap. Further, due to the Steinhaus Theorem [58, p.187] (essentially saying that any regular matrix cannot sum all bounded sequences), the regularity of \( \Lambda \) implies the existence of a sequence \( x \in \ell_\infty \) which is not \( \Lambda \)-summable, i.e. \( \Lambda(x) \not\in c \). Thus, such a sequence \( x \) is in \( \ell_\infty \) but not in \( c^\lambda \). Hence, the inclusion \( \ell_\infty \subset c^\lambda \) does not hold. This concludes the proof. \( \square \)

Finally, we conclude this section by the following result:

**Theorem 3.2.12** If \( \liminf_{n \to \infty} \lambda_{n+1}/\lambda_n = 1 \), then the following hold:

(a) Neither of the spaces \( c_0^\lambda \) and \( c \) includes the other.

(b) Neither of the spaces \( c_0^\lambda \) and \( \ell_\infty \) includes the other.

(c) Neither of the spaces \( c^\lambda \) and \( \ell_\infty \) includes the other.

**Proof.** For (a), it has been shown in Lemma 3.2.11 (a) that the inclusion \( c \subset c_0^\lambda \) does not hold. Further, if \( \liminf_{n \to \infty} \lambda_{n+1}/\lambda_n = 1 \) then the converse inclusion is also not held. For example, the sequence \( y \) defined by (3.2.5), in the proof of Theorem 3.2.9, belongs to the set \( c_0^\lambda \setminus c \). Hence, part (a) follows.

To prove (b), we deduce from Lemma 3.2.11 that the inclusion \( \ell_\infty \subset c_0^\lambda \) does not hold. Moreover, we are going to show that the converse inclusion does not hold if \( \liminf_{n \to \infty} \lambda_{n+1}/\lambda_n = 1 \). For this, suppose that \( \liminf_{n \to \infty} \lambda_{n+1}/\lambda_n = 1 \). Then, as we have seen in the proof of Theorem 3.2.9, there is a subsequence \( (\lambda_{k_r})_{r=0}^{\infty} \) of the sequence \( \lambda = (\lambda_k)_{k=0}^{\infty} \) such that (3.2.4) holds and \( k_{r+1} - k_r \geq 2 \) for all \( r \in \mathbb{N} \).

Now, let \( 0 < \alpha < 1 \) and define the sequence \( x = (x_k)_{k=0}^{\infty} \) by

\[
x_k = \begin{cases} 
\left( \frac{\lambda_k}{\lambda_k - \lambda_{k-1}} \right)^{\alpha} & \text{if } k = k_r, \\
-\left( \frac{\lambda_k - \lambda_{k-2}}{\lambda_k - \lambda_{k-1}} \right)x_{k-1} & \text{if } k = k_r + 1, \quad (r \in \mathbb{N}) \\
0 & \text{otherwise}
\end{cases}
\]

for all \( k \in \mathbb{N} \). Then \( x \not\in \ell_\infty \) by (3.2.4). On the other hand, the straightforward computations yield that
\[
\sum_{k=0}^{n} (\lambda_k - \lambda_{k-1})x_k = \begin{cases}
(\lambda_n - \lambda_{n-1})\left(\frac{\lambda_n}{\lambda_n - \lambda_{n-1}}\right)^\alpha; & (n = k_r), \\
0; & (n \neq k_r)
\end{cases} \quad (r \in \mathbb{N})
\]

for every \( n \in \mathbb{N} \) and hence

\[
\Lambda_n(x) = \begin{cases}
(\frac{\lambda_n - \lambda_{n-1}}{\lambda_n})^{1-\alpha}; & (n = k_r), \\
0; & (n \neq k_r)
\end{cases} \quad (r \in \mathbb{N})
\]

This, together with (3.2.4), implies that \( \Lambda(x) \in c_0 \). Thus, the sequence \( x \) is in \( c_0^\lambda \) but not in \( \ell_\infty \). Consequently, the inclusion \( c_0^\lambda \subset \ell_\infty \) fails.

Finally, part (c) is immediate by combining part (b) and Lemma 3.2.11 (b). \( \square \)

**Remark 3.2.13** The results of this section may extend to the spaces \( Z(u, v; c_0) \), \( Z(u, v; c) \) and \( Z(u, v; \ell_\infty) \) of generalized weighted means with some conditions on the sequences \( u \) and \( v \) (see [74]).

### 3.3 The sequence spaces \( \ell_p^\lambda \) (1 \( \leq \) \( p \) \( < \) \( \infty \))

For \( 1 \leq p < \infty \), we define the sequence space \( \ell_p^\lambda \) as the set of all sequences whose \( \Lambda \)-transforms are in the space \( \ell_p \), that is

\[
\ell_p^\lambda = \left\{ x = (x_k) \in w : \sum_{n=0}^{\infty} \left| \frac{1}{\lambda_n} \sum_{k=0}^{n}(\lambda_k - \lambda_{k-1})x_k \right|^p < \infty \right\}
\]

which can be redefined as the matrix domain of the triangle \( \Lambda \) in the space \( \ell_p \), that is

\[
\ell_p^\lambda = (\ell_p)_\Lambda; \quad (1 \leq p < \infty).
\]

Thus, it is obvious by Remark 3.1.6 that \( \ell_p^\lambda \) is the special case \( r = \lambda \), \( s = e \) and \( t = \Delta \lambda \) of the space \( \ell_p(r, s, t) \) of generalized means. Therefore, it follows by part (b) of Corollary 2.2.4 that \( \ell_p^\lambda (1 \leq p < \infty) \) is a \( BK \) space with the norm given by

\[
\|x\|_{\ell_p^\lambda} = \|\Lambda(x)\|_{\ell_p} = \left( \sum_{n=0}^{\infty} |\Lambda_n(x)|^p \right)^{1/p}; \quad (x \in \ell_p^\lambda).
\]

We refer the reader to [89, 90] for further studies concerning the basis, duals and matrix mappings on the space \( \ell_p^\lambda \) which can also be obtained from Corollaries 2.2.7, 2.3.3, 2.3.6, 2.4.4 and 2.4.5. Further, some imbedding relations concerning the space \( \ell_p^\lambda \) can be found in [91]. Here, we shall derive some inclusion relations for the space \( \ell_p^\lambda \), where \( 1 \leq p < \infty \).
Theorem 3.3.1 If $1 \leq p < q < \infty$, then the inclusion $\ell_p^\lambda \subset \ell_q^\lambda$ strictly holds.

