Chapter 2
SEQUENCE SPACES OF GENERALIZED MEANS
The approach constructing a new sequence space by means of the matrix domain of a particular triangle has recently been employed by several authors in many research papers (see for example [8, 10, 17, 51, 60, 74, 95, 105, 113]). In the present chapter, we introduce the notion of generalized means and define some new sequence spaces of generalized means which include the most known sequence spaces as special cases. Further, we study some topological properties of the spaces of generalized means and construct their bases. Finally, we determine the $\alpha$, $\beta$- and $\gamma$-duals of these spaces and characterize some related matrix classes. The most important materials in this chapter can be found in [87].

Throughout, we shall use the following notations:

- By $\mathcal{U}$ and $\mathcal{U}_o$, we denote the sets of all sequences with non-zero terms and non-zero first terms, respectively; that is
  $$\mathcal{U} = \left\{ u = (u_k) \in w : u_k \neq 0 \text{ for all } k \right\} \text{ and } \mathcal{U}_o = \left\{ u = (u_k) \in w : u_0 \neq 0 \right\}.$$

- For simplicity in notation, we may write $xy = (x_k y_k)_{k=0}^{\infty}$ for any $x, y \in w$. Also, if $y \in \mathcal{U}$ then we put $x/y = (x_k/y_k)_{k=0}^{\infty}$ and $1/y = (1/y_k)_{k=0}^{\infty}$.

- For any sequences $s, t \in w$, the convolution $s \ast t$ is a sequence defined by
  $$(s \ast t)_n = \sum_{k=0}^{n} s_{n-k} t_k; \quad (n \in \mathbb{N}).$$

- For $k \in \mathbb{N}$ and $a \in \mathbb{R}$ with $a \notin \{-1, -2, -3, \ldots\}$, we define the binomial coefficients by
  $$\binom{a}{k} = \begin{cases} 
  a(a-1) \cdots (a-k+1)/k!; & (k \neq 0), \\
  1; & (k = 0), 
  \end{cases}$$
  that is
  $$\binom{a}{k} = \frac{\Gamma(a+1)}{k! \Gamma(a-k+1)}$$
  and their basic properties can be found in [21, p.101]. For example, if $a \in \mathbb{N}$ then we have
  $$\binom{a}{k} = \begin{cases} 
  \frac{a!}{k!(a-k)!}; & (0 \leq k \leq a), \\
  0; & (k > a). 
  \end{cases}$$
2.1 Notion of generalized means

Throughout this chapter, let \( r, t \in \mathcal{U} \) and \( s \in \mathcal{U}_s \). For any sequence \( x = (x_n) \in w \), we write \( \bar{x} = (\bar{x}_n) \) for the sequence of generalized means of \( x \), and we define it by

\[
\bar{x}_n = \frac{1}{r_n} \sum_{k=0}^{n} s_{n-k} t_k x_k; \quad (n \in \mathbb{N}),
\]

(2.1.1)

that is \( \bar{x}_n = (s * t x)_n / r_n \) for all \( n \in \mathbb{N} \). The term \( \bar{x}_n \) in (2.1.1) is called the generalized mean of \( x_0, x_1, \ldots, x_n \) for every \( n \in \mathbb{N} \).

Further, we define the infinite matrix \( \bar{A}(r, s, t) \) by

\[
(\bar{A}(r, s, t))_{nk} = \begin{cases} 
\frac{s_{n-k} t_k}{r_n}; & (0 \leq k \leq n), \\
0; & (k > n)
\end{cases}
\]

(2.1.2)

for all \( n, k \in \mathbb{N} \), and we call it the matrix of generalized means. Then, by using the notation of (1.1.1), it follows by (2.1.1) that \( \bar{x} = (\bar{A}(r, s, t))x \) for all \( x \in w \).

Remark 2.1.1 It is obvious by (2.1.2) that \( \bar{A}(r, s, t) \) is a triangle. Also, it can easily be seen that \( \bar{A}(r, s, t) \) is regular if and only if \( s_{n-i} / r_n \to 0 \) (\( n \to \infty \)) for each \( i \in \mathbb{N} \), \( \sum_{k=0}^{n} |s_{n-k} t_k| = O(|r_n|) \) and \( (s * t)_n / r_n \to 1 \) as \( n \to \infty \) (cf. [111, p.1478]).

It is worth mentioning that the matrix \( \bar{A}(r, s, t) \) defines a new method of summability which includes the most classical methods of summability as special cases. For instance, we will show, in the following example, that the general forms of the well-known matrices of Nörlund, Cesàro, Euler and weighted means, and some other matrices can be obtained as special cases of the matrix \( \bar{A}(r, s, t) \) of generalized means.

Example 2.1.2 The above definition of the matrix \( \bar{A}(r, s, t) \) given by (2.1.2) includes the following special cases:

1. If \( r_n = (s * t)_n \neq 0 \) for all \( n \), then \( \bar{A}(r, s, t) \) is reduced to the matrix \( (N, s, t) \) of generalized Nörlund means [97, 106, 111]. In particular, if \( t = e \) then \( \bar{A}(r, s, t) \) is reduced to the familiar matrix \( (N, s) \) of Nörlund means [22, 77, 113].

2. If \( \alpha > 0 \),

\[
 r_k = \binom{k + \alpha}{k}, \quad s_k = \binom{k + \alpha - 1}{k} \quad \text{and} \quad t_k = 1
\]

for all \( k \), then \( \bar{A}(r, s, t) \) is reduced to the matrix \( (C, \alpha) \) of Cesàro means of order \( \alpha \) [21, 23, 115]. In particular, if \( \alpha = 1 \) then \( \bar{A}(r, s, t) \) is reduced to the famous matrix \( (C, 1) \) of arithmetic means [2, 95, 105].