Proof. Suppose that $1 \leq p < q < \infty$. Then, it is immediate by the inclusion $\ell_p \subset \ell_q$ that the inclusion $\ell_p^\lambda \subset \ell_q^\lambda$ holds. Further, since the inclusion $\ell_p \subset \ell_q$ is strict, there is a sequence $x = (x_k)$ in $\ell_q$ but not in $\ell_p$, i.e., $x \in \ell_q \setminus \ell_p$. Let us now define the sequence $y = (y_k)$ in terms of the sequence $x$ as follows:

$$y_k = \frac{\lambda_k x_k - \lambda_{k-1} x_{k-1}}{\lambda_k - \lambda_{k-1}}; \quad (k \in \mathbb{N}).$$

Then, by using (3.1.2), we have for every $n \in \mathbb{N}$ that

$$\Lambda_n(y) = \frac{1}{\lambda_n} \sum_{k=0}^{n} (\lambda_k x_k - \lambda_{k-1} x_{k-1}) = x_n.$$

This shows that $\Lambda(y) = x$ and hence $\Lambda(y) \in \ell_q \setminus \ell_p$. Thus, the sequence $y$ is in $\ell_q^\lambda$ but not in $\ell_p^\lambda$. Hence, the inclusion $\ell_p^\lambda \subset \ell_q^\lambda$ is strict and this concludes the proof. □

Theorem 3.3.2 The inclusion $\ell_p^\lambda \subset c_0^\lambda$ strictly holds, where $1 \leq p < \infty$.

Proof. It is trivial that the inclusion $\ell_p^\lambda \subset c_0^\lambda$ holds for $1 \leq p < \infty$, since $x \in \ell_p^\lambda$ implies $\Lambda(x) \in \ell_p$ and hence $\Lambda(x) \in c_0$ which means that $x \in c_0^\lambda$. Further, to show that this inclusion is strict, let $1 \leq p < \infty$ and consider the sequence $x = (x_k)$ defined by

$$x_k = \frac{1}{(k+1)^{1/p}}; \quad (k \in \mathbb{N}).$$

Then $x \in c_0$ and hence $x \in c_0^\lambda$ by Lemma 3.2.4. On the other hand, we have for every $n \in \mathbb{N}$ that

$$|\Lambda_n(x)| = \frac{1}{\lambda_n} \sum_{k=0}^{n} \frac{\lambda_k - \lambda_{k-1}}{(k+1)^{1/p}} \geq \frac{1}{\lambda_n (n+1)^{1/p}} \sum_{k=0}^{n} (\lambda_k - \lambda_k - 1) = \frac{1}{(n+1)^{1/p}}$$

which shows that $\Lambda(x) \notin \ell_p$ and hence $x \notin \ell_p^\lambda$. Thus, the sequence $x$ is in $c_0^\lambda$ but not in $\ell_p^\lambda$. Therefore, the inclusion $\ell_p^\lambda \subset c_0^\lambda$ strictly holds and this completes the proof. □

Lemma 3.3.3 Let $1 \leq p < \infty$. Then, the inclusion $\ell_p^\lambda \subset \ell_p$ holds if and only if $S(x) \in \ell_p$ for every sequence $x \in \ell_p^\lambda$.

Proof. Assume that the inclusion $\ell_p^\lambda \subset \ell_p$ holds, where $1 \leq p < \infty$, and take any $x = (x_k) \in \ell_p^\lambda$. Then $x \in \ell_p$ by the assumption. Thus, we obtain from (3.1.3) that

$$\|S(x)\|_{\ell_p} \leq \|x\|_{\ell_p} + \|\Lambda(x)\|_{\ell_p} = \|x\|_{\ell_p} + \|x\|_{\ell_p^\lambda} < \infty$$

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which implies that $S(x) \in \ell_p$.

Conversely, let $x \in \ell^\lambda_p$ be given, where $1 \leq p < \infty$. Then, we have by the hypothesis that $S(x) \in \ell_p$. Again, it follows by (3.1.3) that

$$\|x\|_{\ell_p} \leq \|S(x)\|_{\ell_p} + \|\Lambda(x)\|_{\ell_p} = \|S(x)\|_{\ell_p} + \|x\|_{\ell^\lambda_p} < \infty$$

which shows that $x \in \ell_p$. Hence, the inclusion $\ell^\lambda_p \subset \ell_p$ holds and this concludes the proof.

Although the inclusions $c_0 \subset c^\lambda_0$, $c \subset c^\lambda$ and $\ell_\infty \subset \ell^\lambda_\infty$ always hold, the inclusion $\ell_p \subset \ell^\lambda_p$ need not be held, where $1 \leq p < \infty$. In fact, we are going to show, in the following lemma, that the inclusion $\ell_p \subset \ell^\lambda_p$ fails if $1/\lambda \not\in \ell_p$, where $1/\lambda = (1/\lambda_k)_{k=0}^\infty$.

**Lemma 3.3.4** Let $1 \leq p < \infty$. Then, the spaces $\ell_p$ and $\ell^\lambda_p$ overlap. Furthermore, if $1/\lambda \not\in \ell_p$ then neither of them includes the other.

**Proof.** Let $1 \leq p < \infty$, throughout. Then, it is obvious that the spaces $\ell_p$ and $\ell^\lambda_p$ overlap, since $(\lambda - \lambda_0, -\lambda_0, 0, 0, \ldots) \in \ell_p \cap \ell^\lambda_p$.

Now, suppose that $1/\lambda \not\in \ell_p$ and consider the sequence $x = e^{(0)} \in \ell_p$. Then, we have for every $n \in \mathbb{N}$ that

$$\Lambda_n(x) = \frac{1}{\lambda_n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1}) e^{(0)} = \frac{\lambda_0}{\lambda_n}$$

which shows that $\Lambda(x) \not\in \ell_p$ and hence $x \not\in \ell^\lambda_p$. Thus, the sequence $x$ is in $\ell_p$ but not in $\ell^\lambda_p$. Hence, the inclusion $\ell_p \subset \ell^\lambda_p$ does not hold when $1/\lambda \not\in \ell_p$.

On the other hand, let us define the sequence $y = (y_k)$ by

$$y_k = \begin{cases} 
\frac{1}{\lambda_k}; & (k \text{ is even}), \\
-\frac{1}{\lambda_{k-1}} \left( \frac{\lambda_{k-1} - \lambda_{k-2}}{\lambda_k - \lambda_{k-1}} \right); & (k \text{ is odd})
\end{cases}$$

for all $k \in \mathbb{N}$. Then $y \not\in \ell_p$, since $1/\lambda \not\in \ell_p$. Besides, we have for every $n \in \mathbb{N}$ that

$$\Lambda_n(y) = \begin{cases} 
\frac{1}{\lambda_n} \left( \frac{\lambda_n - \lambda_{n-1}}{\lambda_n} \right); & (n \text{ is even}), \\
0; & (n \text{ is odd})
\end{cases}$$

and hence

$$\sum_{n=0}^\infty |\Lambda_n(y)|^p = \sum_{n=0}^\infty |\Lambda_{2n}(y)|^p$$

$$= \sum_{n=0}^\infty \frac{1}{\lambda_{2n}^p} \left( \frac{\lambda_{2n} - \lambda_{2n-1}}{\lambda_{2n}} \right)^p$$

$$\leq \frac{1}{\lambda_0^p} + \sum_{n=1}^\infty \frac{1}{\lambda_{2n-2}^p} \left( \frac{\lambda_{2n} - \lambda_{2n-2}}{\lambda_{2n}} \right)^p$$

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\[
\begin{align*}
\frac{1}{\lambda_0^p} + \sum_{n=1}^{\infty} \frac{1}{\lambda_{2n-2}^p} \left( \frac{\lambda_{2n}^p - \lambda_{2n-2}^p}{\lambda_{2n}^p} \right) \\
= \frac{1}{\lambda_0^p} + \sum_{n=1}^{\infty} \left( \frac{1}{\lambda_{2n-2}^p} - \frac{1}{\lambda_{2n}^p} \right) \\
= \frac{2}{\lambda_0^p} < \infty.
\end{align*}
\]

This shows that \( \Lambda(y) \in \ell_p \) and so \( y \in \ell^\lambda_p \). Thus, the sequence \( y \) is in \( \ell^\lambda_p \) but not in \( \ell_p \). Therefore, the inclusion \( \ell^\lambda_p \subset \ell_p \) also fails when \( 1/\lambda \notin \ell_p \). Hence, we conclude that if \( 1/\lambda \notin \ell_p \) then neither of the spaces \( \ell_p \) and \( \ell^\lambda_p \) includes the other, where \( 1 \leq p < \infty \). This concludes the proof.

**Lemma 3.3.5** If the inclusion \( \ell_p \subset \ell^\lambda_p \) holds, then \( 1/\lambda \in \ell_p \) for \( 1 \leq p < \infty \).

**Proof.** Suppose that the inclusion \( \ell_p \subset \ell^\lambda_p \) holds, where \( 1 \leq p < \infty \), and consider the sequence \( x = e^{(0)} \in \ell_p \). Then \( x \in \ell^\lambda_p \) and hence \( \Lambda(x) \in \ell_p \). Thus, we obtain that

\[
\lambda_0^p \sum_{n=0}^{\infty} \left( \frac{1}{\lambda_n} \right)^p = \sum_{n=0}^{\infty} |\Lambda_n(x)|^p < \infty
\]

which shows that \( 1/\lambda \in \ell_p \) and this completes the proof. \( \square \)

We shall later show that the condition \( 1/\lambda \in \ell_p \) is not only necessary but also sufficient for the inclusion \( \ell_p \subset \ell^\lambda_p \) to be held, where \( 1 \leq p < \infty \). Before that, by taking into account the definition of the sequence \( \lambda = (\lambda_k) \) given by (3.1.1), we find that

\[
0 < \frac{\lambda_k - \lambda_{k-1}}{\lambda_n} < 1; \quad (0 \leq k \leq n)
\]

for all \( n, k \in \mathbb{N} \) with \( n + k > 0 \). Furthermore, if \( 1/\lambda \in \ell_1 \) then we have the following result which is easy to prove.

**Lemma 3.3.6** If \( 1/\lambda \in \ell_1 \), then

\[
\sup_k \left( (\lambda_k - \lambda_{k-1}) \sum_{n=k}^{\infty} \frac{1}{\lambda_n} \right) < \infty.
\]

Now, we prove the following:

**Theorem 3.3.7** The inclusion \( \ell_1 \subset \ell^\lambda_1 \) holds if and only if \( 1/\lambda \in \ell_1 \).

**Proof.** The necessity is immediate by Lemma 3.3.5. To prove the sufficiency, suppose that \( 1/\lambda \in \ell_1 \). Then, we have by Lemma 3.3.6 that

\[
M = \sup_k \left( (\lambda_k - \lambda_{k-1}) \sum_{n=k}^{\infty} \frac{1}{\lambda_n} \right) < \infty.
\]
Now, let \( x = (x_k) \in \ell_1 \) be given. Then, we have

\[
\|x\|_{\ell^1_\lambda} = \sum_{n=0}^{\infty} |\Lambda_n(x)| \\
\leq \sum_{n=0}^{\infty} \frac{1}{\lambda_n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1}) |x_k| \\
= \sum_{k=0}^{\infty} |x_k| (\lambda_k - \lambda_{k-1}) \sum_{n=k}^{\infty} \frac{1}{\lambda_n} \\
\leq M \sum_{k=0}^{\infty} |x_k| \\
= M \|x\|_{\ell_1} < \infty.
\]

This shows that \( x \in \ell^1_\lambda \). Hence, the inclusion \( \ell_1 \subset \ell^1_\lambda \) holds which concludes the proof. \( \Box \)

**Corollary 3.3.8** If \( 1/\lambda \in \ell_1 \), then the inclusion \( \ell_p \subset \ell^\lambda_p \) holds for \( 1 \leq p < \infty \).

**Proof.** The inclusion trivially holds for \( p = 1 \) by Theorem 3.3.7, above. Thus, let \( 1 < p < \infty \) and take any \( x = (x_k) \in \ell_p \). Then, for every \( n \in \mathbb{N} \), we obtain by applying the Hölder’s inequality that

\[
|\Lambda_n(x)|^p \leq \left[ \sum_{k=0}^{n} \left( \frac{\lambda_k - \lambda_{k-1}}{\lambda_n} \right)^p \right] \sum_{k=0}^{n} \left( \frac{\lambda_k - \lambda_{k-1}}{\lambda_n} \right) |x_k|^p \\
= \frac{1}{\lambda_n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1}) |x_k|^p
\]

which implies that

\[
\|x\|_{\ell^\lambda_p}^p = \sum_{n=0}^{\infty} |\Lambda_n(x)|^p \\
\leq \sum_{n=0}^{\infty} \frac{1}{\lambda_n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1}) |x_k|^p \\
= \sum_{k=0}^{\infty} |x_k|^p (\lambda_k - \lambda_{k-1}) \sum_{n=k}^{\infty} \frac{1}{\lambda_n} \\
\leq M \sum_{k=0}^{\infty} |x_k|^p \\
= M \|x\|_{\ell_p}^p < \infty,
\]

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where \( M = \sup_k \left[ (\lambda_k - \lambda_{k-1}) \sum_{n=k}^{\infty} 1/\lambda_n \right] < \infty \) by Lemma 3.3.6. This shows that \( x \in \ell_p^\lambda \). Hence, the inclusion \( \ell_p \subset \ell_p^\lambda \) also holds for \( 1 < p < \infty \). This completes the proof. \( \square \)

**Corollary 3.3.9** The inclusion \( \ell_p \subset \ell_p^\lambda \) holds if and only if \( 1/\lambda \in \ell_p \), where \( 1 \leq p < \infty \).