3. If \( 0 < \alpha < 1 \), \( r_k = 1 / k! \), \( s_k = (1 - \alpha)^k / k! \) and \( t_k = \alpha^k / k! \) for all \( k \), then \( \bar{A}(r, s, t) \) is reduced to the matrix \( (E, \alpha) \) of Euler means* of order \( \alpha \) [4, 8, 82].

*We may equivalently put \( r_k = (1 + \alpha)^k / k! \), \( s_k = \alpha^k / k! \) and \( t_k = 1 / k! \) for all \( k \), where \( \alpha > 0 \) (see [49, 65, 78]).
(4) If $s = e$, then $\bar{A}(r, s, t)$ is reduced to the matrix $(\bar{\bar{N}}, 1/r, t)$ of generalized weighted means [6, 64, 74]. In particular, if $t_n > 0$ (t is a sequence of positive reals) and $r_n = \sum_{k=0}^{n} t_k$ for all $n$, then $\bar{A}(r, s, t)$ is reduced to the matrix $(\bar{\bar{N}}, t)$ of weighted means [48, 60, 69].

(5) If $\alpha, \beta, \gamma \in \mathbb{C}$ with $\alpha, \beta \neq 0$, $r = t = e$ and $s = (\alpha, \beta, \gamma, 0, 0, \ldots)$, then $\bar{A}(r, s, t)$ is reduced to the matrix $B(\alpha, \beta, \gamma)$ studied in [19, 41]. In particular, if $\gamma = 0$ then $\bar{A}(r, s, t)$ is reduced to the generalized difference matrix $B(\alpha, \beta)$ [5, 20, 31]. Especially, if $r = t = e$ and $s = e^{(0)} - e^{(1)} = (1, -1, 0, 0, \ldots)$ then $\bar{A}(r, s, t)$ is reduced to the band matrix $\Delta$ defining the difference operator [16, 48, 71].

(6) If $0 < \alpha < 1$, $r_k = k + 1$, $s_k = 1$ and $t_k = 1 + \alpha^k$ for all $k$, then $\bar{A}(r, s, t)$ is reduced to the matrix $A^0$ studied in [10, 11, 12].

(7) If $r = e$, $s = e^{(0)}$ and $t_k > 0$ for all $k$, then $\bar{A}(r, s, t)$ is reduced to the diagonal matrix $D_t$ studied by de Malafosse [59, 60, 63].

(8) Finally, if $r = s = t = e$ then $\bar{A}(r, s, t)$ is reduced to the matrix $\Sigma$ defining the partial sum [7, 64, 73].

Now, since $\bar{A}(r, s, t)$ is a triangle, it has a unique inverse $(\bar{A}(r, s, t))^{-1}$ which is also a triangle [70, Proposition 1.1]. More precisely, by making a slight generalization of the work done in [77], we put $D_0^{(s)} = 1/s_0$ and

$$D_n^{(s)} = \frac{1}{s_0^{n+1}} \begin{vmatrix} s_1 & s_0 & 0 & 0 & \cdots & 0 \\ s_2 & s_1 & s_0 & 0 & \cdots & 0 \\ s_3 & s_2 & s_1 & s_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ s_{n-1} & s_{n-2} & s_{n-3} & s_{n-4} & \cdots & s_0 \\ s_n & s_{n-1} & s_{n-2} & s_{n-3} & \cdots & s_1 \end{vmatrix}; \quad (n = 1, 2, 3, \ldots).$$

Then, the entries of $(\bar{A}(r, s, t))^{-1}$ are given by

$$\begin{cases} (-1)^{n-k} D_n^{(s)} r_k / t_n; & (0 \leq k \leq n), \\ 0; & (k > n) \end{cases}$$

for all $n, k \in \mathbb{N}$. Therefore, we have by (2.1.1) that

$$x_n = \frac{1}{t_n} \sum_{k=0}^{n} (-1)^{n-k} D_n^{(s)} r_k \bar{x}_k; \quad (n \in \mathbb{N}).$$

**Remark 2.1.3** Let us remark that the identity matrix $I$ is a matrix of generalized means, that is $I = \bar{A}(e, e^{(0)}, e)$. Also, we may note by (2.1.2) and (2.1.3) that $(\bar{A}(r, s, t))^{-1} = \bar{A}(t, s', r)$, where $s' = (s'_n)$ with $s'_n = (-1)^n D_n^{(s)}$ for all $n \in \mathbb{N}$, which shows that the inverse of a matrix of generalized means is also a matrix of generalized means. This leads us to another idea concerning the class $\bar{A} = \{ \bar{A}(r, s, t) : r, t \in \mathcal{U} \text{ and } s \in \mathcal{U}_s \}$ of all matrices of generalized means. For example, we may ask the
natural question: What about the other algebraic properties of the class $\mathcal{A}$? The answer is left as an open problem for future research.

**Remark 2.1.4** We should note that $(\bar{A}(r,s,t))^{-1}_{nk}$ in (2.1.3) is just a notation for the entries of $(\bar{A}(r,s,t))^{-1}$, and not for $1/(\bar{A}(r,s,t))_{nk}$.

### 2.2 The sequence spaces of generalized means

For an arbitrary subset $X$ of $\mathfrak{w}$, we define the set $X(r,s,t)$ as the matrix domain of the triangle $\bar{A}(r,s,t)$ in $X$, that is

$$X(r,s,t) = \left\{ x = (x_k) \in \mathfrak{w} : \bar{x} = \left( \frac{1}{r_n} \sum_{k=0}^{n} s_{n-k} t_k x_k \right)_{n=0}^{\infty} \in X \right\}. \quad (2.2.1)$$

Then, it is obvious that $X(r,s,t)$ is a sequence space whenever $X$ is a sequence space, and we call it the sequence space of generalized means. In particular, if $r_n = (s \ast t)_n \neq 0$ for all $n \in \mathbb{N}$, then we write $X(s,t)$ instead of $X(r,s,t)$ and we call it the sequence space of generalized Nörlund means.