**Proof.** The necessity is immediate by Lemma 3.3.5. Thus, we turn to the sufficiency. For, suppose that \( 1/\lambda \in \ell_p \), where \( 1 \leq p < \infty \). Then \( 1/\lambda^p = (1/\lambda_k^p) \in \ell_1 \). Thus, it follows by Lemma 3.3.6 that

\[
\sup_k \left( (\lambda_k - \lambda_{k-1})^p \sum_{n=k}^{\infty} \frac{1}{\lambda_n^p} \right) \leq \sup_k \left( (\lambda_k^p - \lambda_{k-1}^p) \sum_{n=k}^{\infty} \frac{1}{\lambda_n^p} \right) < \infty.
\]

On the other hand, we have for every fixed \( k \in \mathbb{N} \) that

\[
\Lambda_n(e^{(k)}) = \begin{cases} \frac{\lambda_k - \lambda_{k-1}}{\lambda_n}; & (n \geq k), \\ 0; & (n < k). \end{cases}
\]

Thus, we obtain that

\[
\|e^{(k)}\|_{\ell_p^\lambda}^p = (\lambda_k - \lambda_{k-1})^p \sum_{n=k}^{\infty} \frac{1}{\lambda_n^p} < \infty; \quad (k \in \mathbb{N})
\]

which yields that \( e^{(k)} \in \ell_p^\lambda \) for every \( k \in \mathbb{N} \), i.e., every basis element of the space \( \ell_p \) is in \( \ell_p^\lambda \). Therefore, we deduce that the space \( \ell_p^\lambda \) contains the Schauder basis of the space \( \ell_p \) such that

\[
\sup_k \|e^{(k)}\|_{\ell_p^\lambda} < \infty.
\]

This leads us, with the fact that \( \ell_p \) has \( AK \), to the consequence that the inclusion \( \ell_p \subset \ell_p^\lambda \) holds for \( 1 \leq p < \infty \) which concludes the proof. \( \square \)

Now, in the following example, we give an important special case of the space \( \ell_p^\lambda \), where \( 1 \leq p < \infty \).

**Example 3.3.10** Consider the special case of the sequence \( \lambda = (\lambda_k) \) given by \( \lambda_k = k+1 \) for all \( k \in \mathbb{N} \). Then \( 1/\lambda \notin \ell_1 \) while \( 1/\lambda \in \ell_p \) for \( 1 < p < \infty \). Hence, the inclusion \( \ell_1 \subset \ell_1^\lambda \) does not hold by Lemma 3.3.5.

On the other hand, by applying the Hardy’s inequality (see [45, p.239]), we obtain that

\[
\sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} \frac{|x_k|}{n+1} \right)^p < \left( \frac{p}{p-1} \right)^p \sum_{n=0}^{\infty} |x_n|^p; \quad (1 < p < \infty)
\]

which implies that

\[
\|x\|_{\ell_p^\lambda} < \left( \frac{p}{p-1} \right) \|x\|_{\ell_p};
\]

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for all \( x \in \ell_p \) and this shows that the inclusion \( \ell_p \subset \ell^\lambda_p \) holds for \( 1 < p < \infty \). Further, this inclusion is strict. For example, the sequence \( y = \{(-1)^k\} \) is not in \( \ell_p \) but in \( \ell^\lambda_p \), since
\[
\sum_{n=0}^\infty |\lambda_n(y)|^p = \sum_{n=0}^\infty \left| \frac{1}{n+1} \sum_{k=0}^n (-1)^k \right|^p = \sum_{n=0}^\infty \frac{1}{(2n+1)^p} < \infty; \quad (1 < p < \infty).
\]

**Remark 3.3.11** In the special case of the sequence \( \lambda = (\lambda_k) \) given in Example 3.3.10, i.e., \( \lambda_k = k + 1 \) for all \( k \in \mathbb{N} \), we may note that the space \( \ell^\lambda_p \) is reduced to the Cesàro sequence space \( X_p \) of non-absolute type, where \( 1 \leq p < \infty \) (see [95]).

Now, let \( x = (x_k) \) be a null sequence of positive reals, that is
\[
x_k > 0 \text{ for all } k \in \mathbb{N} \text{ and } x_k \to 0 \text{ as } k \to \infty.
\]

Then, as is easy to see, for every positive integer \( m \) there is a subsequence \( (x_{kr})_{r=0}^\infty \) of the sequence \( x \) such that
\[
x_{kr} = O \left( \frac{1}{(r+1)^{m+1}} \right)
\]
and hence
\[
(r+1)x_{kr} = O \left( \frac{1}{(r+1)^m} \right).
\]

Further, this subsequence can be chosen such that \( k_{r+1} - kr \geq 2 \) for all \( r \in \mathbb{N} \).

In general, if \( x = (x_k) \) is a sequence of positive reals such that \( \liminf_{k \to \infty} x_k = 0 \), then there is a subsequence \( x' = (x_{kr})_{r=0}^\infty \) of the sequence \( x \) such that \( \lim_{r \to \infty} x_{kr} = 0 \). Thus \( x' \) is a null sequence of positive reals. Hence, as we have seen above, for every positive integer \( m \) there is a subsequence \( (x_{kr})_{r=0}^\infty \) of the sequence \( x' \), and hence of the sequence \( x \), such that \( k_{r+1} - kr \geq 2 \) for all \( r \in \mathbb{N} \) and
\[
(r+1)x_{kr} = O \left( \frac{1}{(r+1)^m} \right),
\]
where \( k_r = k'_{d(r)} \) and \( \theta : \mathbb{N} \to \mathbb{N} \) is a suitable increasing function.

Now, let \( 0 < p < \infty \). Then, we can choose a positive integer \( m \) such that \( mp > 1 \). In this situation, the sequence \( (x_{kr})_{r=0}^\infty \) is in the space \( \ell_p \).

Also, it is obvious that the subsequence \( (x_{kr})_{r=0}^\infty \) depends on the positive integer \( m \) which is, in turn, depending on \( p \). Thus, our subsequence depends on \( p \).

Hence, from the above discussion, we conclude the following result:

**Lemma 3.3.12** Let \( x = (x_k) \) be a sequence of positive reals such that \( \liminf_{k \to \infty} x_k = 0 \). Then, for every positive number \( 0 < p < \infty \) there is a subsequence \( x^{(p)} = (x_{kr})_{r=0}^\infty \) of \( x \), depending on \( p \), such that \( k_{r+1} - kr \geq 2 \) for all \( r \in \mathbb{N} \) and
\[
\sum_r |(r+1)x_{kr}|^p < \infty.
\]

Now, we prove the following result which gives necessary and sufficient conditions for the matrix \( \Lambda \) to be stronger than \( p \)-absolute convergence, i.e., for the inclusion \( \ell_p \subset \ell^\lambda_p \) to be strict, where \( 1 \leq p < \infty \).
Theorem 3.3.13 Let $1 \leq p < \infty$. Then, the inclusion $\ell_p \subset \ell_p^\lambda$ strictly holds if and only if $1/\lambda \in \ell_p$ and $\liminf_{n \to \infty} \lambda_{n+1}/\lambda_n = 1$.

Proof. Suppose that the inclusion $\ell_p \subset \ell_p^\lambda$ is strict, where $1 \leq p < \infty$. Then, the necessity of the first condition is immediate by Lemma 3.3.5. Further, since the inclusion $\ell_p^\lambda \subset \ell_p$ cannot be held, Lemma 3.3.3 implies the existence of a sequence $x \in \ell_p^\lambda$ such that $S(x) = (S_n(x))_{n=0}^\infty \notin \ell_p$. Since $x \in \ell_p^\lambda$, we have $\sum_{n=0}^\infty |\Lambda_n(x)|^p < \infty$. Thus, it follows by applying the Minkowski’s inequality that $\sum_{n=0}^\infty |\Lambda_n(x) - \Lambda_{n-1}(x)|^p < \infty$. This means that $(\Lambda_n(x) - \Lambda_{n-1}(x))_{n=0}^\infty \in \ell_p$ and since $(S_n(x))_{n=0}^\infty \notin \ell_p$, it follows by the relation (3.2.3) that $(\lambda_{n-1} - (\lambda_n - \lambda_{n-1}))_{n=0}^\infty \notin \ell_\infty$ and hence $(\lambda_n/\lambda_{n-1})_{n=0}^\infty \notin \ell_\infty$.

This leads us with Lemma 3.2.7 (a) to the necessity of the second condition.

Conversely, since $1/\lambda \in \ell_p$, we have by Corollary 3.3.9 that the inclusion $\ell_p \subset \ell_p^\lambda$ holds. Further, since $\liminf_{k \to \infty} \lambda_{k+1}/\lambda_k = 1$, we obtain by Lemma 3.2.7 (a) that

$$\liminf_{k \to \infty} \left(\frac{\lambda_k - \lambda_{k-1}}{\lambda_k}\right) = 0.$$  

Thus, it follows by Lemma 3.3.12 that there is a subsequence $\lambda^{(p)} = (\lambda_{k_r})_{r=0}^\infty$ of the sequence $\lambda = (\lambda_k)$, depending on $p$, such that $k_{r+1} - k_r \geq 2$ for all $r \in \mathbb{N}$ and

$$\sum_{r=0}^\infty \left(\begin{array}{c} (r+1) \left(\frac{\lambda_{k_r} - \lambda_{k_r-1}}{\lambda_{k_r}}\right) \end{array}\right)^p < \infty.$$  

Let us now define the sequence $y = (y_k)$ for every $k \in \mathbb{N}$ by

$$y_k = \begin{cases} r+1; & (k = k_r), \\ -(r+1) \left(\frac{\lambda_{k_r-1} - \lambda_{k_r-2}}{\lambda_k - \lambda_{k_r-1}}\right); & (k = k_r+1), \\ 0; & \text{(otherwise)}. \end{cases}$$  

Then, it is clear that $y \notin \ell_p$. On the other hand, we have for every $n \in \mathbb{N}$ that

$$\Lambda_n(y) = \begin{cases} (r+1) \left(\frac{\lambda_n - \lambda_{n-1}}{\lambda_n}\right); & (n = k_r), \\ 0; & \text{(otherwise)} \end{cases}$$  

This and (3.3.4) imply that $\Lambda(y) \in \ell_p$ and hence $y \in \ell_p^\lambda$. Thus, the sequence $y$ is in $\ell_p^\lambda$ but not in $\ell_p$. Therefore, we deduce by combining this with the inclusion $\ell_p \subset \ell_p^\lambda$ that this inclusion is strict, where $1 \leq p < \infty$. This completes the proof. \(\square\)

Now, as an immediate consequence of Theorem 3.3.13, the following corollary presents the necessary and sufficient condition for the matrix $\Lambda$ to be equivalent to $p$-absolute convergence, where $1 \leq p < \infty$.

Corollary 3.3.14 The equality $\ell_p^\lambda = \ell_p$ holds if and only if $\liminf_{n \to \infty} \lambda_{n+1}/\lambda_n > 1$, where $1 \leq p < \infty$. 

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Proof. The necessity follows from Theorem 3.3.13. For, if the equality holds, then the inclusion \( \ell_p \subseteq \ell_p^\lambda \) holds and hence \( 1/\lambda \in \ell_p \) by Lemma 3.3.5. Further, since the inclusion \( \ell_p \subseteq \ell_p^\lambda \) cannot be strict, we have by Theorem 3.3.13 that \( \liminf_{n \to \infty} \lambda_{n+1}/\lambda_n \neq 1 \) and hence \( \liminf_{n \to \infty} \lambda_{n+1}/\lambda_n > 1 \).

To prove the sufficiency, suppose that \( \liminf_{n \to \infty} \lambda_{n+1}/\lambda_n > 1 \). Then, there exists a constant \( a > 1 \) such that \( \lambda_{n+1}/\lambda_n \geq a \) and hence \( \lambda_n \geq \lambda_0 a^n \) for all \( n \in \mathbb{N} \). This shows that \( 1/\lambda \in \ell_1 \) which leads us with Corollary 3.3.8 to the consequence that the inclusion \( \ell_p \subseteq \ell_p^\lambda \) holds for \( 1 \leq p < \infty \).