**Remark 2.2.1** It is worth mentioning that $X(r,s,t)$ is reduced to $X$ in the special case $r = t = e$ and $s = e^{(0)}$, that is $X(e,e^{(0)},e) = X$, which allows us to deal with any sequence space as a sequence space of generalized means. In particular, we may use this fact for the classical sequence spaces. Further, we have $\ell_{\infty}(e,e,e) = bs$, $c(e,e,e) = cs$, $c_0(e,e,e) = cs_0$ and $\ell_1(e,e^{(0)} - e^{(1)},e) = bv$. Moreover, we may note by Example 2.1.2 that all of the known sequence spaces studied in [4, 8, 10, 11, 12, 16, 31, 48, 51, 60, 63, 64, 69, 71, 74, 82, 84, 95, 105] and [113] are special cases of the space $X(r,s,t)$ of generalized means.

Now, we may begin with the following result which is essential in the text.

**Theorem 2.2.2** Let $X \subset \mathfrak{w}$ and $\bar{X} = X(r,s,t)$. Then, we have

(a) If $(X,d)$ is a linear metric space, then $\bar{X}$ is also a linear metric space with the metric

$$\bar{d}(x,y) = d(\bar{x}, \bar{y}); \quad (x, y \in \bar{X}). \quad (2.2.2)$$

Furthermore, if $X$ is an FK space then $\bar{X}$ is an FK space too.

(b) If $X$ is a normed space, then so is $\bar{X}$ with the norm

$$\|x\|_X = \|\bar{x}\|_X; \quad (x \in \bar{X}). \quad (2.2.3)$$

(c) If $X$ is a BK space, then $\bar{X}$ is a BK space with the norm given by (2.2.3). Moreover, if $Y$ is a closed subspace of $X$, then $\bar{Y} = Y(r,s,t)$ is a closed subspace of $\bar{X}$.

**Proof.** Part (a) is immediate by [117, Theorem 4.3.12], part (b) follows from [68, Lemma 3.2], and part (c) is obtained by [49, Theorem 2.2].

As immediate consequences of Theorem 2.2.2, we have the following corollaries:
Corollary 2.2.3 Let \( \bar{w} = w(r, s, t) \). Then \((\bar{w}, d_{\bar{w}})\) is an FK space which is not a BK space, where

\[
d_{\bar{w}}(x, y) = d_{\bar{w}}(\bar{x}, \bar{y}) = \sum_{n=0}^{\infty} \frac{1}{2^n} \left( \frac{|\bar{x}_n - \bar{y}_n|}{1 + |\bar{x}_n - \bar{y}_n|} \right); \quad (x, y \in \bar{w}).
\]

Corollary 2.2.4 Let \( \bar{c}_0 = c_0(r, s, t) \), \( \bar{c} = c(r, s, t) \), \( \bar{\ell}_\infty = \ell_\infty(r, s, t) \) and \( \bar{\ell}_p = \ell_p(r, s, t) \) for \( 1 \leq p < \infty \). Then, we have

(a) The spaces \( \bar{c}_0, \bar{c} \) and \( \bar{\ell}_\infty \) are BK spaces with the same norm given by

\[
\|x\|_{\bar{\ell}_\infty} = \|\bar{x}\|_{\ell_\infty} = \sup_n \left| \frac{1}{r_n} \sum_{k=0}^{n} s_{n-k} t_k x_k \right|.
\]

Furthermore, the spaces \( \bar{c}_0 \) and \( \bar{c} \) are closed subspaces of \( \bar{\ell}_\infty \).

(b) The space \( \bar{\ell}_p \) (\( 1 \leq p < \infty \)) is a BK space with the norm

\[
\|x\|_{\bar{\ell}_p} = \|\bar{x}\|_{\ell_p} = \left( \sum_{n=0}^{\infty} \left| \frac{1}{r_n} \sum_{k=0}^{n} s_{n-k} t_k x_k \right|^p \right)^{1/p}.
\]

Moreover, we have the following results:

Theorem 2.2.5 Let \( X \subset w \). Then, we have

(a) If \( X \) is a linear space, then \( X(r, s, t) \) is linearly isomorphic to \( X \).

(b) If \( X \) is a linear metric space, then \( X(r, s, t) \) is isometrically isomorphic to \( X \).

(c) If \( X \) is a normed space, then \( X(r, s, t) \) is norm isomorphic to \( X \).

Proof. Let \( X \subset w \) be a linear space and define the linear operator \( L : X(r, s, t) \to X \) by \( L(x) = \bar{x} \), i.e., \( L(x) = (\bar{A}(r, s, t))x \) for all \( x \in X(r, s, t) \). Then, it is obvious by (2.1.1), (2.1.4) and (2.2.1) that \( L \) is bijective. This proves (a). Further, if \( (X, d) \) is a linear metric space then it follows by (2.2.2) that \( \bar{d}(x, y) = d(L(x), L(y)) \) for all \( x, y \in X(r, s, t) \), i.e., \( L \) is isometry and hence (b) follows. Moreover, if \( X \) is a norm space then \( L \) is norm preserving by (2.2.3). This yields (c) and completes the proof. \( \square \)

Theorem 2.2.6 Let \( X \subset w \) be a linear metric space with Schauder basis \( (a^{(k)})_{k=0}^{\infty} \) and define the sequence \( b^{(k)} = (b^{(k)}_n)_{n=0}^{\infty} \in X(r, s, t) \) for every fixed \( k \in \mathbb{N} \) by

\[
b^{(k)}_n = \frac{1}{t_n} \sum_{j=0}^{n} (-1)^{n-j} D_{n-j}^{(s)} r_j a^{(k)}_j; \quad (n \in \mathbb{N}).
\]

Then, the sequence \( (b^{(k)})_{k=0}^{\infty} \) is a Schauder basis for \( X(r, s, t) \) and every \( x \in X(r, s, t) \) has a unique representation \( x = \sum_{k=0}^{\infty} \alpha_k(\bar{x}) b^{(k)} \), where \( (\alpha_k(\bar{x}))_{k=0}^{\infty} \) is the sequence of coefficients of \( \bar{x} = (\bar{A}(r, s, t))x \) with respect to the basis \( (a^{(k)})_{k=0}^{\infty} \).