On the other hand, we have by Lemma 3.2.7 (b) that \( (\lambda_n/(\lambda_n-\lambda_{n-1}))_{n=0}^\infty \in \ell_\infty \) and hence \( (\lambda_{n-1}/(\lambda_n-\lambda_{n-1}))_{n=0}^\infty \in \ell_\infty \).

Now, let \( x \in \ell_p^\lambda \). Then \( \Lambda(x) = (\Lambda_n(x))_{n=0}^\infty \in \ell_p \) and hence \( (\Lambda_n(x)-\Lambda_{n-1}(x))_{n=0}^\infty \in \ell_p \). Thus, we obtain by (3.2.3) that \( (S_n(x))_{n=0}^\infty \in \ell_p \), i.e., \( S(x) \in \ell_p \) for every \( x \in \ell_p^\lambda \). Therefore, we deduce by Lemma 3.3.3 that the inclusion \( \ell_p^\lambda \subseteq \ell_p \) also holds. Hence, by combining the inclusions \( \ell_p \subseteq \ell_p^\lambda \) and \( \ell_p^\lambda \subseteq \ell_p \), we get the equality \( \ell_p^\lambda = \ell_p \), where \( 1 \leq p < \infty \). This concludes the proof.

Finally, we end this section with the following corollary:

**Corollary 3.3.15** Although the spaces \( \ell_p^\lambda \), \( c_0 \), \( c \) and \( \ell_\infty \) overlap, the space \( \ell_p^\lambda \) does not include any of the other spaces. Furthermore, if \( \liminf_{n \to \infty} \lambda_{n+1}/\lambda_n = 1 \), then none of the spaces \( c_0 \), \( c \) and \( \ell_\infty \) includes the space \( \ell_p^\lambda \), where \( 1 \leq p < \infty \).

**Proof.** Let \( 1 \leq p < \infty \). Then, it is obvious that the spaces \( \ell_p^\lambda \), \( c_0 \), \( c \) and \( \ell_\infty \) overlap, since the sequence \( (\lambda_1-\lambda_0,-\lambda_0,0,0,\ldots) \) belongs to all these spaces.

Further, the space \( \ell_p^\lambda \) does not include the space \( c_0 \), since the sequence \( x \) defined by (3.3.3), in the proof of Theorem 3.3.2, is in \( c_0 \) but not in \( \ell_p^\lambda \). Hence, the space \( \ell_p^\lambda \) does not include any one of the spaces \( c_0 \), \( c \) and \( \ell_\infty \).

Moreover, if \( \liminf_{n \to \infty} \lambda_{n+1}/\lambda_n = 1 \) then the space \( \ell_\infty \) does not include the space \( \ell_p^\lambda \). To see this, Lemma 3.3.12 implies that the sequence \( y \) defined by (3.3.5), in the proof of Theorem 3.3.13, is in \( \ell_p^\lambda \) but not in \( \ell_\infty \). Therefore, none of the spaces \( c_0 \), \( c \) and \( \ell_\infty \) includes the space \( \ell_p^\lambda \) when \( \liminf_{n \to \infty} \lambda_{n+1}/\lambda_n = 1 \). This completes the proof.

### 3.4 The difference spaces \( \ell_\infty^\lambda(\Delta) \), \( c^\lambda(\Delta) \) and \( c_0^\lambda(\Delta) \)

Recently, many difference sequence spaces have been introduced in different ways (see for example [11, 37, 38, 53, 61, 67, 99]). In this last section, following [11, 16] and [71], we treat slightly more different than Kizmaz [53] and other authors following him, and employ the technique obtaining a new sequence space by means of the matrix domain of a triangle. Thus, we define the spaces \( \ell_\infty^\lambda(\Delta) \), \( c^\lambda(\Delta) \) and \( c_0^\lambda(\Delta) \) of difference sequences as follows:

\[
\ell_\infty^\lambda(\Delta) = \left\{ x = (x_k) \in w : \sup_n \left| \frac{1}{\lambda_n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1})(x_k - x_{k-1}) \right| < \infty \right\},
\]

\[
c^\lambda(\Delta) = \left\{ x = (x_k) \in w : \lim_{n \to \infty} \left( \frac{1}{\lambda_n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1})(x_k - x_{k-1}) \right) \right\}.
\]
\[ c_0^\lambda(\Delta) = \left\{ x = (x_k) \in w : \lim_{n \to \infty} \left( \frac{1}{\lambda_n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1})(x_k - x_{k-1}) \right) = 0 \right\}. \]

Then, it follows by (1.2.4) that
\[ \ell^\lambda_\infty(\Delta) = (\ell^\lambda_\infty)_\Delta, \quad c^\lambda(\Delta) = (c^\lambda)_\Delta \quad \text{and} \quad c_0^\lambda(\Delta) = (c_0^\lambda)_\Delta, \tag{3.4.1} \]
where \( \Delta \) denotes the band matrix defining the difference operator, that is \( \Delta x = (x_k - x_{k-1})_{k=0}^{\infty} \) for all \( x = (x_k) \in w \).

On the other hand, we define the infinite matrix \( \bar{\Lambda} = (\bar{\lambda}_{nk}) \) for all \( n, k \in \mathbb{N} \) by
\[
\bar{\lambda}_{nk} = \begin{cases} 
\frac{(\lambda_k - \lambda_{k-1}) - (\lambda_{k+1} - \lambda_k)}{\lambda_n}; & (k < n), \\
\frac{\lambda_n - \lambda_{n-1}}{\lambda_n}; & (k = n), \\
0; & (k > n).
\end{cases} \tag{3.4.2}
\]

Then, it can easily be seen that
\[ \bar{\Lambda}_n(x) = \frac{1}{\lambda_n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1}) (x_k - x_{k-1}); \quad (n \in \mathbb{N}) \tag{3.4.3} \]
for all \( x = (x_k) \in w \) which yields that \( \bar{\Lambda} = \Delta \Lambda \). Therefore, it follows by (3.2.1) and (3.4.1) that the difference spaces \( \ell^\lambda_\infty(\Delta) \), \( c^\lambda(\Delta) \) and \( c_0^\lambda(\Delta) \) are the matrix domains of the triangle \( \bar{\Lambda} \) in the spaces \( \ell_\infty, c \) and \( c_0 \), respectively; that is
\[ \ell^\lambda_\infty(\Delta) = (\ell_\infty)_{\bar{\Lambda}}, \quad c^\lambda(\Delta) = c_{\bar{\Lambda}} \quad \text{and} \quad c_0^\lambda(\Delta) = (c_0)_{\bar{\Lambda}}. \tag{3.4.4} \]

Further, it is obvious by (3.4.4) that \( \ell^\lambda_\infty(\Delta) \), \( c^\lambda(\Delta) \) and \( c_0^\lambda(\Delta) \) are BK spaces with the same norm given by
\[ \|x\|_{\ell^\lambda_\infty(\Delta)} = \|\bar{\Lambda}(x)\|_{\ell_\infty} = \sup_n |\bar{\Lambda}_n(x)|. \tag{3.4.5} \]

For brevity, we shall omit those results concerning the bases, duals and matrix mappings on the spaces \( c_0^\lambda(\Delta) \), \( c^\lambda(\Delta) \) and \( \ell^\lambda_\infty(\Delta) \) which can be found in our research paper [86], and we shall continue to establish some inclusion relations between these spaces.