Proof. Since the operator \( L : X(r, s, t) \to X \) defined in the proof of Theorem 2.2.5 (b) is an isometric isomorphism, the inverse image of the Schauder basis of \( X \) is the
Schauder basis for $X(r, s, t)$. Also, we have by (2.1.3) that $L(b^{(k)}) = (\bar{A}(r, s, t))b^{(k)} = a^{(k)}$ for all $k$. Therefore, the sequence $(b^{(k)})_{k=0}^{\infty}$ is a Schauder basis for $X(r, s, t)$ (see [70, Proposition 2.1]). Further, let $x \in X(r, s, t)$ be given. Then $\bar{x} = L(x) \in X$, where $\bar{x} = (\bar{A}(r, s, t))x$. Thus, if $(\alpha_k(\bar{x}))_{k=0}^{\infty}$ is the sequence of coefficients of $\bar{x}$ with respect to the basis $(a^{(k)})_{k=0}^{\infty}$, then we have $d(\bar{x}, \bar{x}(m)) \to 0 \ (m \to \infty)$, where $d$ denotes the metric on $X$ and $\bar{x}(m) = \sum_{k=0}^{m} \alpha_k(\bar{x}) a^{(k)}$ for all $m \in \mathbb{N}$.

Now, let $x^{(m)} = \sum_{k=0}^{m} \alpha_k(\bar{x}) b^{(k)}$ for all $m \in \mathbb{N}$. Then, we have for every $m \in \mathbb{N}$ that $L(x^{(m)}) = \sum_{k=0}^{m} \alpha_k(\bar{x}) a^{(k)} = \bar{x}(m)$. Therefore, we obtain by (2.2.2) that

$$d(x, x^{(m)}) = d(L(x), L(x^{(m)})) = d(\bar{x}, \bar{x}(m)) \to 0 \ (m \to \infty).$$

Hence, the sequence $x$ is uniquely represented by $x = \sum_{k=0}^{\infty} \alpha_k(x) a^{(k)}$ and this concludes the proof.

**Corollary 2.2.7** Let $X$ be an FK space with $AK$ and $b^{(k)} = (b_n^{(k)})_{n=0}^{\infty} \in X(r, s, t)$ be defined for every fixed $k \in \mathbb{N}$ by

$$b_n^{(k)} = \begin{cases} 0; & (n < k), \\ (-1)^{n-k}D_{n-k}^{(s)}r_k/t_n; & (n \geq k). \end{cases}$$

(2.2.4)

Then, we have

(a) The sequence $(b^{(k)})_{k=0}^{\infty}$ is a Schauder basis for $X(r, s, t)$ and every $x \in X(r, s, t)$ has a unique representation $x = \sum_{k=0}^{\infty} \bar{x} b^{(k)}$, where $\bar{x} = (\bar{A}(r, s, t))x$.

(b) Let $Y = X \oplus e$ and define the sequence $b = (b_n)_{n=0}^{\infty} \in Y(r, s, t)$ by

$$b_n = \frac{1}{t_n} \sum_{j=0}^{n} (-1)^{n-j}D_{n-j}^{(s)}r_j; \quad (n \in \mathbb{N}).$$

Then, the sequence $(b, b^{(0)}, b^{(1)}, b^{(2)}, \ldots)$ is a Schauder basis for $Y(r, s, t)$ and every $y \in Y(r, s, t)$ has a unique representation $y = lb + \sum_{k=0}^{\infty} (\bar{y}_k - l) b^{(k)}$, where $\bar{y} = (\bar{A}(r, s, t))y$ and $l$ is the uniquely determined complex number such that $\bar{y} - le \in X$.

**Proof.** It is clear by (2.1.4) that $(\bar{A}(r, s, t))b^{(k)} = e^{(k)}$ for all $k$ and $(\bar{A}(r, s, t))b = e$. Thus, part (a) follows from Theorem 2.2.6, and part (b) is immediate by part (a).

**Remark 2.2.8** We may note that part (a) of Corollary 2.2.7 includes the finding of Schauder basis for the spaces $c_0(r, s, t)$ and $\ell_p(r, s, t)$, where $1 \leq p < \infty$. Also, if we put $X = c_0$ in part (b) of Corollary 2.2.7, then $Y = c$ and so we get the basis for $c(r, s, t)$ and the unique representation $x = lb + \sum_{k=0}^{\infty} (\bar{x}_k - l) b^{(k)}$ of any $x \in c(r, s, t)$, where $\bar{x} = (\bar{A}(r, s, t))x$ and $l = \lim_{k \to \infty} \bar{x}_k$.

Finally, we conclude this section by the following corollary which is immediate by combining Theorems 2.2.2 (b) and 2.2.6.

**Corollary 2.2.9** If $X$ is a normed sequence space with a Schauder basis, then the space $X(r, s, t)$ is separable.
2.3 \(\alpha-, \beta-\) and \(\gamma\)-duals of the space \(X(r, s, t)\)

In the present section, we essentially determine the \(\beta\)-dual of the space \(X(r, s, t)\)
of generalized means. Further, we similarly deduce some results concerning the \(\alpha\)- and
\(\gamma\)-duals of the space \(X(r, s, t)\).

In the beginning, let us recall that the inverse of our triangle \(\bar{A}(r, s, t)\) is that
triangle defined by (2.1.3). Then, for any given \(a = (a_k)_{k=0}^{\infty} \in w\), we define the
sequence \(\tilde{a} = (\tilde{a}_k)_{k=0}^{\infty}\) and the triangle \(A' = (a'_{mk})_{m,k=0}^{\infty}\), which will frequently be used,
as follows:

\[
\tilde{a}_k = \sum_{j=k}^{\infty} (-1)^{j-k} D_{j-k}^{(s)} r_k a_j / t_j; \quad (k \in \mathbb{N}) \tag{2.3.1}
\]

and

\[
a'_m k = \sum_{j=m}^{\infty} (-1)^{j-k} D_{j-k}^{(s)} r_k a_j / t_j; \quad (0 \leq k \leq m, m \in \mathbb{N}) \tag{2.3.2}
\]

provided the series on the right hand sides converge for all \(k, m \in \mathbb{N}\).

With the notations of (2.3.1) and (2.3.2), we have the following results determining
the \(\beta\)-dual of the space \(X(r, s, t)\) of generalized means.

**Theorem 2.3.1** Let \(X\) be an \(FK\) space with \(AK\) or \(X = \ell_\infty\). Then, we have \(a \in (X(r, s, t))^\beta\) if and only if

\[
\sum_{j=k}^{\infty} (-1)^{j-k} D_{j-k}^{(s)} a_j / t_j \text{ converges for every } k \in \mathbb{N}, \tag{2.3.3}
\]

\[
\tilde{a} = (\tilde{a}_k) \in X^\beta, \tag{2.3.4}
\]

\[
A' \in (X, c_0). \tag{2.3.5}
\]

Furthermore, if \(a \in (X(r, s, t))^\beta\) then we have

\[
\sum_{k=0}^{\infty} a_k x_k = \sum_{k=0}^{\infty} \tilde{a}_k \bar{x}_k \tag{2.3.6}
\]

for all \(x = (x_k) \in X(r, s, t)\) with \(\bar{x} = (\bar{A}(r, s, t))x\).

**Proof.** Suppose that \(a \in (X(r, s, t))^\beta\). Then \(ax \in cs\) for all \(x \in X(r, s, t)\). Thus, we
have for every fixed \(k \in \mathbb{N}\) that \(ab^{(k)} \in cs\), where \(b^{(k)} \in X(r, s, t)\) is the sequence defined
by (2.2.4) for each \(k \in \mathbb{N}\). This shows that (2.3.3) holds. Therefore, it follows by (2.3.1)
and (2.3.2) that the sequence \(\tilde{a} = (\tilde{a}_k)\) and the triangle \(A' = (a'_m k)\) are well-defined.
This leads us with Lemma 1.2.9 to the consequence that (2.3.4) and (2.3.5) hold.

Conversely, suppose that conditions (2.3.3), (2.3.4) and (2.3.5) are satisfied for
a given sequence \(a \in w\). Then, we deduce by Lemma 1.2.9 and Remark 1.2.11 that
\(a \in (X(r, s, t))^\beta\).

Finally, if \(a \in (X(r, s, t))^\beta\) then we get (2.3.6) from (1.2.6) of Lemma 1.2.9. This
completes the proof. \(\square\)
Theorem 2.3.2  We have \( a \in (c(r, s, t))^{\beta} \) if and only if (2.3.3) holds and

\[
\tilde{a} = (\tilde{a}_k) \in \ell_1, \quad \tag{2.3.7}
\]
\[
\sup_m \left( \sum_{k=0}^{m} |a'_{mk}| \right) < \infty, \quad \tag{2.3.8}
\]
\[
\lim_{m \to \infty} \left( \sum_{k=0}^{m} a'_{mk} \right) = \alpha. \quad \tag{2.3.9}
\]

Furthermore, if \( a \in (c(r, s, t))^{\beta} \) then we have

\[
\sum_{k=0}^{\infty} a_k x_k = \sum_{k=0}^{\infty} \tilde{a}_k \bar{x}_k - l\alpha \quad \tag{2.3.10}
\]

for all \( x = (x_k) \in c(r, s, t) \) with \( \bar{x} = (\bar{A}(r, s, t))x \) and \( l = \lim_{k \to \infty} \bar{x}_k \).

**Proof.** By following the same technique used in the proof of Theorem 2.3.1 with Lemma 1.2.10 instead of Lemma 1.2.9, we obtain that \( a \in (c(r, s, t))^{\beta} \) if and only if (2.3.3) and (2.3.7) hold and \( A' = (a'_{mk}) \) is the triangle defined by (2.3.2). Further, it follows by Lemma 1.1.2 (a) that \( A' \in (c, c) \) if and only if (2.3.8) and (2.3.9) hold and

\[
\lim_{m \to \infty} a'_{mk} \text{ exists for every } k \in \mathbb{N}. \quad \tag{2.3.11}
\]

On the other hand, let \( k \in \mathbb{N} \) be given. For any \( m > k \), we derive from (2.3.1) and (2.3.2) that

\[
\tilde{a}_k = \sum_{j=k}^{m-1} (-1)^{j-k} D^{(s)}_{j-k} r_k a_j / t_j = a'_{mk}. \quad \tag{2.3.12}
\]

Also, it is obvious by (2.3.3) that

\[
\lim_{m \to \infty} \left( \sum_{j=k}^{m-1} (-1)^{j-k} D^{(s)}_{j-k} r_k a_j / t_j \right) = \tilde{a}_k; \quad (k \in \mathbb{N}).
\]

Therefore, we obtain by (2.3.12) that

\[
\lim_{m \to \infty} a'_{mk} = 0 \text{ for all } k \in \mathbb{N}. \quad \tag{2.3.13}
\]

This just shows that (2.3.11) is redundant which concludes the proof, since (2.3.10) is immediate by (1.2.7) of Lemma 1.2.10.