**Theorem 3.4.1** The inclusions \( c_0^\lambda(\Delta) \subset c^\lambda(\Delta) \subset \ell^\lambda_\infty(\Delta) \) strictly hold.

**Proof.** The inclusions \( c_0^\lambda(\Delta) \subset c^\lambda(\Delta) \subset \ell^\lambda_\infty(\Delta) \) trivially hold. To show that these inclusions are strict, let us firstly consider the sequence \( x = (x_k) \) defined by \( x_k = k + 1 \) for all \( k \in \mathbb{N} \). Then, we obtain by (3.4.3) that
\[ \bar{\Lambda}_n(x) = \frac{1}{\lambda_n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1}) = 1; \quad (n \in \mathbb{N}) \]
which shows that \( \bar{\Lambda}(x) = e \) and hence \( \bar{\Lambda}(x) \in c \setminus c_0 \). Thus, the sequence \( x \) is in \( c^\lambda(\Delta) \) but not in \( c_0^\lambda(\Delta) \). Hence, the inclusion \( c_0^\lambda(\Delta) \subset c^\lambda(\Delta) \) is strict.
Similarly, let us define the sequence $y = (y_k)$ by

$$y_k = \sum_{j=0}^{k} (-1)^j \frac{\lambda_j + \lambda_{j-1}}{\lambda_j - \lambda_{j-1}}; \quad (k \in \mathbb{N}).$$

Then, we have for every $k \in \mathbb{N}$ that

$$y_k - y_{k-1} = (-1)^k \frac{\lambda_k + \lambda_{k-1}}{\lambda_k - \lambda_{k-1}}$$

which leads us with (3.4.3) to the consequence that $\tilde{A}_n(y) = (-1)^n$ for all $n \in \mathbb{N}$ (as we have shown in the proof of Theorem 3.2.3) and hence $\tilde{A}_n(y) \in \ell_\infty \setminus c$. Thus, the sequence $y$ is in $\ell^\lambda_\infty(\Delta)$ but not in $c^\lambda(\Delta)$. Therefore, the inclusion $c^\lambda(\Delta) \subset \ell^\lambda_\infty(\Delta)$ strictly holds. This completes the proof.

**Theorem 3.4.2** The inclusions $c_0 \subset c^0(\Delta)$, $c \subset c^\lambda(\Delta)$ and $\ell_\infty \subset \ell^\lambda_\infty(\Delta)$ strictly hold.

**Proof.** It is trivial that these inclusions hold. To show that these inclusions are strict, we define the sequence $x = (x_k)$ by

$$x_k = \sqrt{k + 1}; \quad (k \in \mathbb{N}). \quad (3.4.6)$$

Then $x \not\in \ell_\infty$. On the other hand, it can easily be seen that $\Delta x = (x_k - x_{k-1}) \in c_0$ and hence $\Delta x \in c^0_0$ by Lemma 3.2.4. Therefore, we obtain by (3.4.1) that $x \in c^0_0(\Delta)$. Thus, the sequence $x$ is in all of the spaces $c^0_0(\Delta)$, $c^\lambda(\Delta)$ and $\ell^\lambda_\infty(\Delta)$ but not in any of the spaces $c_0$, $c$ or $\ell_\infty$. Hence, we deduce that the inclusions $c_0 \subset c^0_0(\Delta)$, $c \subset c^\lambda(\Delta)$ and $\ell_\infty \subset \ell^\lambda_\infty(\Delta)$ are strict. This concludes the proof.

**Theorem 3.4.3** The inclusion $c \subset c^0_0(\Delta)$ strictly holds.

**Proof.** Let $x \in c$ be given. Then $\Delta x \in c_0$ and hence $\Delta x \in c^0_0$ which means that $x \in c^0_0(\Delta)$. Consequently, the inclusion $c \subset c^0_0(\Delta)$ holds. Further, this inclusion is strict, since the sequence $x$ defined by (3.4.6) is in $c^0_0(\Delta)$ but not in $c$.

Now, by combining Theorems 3.4.3 with the proof of Theorem 3.4.2, we have the following corollary:

**Corollary 3.4.4** Although the spaces $\ell_\infty$ and $c^0_0(\Delta)$ overlap, the space $\ell_\infty$ does not include the space $c^0_0(\Delta)$.

**Theorem 3.4.5** The inclusion $\ell_\infty \subset c^0_0(\Delta)$ strictly holds if and only if $z \in c^0_0$, where $z = (z_k)$ is the sequence defined by

$$z_k = \left| 1 - \frac{\lambda_{k+1} - \lambda_k}{\lambda_k - \lambda_{k-1}} \right|; \quad (k \in \mathbb{N}).$$

**Proof.** Suppose that the inclusion $\ell_\infty \subset c^0_0(\Delta)$ holds. Then $\tilde{A}(x) \in c_0$ for all $x \in \ell_\infty$ which means that the matrix $\tilde{A} = (\tilde{A}_{nk})$ is in the class $(\ell_\infty, c_0)$. Thus, it follows by part (c) of Lemma 1.1.4 that

$$\lim_{n \to \infty} \left( \sum_{k=0}^{\infty} |\tilde{A}_{nk}| \right) = 0. \quad (3.4.7)$$

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Further, by taking into account the definition of the matrix \( \bar{\Lambda} \) given by (3.4.2), we have for every \( n \geq 1 \) that
\[
\sum_{k=0}^{\infty} |\bar{\lambda}_{nk}| = \frac{1}{\lambda_n} \sum_{k=0}^{n-1} |(\lambda_k - \lambda_{k+1} - \lambda_k)| + \frac{\lambda_n - \lambda_{n-1}}{\lambda_n}.
\](3.4.8)