Now, by taking into account that (2.3.3) implies (2.3.13), we have the following corollaries which are immediate by combining Theorem 2.3.1 with Lemmas 1.1.4 and 1.2.6 (c), respectively.

**Corollary 2.3.3**  Let \( 1 < p < \infty \) and \( q = p/(p - 1) \). Then, we have
(a) $a \in (\ell_p(r, s, t))^\beta$ if and only if (2.3.3) holds and
\[
\tilde{a} = (\tilde{a}_k) \in \ell_q,
\]
\[
\sup_m \left( \sum_{k=0}^m |\tilde{a}'_{mk}|^q \right) < \infty.
\]
(b) $a \in (\ell_1(r, s, t))^\beta$ if and only if (2.3.3) holds and
\[
\tilde{a} = (\tilde{a}_k) \in \ell_\infty,
\]
\[
\sup_m \left( \max_{0 \leq k \leq m} |\tilde{a}'_{mk}| \right) < \infty.
\]
(c) $a \in (\ell_\infty(r, s, t))^\beta$ if and only if (2.3.3) and (2.3.7) hold and
\[
\lim_{m \to \infty} \left( \sum_{k=0}^m |\tilde{a}'_{mk}| \right) = 0.
\]
(d) $a \in (c_0(r, s, t))^\beta$ if and only if (2.3.3), (2.3.7) and (2.3.8) hold.

**Corollary 2.3.4** Let $X$ be a BK space with AK. Then, we have $a \in (X(r, s, t))^\beta$ if and only if (2.3.3) and (2.3.4) hold and
\[
\sup_m \|A'_m\|_X^* < \infty,
\]
where $A'_m = (a'_{m0}, a'_{m1}, \ldots, a'_{mm}, 0, 0, \ldots)$ for every $m \in \mathbb{N}$.

Now, let us turn to the $\alpha$- and $\gamma$-duals of the space $X(r, s, t)$. Then, we have the following result which can be proved similarly as the proofs of Theorems 2.3.1 and 2.3.2, above. Thus, we shall omit its proof.

**Theorem 2.3.5** Let $X$ be an FK space. Then, we have
(a) $a \in (X(r, s, t))^\alpha$ if and only if (2.3.3) holds and
\[
\tilde{a} = (\tilde{a}_k) \in X^a,
\]
\[
A' \in (X, \ell_1).
\]
(b) $a \in (X(r, s, t))^\gamma$ if and only if (2.3.3) holds and
\[
\tilde{a} = (\tilde{a}_k) \in X^\gamma,
\]
\[
A' \in (X, \ell_\infty).
\]

Further, by using Lemma 1.2.6, we deduce the following corollary:

**Corollary 2.3.6** Let $X$ be a BK space with AK. Then, we have
(a) \( a \in (X(r,s,t))^\alpha \) if and only if (2.3.3) and (2.3.17) hold and
\[
\sup_{N \in \mathcal{F}} \left\| \sum_{m \in N} A_m' \right\|_X < \infty.
\]
(b) \( a \in (X(r,s,t))^\gamma \) if and only if (2.3.3), (2.3.16) and (2.3.18) hold.

Finally, it is obvious that the special case of Theorem 2.3.5 when \( X \) is any of the classical sequence spaces can be obtained similarly as in Corollary 2.3.3 by using Lemmas 1.1.1, 1.1.5, 1.1.6 and 1.1.7. For example, we shall end this section with the following corollary:

**Corollary 2.3.7** We have the following:
(a) \( a \in (\ell_1(r,s,t))^\alpha \) if and only if (2.3.3) and (2.3.14) hold and
\[
\sup_k \left( \sum_{m=k}^\infty |a_{mk}'| \right) < \infty.
\]
(b) \( a \in (\ell_1(r,s,t))^\gamma \) if and only if (2.3.3), (2.3.14) and (2.3.15) hold.

**Remark 2.3.8** Let us consider the special case \( s = e^{(0)} \). Then, it follows by the definition of \( D_n^{(s)} \) that \( D_n^{(s)} = 0 \) for all \( n \geq 1 \). Similarly, if \( s = e \) then we have \( D_n^{(s)} = 0 \) for all \( n \geq 2 \). Thus, we may note, in such cases, that (2.3.3) trivially holds for any \( a \in w \). This means that there exist some \( s \in \mathcal{U}_o \) such that the condition (2.3.3) is redundant in all our results of this section.

### 2.4 Matrix mappings on the spaces of generalized means

In this last section, we characterize some matrix mappings on the sequence spaces of generalized means.

For an infinite matrix \( A = (a_{nk})_{n,k=0}^\infty \), we assume that
\[
\sum_{j=k}^\infty (-1)^{j-k} D_j^{(s)} a_{nj} / t_j \text{ converge for all } n, k \in \mathbb{N}.
\]  

(2.4.1)

Then, we define the *associated matrix* \( \tilde{A} = (\tilde{a}_{nk})_{n,k=0}^\infty \) by
\[
\tilde{a}_{nk} = \sum_{j=k}^\infty (-1)^{j-k} D_j^{(s)} r_k a_{nj} / t_j; \quad (n, k \in \mathbb{N}).
\]  

(2.4.2)

Also, for each \( n \in \mathbb{N} \), we define the triangle \( A^{(n)} = (a_{mk}^{(n)})_{m,k=0}^\infty \) by
\[
a_{mk}^{(n)} = \sum_{j=m}^\infty (-1)^{j-k} D_j^{(s)} r_k a_{nj} / t_j; \quad (0 \leq k \leq m, m \in \mathbb{N}),
\]  

(2.4.3)

where \( a_{mk}^{(n)} = 0 \) for all \( k > m \) (\( m, n \in \mathbb{N} \)).
With the above notations, we have the following results on matrix transformations (see [70, Theorem 3.4; Remark 3.5]).

**Theorem 2.4.1** Let $X$ be an FK space with $AK$ or $X = l_\infty$, $\tilde{X} = X(r, s, t)$ and $Y$ a sequence space. Then, we have $A \in (\tilde{X}, Y)$ if and only if (2.4.1) holds, $\tilde{A} \in (X, Y)$ and $A^{(n)} \in (X, c_0)$ for all $n \in \mathbb{N}$. Moreover, if $A \in (\tilde{X}, Y)$ then we have

$$
\sum_{k=0}^{\infty} a_{nk} x_k = \sum_{k=0}^{\infty} \tilde{a}_{nk} \bar{x}_k; \quad (n \in \mathbb{N})
$$

(2.4.4)

for all $x = (x_k) \in \tilde{X}$ with $\bar{x} = (\tilde{A}(r, s, t))x$.

**Proof.** Assume that $A \in (\tilde{X}, Y)$. Then $A_n \in \tilde{X}^\beta$ for all $n \in \mathbb{N}$. Thus, it follows by Theorem 2.3.1 that (2.4.1) holds, $\tilde{A}_n \in X^\beta$ and $A^{(n)} \in (X, c_0)$ for all $n \in \mathbb{N}$. Further, since $A_n \in \tilde{X}^\beta$ for all $n \in \mathbb{N}$, we have by (2.3.6) that the equality (2.4.4) holds for every $x \in \tilde{X}$ with $\bar{x} = (\tilde{A}(r, s, t))x$.

Now, let $\bar{x} = (\bar{x}_k) \in X$ be given and let us define the sequence $x = (x_k)$ by (2.1.4). Then $\bar{x} = (\tilde{A}(r, s, t))x$ and so $x \in \tilde{X}$. Therefore, we obtain by (2.4.4) that $Ax = \tilde{A} \bar{x}$. This leads us with our assumption to the consequence that $\tilde{A} \bar{x} \in Y$ for all $\bar{x} \in X$ and hence $\tilde{A} \in (X, Y)$, since $\tilde{A}_n \in X^\beta$ for all $n \in \mathbb{N}$ (as we have already shown).

Conversely, suppose that (2.4.1) holds, $\tilde{A} \in (X, Y)$ and $A^{(n)} \in (X, c_0)$ for all $n \in \mathbb{N}$. Since $\tilde{A} \in (X, Y)$ implies $\tilde{A}_n \in X^\beta (n \in \mathbb{N})$, we deduce from Theorem 2.3.1 that $A_n \in \tilde{X}^\beta$ for all $n \in \mathbb{N}$ and so the equality (2.4.4) holds for all sequences $x \in \tilde{X}$ and $\bar{x} \in X$ which are connected by the relation $\bar{x} = (\tilde{A}(r, s, t))x$. Further, since $\tilde{A} \bar{x} \in Y$ for all $\bar{x} \in X$, we obtain by (2.4.4) that $Ax \in Y$ for all $x \in \tilde{X}$. Hence $A \in (\tilde{X}, Y)$ and this completes the proof. \(\square\)

**Theorem 2.4.2** Let $Y$ be a sequence space and $\bar{c} = c(r, s, t)$. Then, we have $A \in (\bar{c}, Y)$ if and only if (2.4.1) holds, $\tilde{A} \in (c_0, Y)$ and

$$
\sup_{m} \left( \sum_{k=0}^{m} |a_{mk}^{(n)}| \right) < \infty \text{ for every } n \in \mathbb{N},
$$

(2.4.5)

$$
\lim_{m \to \infty} \left( \sum_{k=0}^{m} a_{mk}^{(n)} \right) = \alpha_n \text{ for all } n \in \mathbb{N},
$$

(2.4.6)

$$
\left( \sum_{k=0}^{\infty} \tilde{a}_{nk} - \alpha_n \right)_{n=0} \in Y.
$$

(2.4.7)

Moreover, if $A \in (\bar{c}, Y)$ then we have

$$
\sum_{k=0}^{\infty} a_{nk} x_k = \sum_{k=0}^{\infty} \tilde{a}_{nk} \bar{x}_k - l \alpha_n; \quad (n \in \mathbb{N})
$$

(2.4.8)

for all $x = (x_k) \in \bar{c}$ with $\bar{x} = (\tilde{A}(r, s, t))x$ and $l = \lim_{k \to \infty} \bar{x}_k$.

**Proof.** Suppose that $A \in (\bar{c}, Y)$. Then, we have $A \in (\bar{c}_0, Y)$, where $\bar{c}_0 = c_0(r, s, t)$, and hence (2.4.1) holds and $\tilde{A} \in (c_0, Y)$ by Theorem 2.4.1. Further, we have by the
hypothesis that \( A_n \in \ell c^\beta \) for all \( n \in \mathbb{N} \). Thus, we deduce by Theorem 2.3.2 that (2.4.5) and (2.4.6) hold and the equality (2.4.8) holds for every \( x \in \bar{c} \) with \( \bar{x} \in c \) such that \( \bar{x} = (\bar{A}(r, s, t))x \) and \( l = \lim_{k \to \infty} \bar{x}_k \). Moreover, let us rewrite (2.4.8) as follows:

\[
\sum_{k=0}^{\infty} a_{nk} x_k = \sum_{k=0}^{\infty} \bar{a}_{nk} (\bar{x}_k - l) + l \left( \sum_{k=0}^{\infty} \bar{a}_n - \alpha_n \right); \quad (n \in \mathbb{N}). \tag{2.4.9}
\]

Then, it is immediate by (2.4.9) that (2.4.7) holds, since \( A \in (\bar{c}, Y) \) and \( \bar{A} \in (c_0, Y) \).