Thus, the condition (3.4.7) implies both
\[
\lim_{n \to \infty} \left( \frac{\lambda_n - \lambda_{n-1}}{\lambda_n} \right) = 0
\](3.4.9)
and
\[
\lim_{n \to \infty} \left( \frac{1}{\lambda_n} \sum_{k=0}^{n-1} |(\lambda_k - \lambda_{k+1} - \lambda_k)| \right) = 0.
\](3.4.10)

On the other hand, we have for every \( n \geq 1 \) that
\[
\frac{1}{\lambda_n} \sum_{k=0}^{n-1} |(\lambda_k - \lambda_{k+1} - \lambda_k)| = \frac{\lambda_{n-1}}{\lambda_n} \left( \frac{1}{\lambda_{n-1}} \sum_{k=0}^{n-1} (\lambda_k - \lambda_{k-1}) z_k \right)
\]
and since \( \lim_{n \to \infty} \lambda_{n-1}/\lambda_n = 1 \) by (3.4.9), we obtain by (3.4.10) that
\[
\lim_{n \to \infty} \left( \frac{1}{\lambda_{n-1}} \sum_{k=0}^{n-1} (\lambda_k - \lambda_{k-1}) z_k \right) = 0
\](3.4.11)
which shows that \( z = (z_k) \in c_0^\lambda \).

Conversely, suppose that \( z = (z_k) \in c_0^\lambda \). Then (3.4.11) holds. Further, we have for every \( n \geq 1 \) that
\[
\frac{1}{\lambda_n} \sum_{k=0}^{n-1} |(\lambda_k - \lambda_{k+1} - \lambda_k)| \leq \frac{1}{\lambda_{n-1}} \sum_{k=0}^{n-1} (\lambda_k - \lambda_{k-1}) z_k.
\]

This and (3.4.11) together imply that (3.4.10) holds. On the other hand, we have for every \( n \geq 1 \) that
\[
\frac{\lambda_n - \lambda_{n-1} - \lambda_0}{\lambda_n} = \frac{\lambda_{n-1} - (\lambda_n - \lambda_0)}{\lambda_n}
\]
\[
= \left| \frac{1}{\lambda_n} \sum_{k=0}^{n-1} [(\lambda_k - \lambda_{k-1}) - (\lambda_{k+1} - \lambda_k)] \right|
\]
\[
\leq \frac{1}{\lambda_n} \sum_{k=0}^{n-1} \left| (\lambda_k - \lambda_{k-1}) - (\lambda_{k+1} - \lambda_k) \right|.
\]
Therefore, it follows by (3.4.10) that

$$\lim_{n \to \infty} \left( \frac{\lambda_n - \lambda_{n-1}}{\lambda_n} \right) = \lim_{n \to \infty} \left( \frac{\lambda_n - \lambda_{n-1} - \lambda_0}{\lambda_n} \right) = 0$$

which just shows that (3.4.9) holds. Thus, by using (3.4.9) and (3.4.10), we deduce from (3.4.8) that (3.4.7) holds. This leads us with Lemma 1.1.4 (c) to the consequence that $\bar{\Lambda} \in (\ell_\infty, c_0)$. Hence, the inclusion $\ell_\infty \subset c_0^\lambda(\Delta)$ holds which is a strict inclusion by Corollary 3.4.4. This completes the proof. \(\square\)

As an immediate consequence of Theorem 3.4.5, we have the following corollary:

**Corollary 3.4.6** If $\lim_{n \to \infty} \frac{\lambda_{n+1} - \lambda_n}{\lambda_n - \lambda_{n-1}} = 1$, then the inclusion $\ell_\infty \subset c_0^\lambda(\Delta)$ strictly holds.

Now, we give an example of a sequence $\lambda$ such that the inclusion $\ell_\infty \subset c_0^\lambda(\Delta)$ does not hold.

**Example 3.4.7** Consider the sequence $\lambda = (\lambda_k)$ given by $\lambda_k = 2^{k+1}$ for all $k \in \mathbb{N}$. Then, we obtain by (3.4.3) that

$$\bar{\Lambda}_n(x) = \frac{1}{2^{n+1}} \left( 2x_0 + \sum_{k=1}^{n} 2^k (x_k - x_{k-1}) \right); \quad (n \geq 1).$$

Therefore, by taking $x = (x_k) \in \ell_\infty$ with $x_k = (-1)^k$ for all $k \in \mathbb{N}$, we have for every $n \geq 1$ that

$$\bar{\Lambda}_n(x) = \frac{1}{2^n} \left( 1 + \sum_{k=1}^{n} (-2)^k \right) = \frac{1}{3} \left[ 2(-1)^n + \frac{1}{2^n} \right]$$

which shows that $\bar{\Lambda}(x) \notin c_0$ and so $x \notin c_0^\lambda(\Delta)$. Thus, the inclusion $\ell_\infty \subset c_0^\lambda(\Delta)$ fails.

Finally, let $c_0(\Delta)$, $c(\Delta)$ and $\ell_\infty(\Delta)$ be the difference spaces defined in [53]. Then, we conclude this chapter by the following corollaries which are immediate by Lemmas 3.2.4 and 3.2.5, Theorems 3.2.6 and 3.2.9, and Corollary 3.2.10.

**Corollary 3.4.8** The inclusions $c_0(\Delta) \subset c_0^\lambda(\Delta)$, $c(\Delta) \subset c^\lambda(\Delta)$ and $\ell_\infty(\Delta) \subset \ell_\infty^\lambda(\Delta)$ hold.

**Corollary 3.4.9** The equality $c_0^\lambda(\Delta) \cap c(\Delta) = c_0(\Delta)$ holds.

**Corollary 3.4.10** We have the following:

(a) The inclusions $c_0(\Delta) \subset c_0^\lambda(\Delta)$, $c(\Delta) \subset c^\lambda(\Delta)$ and $\ell_\infty(\Delta) \subset \ell_\infty^\lambda(\Delta)$ strictly hold if and only if $\liminf_{n \to \infty} \frac{\lambda_{n+1}}{\lambda_n} = 1$.

(b) The equalities $c_0^\lambda(\Delta) = c_0(\Delta)$, $c^\lambda(\Delta) = c(\Delta)$ and $\ell_\infty^\lambda(\Delta) = \ell_\infty(\Delta)$ hold if and only if $\liminf_{n \to \infty} \frac{\lambda_{n+1}}{\lambda_n} > 1$.

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