Conversely, suppose that (2.4.1), (2.4.5), (2.4.6) and (2.4.7) hold and \( \bar{A} \in (c_0, Y) \). Since \( \bar{A} \in (c_0, Y) \) implies \( \bar{A}_n \in \ell_1 = c_0^\beta \) (\( n \in \mathbb{N} \)), we deduce from (2.4.1), (2.4.5), (2.4.6) and Theorem 2.3.2 that \( A_n \in \ell c^\beta \) (\( n \in \mathbb{N} \)) and so the equality (2.4.8) holds which can be written as in (2.4.9) for all sequences \( x \in \bar{c} \) and \( \bar{x} \in c \) such that \( \bar{x} = (\bar{A}(r, s, t))x \) and \( l = \lim_{k \to \infty} \bar{x}_k \). Therefore, by bearing in mind that (2.4.7) holds and \( \bar{A} \in (c_0, Y) \), we obtain by (2.4.9) that \( Ax \in Y \) for all \( x \in \bar{c} \) and hence \( A \in (\bar{c}, Y) \). This concludes the proof. \( \square \)

Now, suppose that (2.4.1) holds and let \( n, k \in \mathbb{N} \) be given. For any \( m > k \), we derive from (2.4.2) and (2.4.3) that

\[
\bar{a}_{nk} - \sum_{j=k}^{m-1} (-1)^{j-k} D_j^{(s)} r_k a_{nj}/t_j = a_{mk}^{(n)}
\]

and since

\[
\lim_{m \to \infty} \left( \sum_{j=k}^{m-1} (-1)^{j-k} D_j^{(s)} r_k a_{nj}/t_j \right) = \bar{a}_{nk}; \quad (n, k \in \mathbb{N})
\]

by (2.4.1), we deduce from the above equality that

\[
\lim_{m \to \infty} a_{mk}^{(n)} = 0; \quad (n, k \in \mathbb{N}). \tag{2.4.10}
\]

Therefore, by using the fact that (2.4.1) implies (2.4.10), we have the following corollary which is immediate by combining Theorem 2.4.1 and Lemma 1.2.6.

**Corollary 2.4.3** Let \( X \) be a BK space with \( AK \) and \( \bar{X} = X(r, s, t) \). Then, we have

(a) \( A \in (\bar{X}, \ell_\infty) \) if and only if (2.4.1) holds and

\[
\sup_{n} \| \tilde{A}_n \|_X^* < \infty, \tag{2.4.11}
\]

\[
\sup_{m} \| A_m^{(n)} \|_X^* < \infty \quad \text{for every} \quad n \in \mathbb{N}, \tag{2.4.12}
\]

where \( A_m^{(n)} = (a_{m0}^{(n)}, a_{m1}^{(n)}, \ldots, a_{mn}^{(n)}, 0, 0, \ldots) \) for all \( m, n \in \mathbb{N} \).

(b) \( A \in (\bar{X}, c) \) if and only if (2.4.1), (2.4.11) and (2.4.12) hold and

\[
\lim_{n \to \infty} \bar{a}_{nk} = \bar{a}_k \quad \text{for all} \quad k \in \mathbb{N}. \tag{2.4.13}
\]
Furthermore, if \( A \in (\bar{X}, c) \) then we have

\[
\lim_{n \to \infty} A_n(x) = \sum_{k=0}^{\infty} \tilde{a}_k \bar{x}_k.
\]  

(2.4.14)

for all \( x \in \bar{X} \) with \( \bar{x} = (\bar{A}(r, s, t))x \).

(c) \( A \in (\bar{X}, c_0) \) if and only if (2.4.1), (2.4.11) and (2.4.12) hold and

\[
\lim_{n \to \infty} \tilde{a}_{nk} = 0 \quad \text{for all } k \in \mathbb{N}.
\]  

(2.4.15)

(d) \( A \in (\bar{X}, \ell_1) \) if and only if (2.4.1) and (2.4.12) hold and

\[
sup_{N \in \mathcal{F}} \left\| \sum_{n \in N} \bar{A}_n \right\|_X^* < \infty.
\]

Similarly, by combining Theorem 2.4.1 with Lemmas 1.1.1–1.1.7, we deduce the following corollaries:

**Corollary 2.4.4** Let \( 1 < p < \infty \) and \( q = p/(p - 1) \). Then, by writing \( \tilde{\ell}_p = \ell_p(r, s, t) \), we have

(a) \( A \in (\tilde{\ell}_p, \ell_\infty) \) if and only if (2.4.1) holds and

\[
sup_{n} \left( \sum_{k=0}^{\infty} |\tilde{a}_{nk}|^q \right) < \infty,
\]  

(2.4.16)

\[
sup_{m} \left( \sum_{k=0}^{m} |a_{mk}|^q \right) < \infty \quad \text{for every } n \in \mathbb{N}.
\]  

(2.4.17)

(b) \( A \in (\tilde{\ell}_p, c) \) if and only if (2.4.1), (2.4.13), (2.4.16) and (2.4.17) hold. Furthermore, if \( A \in (\tilde{\ell}_p, c) \) then (2.4.14) holds for every \( x \in \tilde{\ell}_p \) with \( \bar{x} = (\bar{A}(r, s, t))x \).

(c) \( A \in (\tilde{\ell}_p, c_0) \) if and only if (2.4.1), (2.4.15), (2.4.16) and (2.4.17) hold.

(d) \( A \in (\tilde{\ell}_p, \ell_1) \) if and only if (2.4.1) and (2.4.17) hold and

\[
sup_{N \in \mathcal{F}} \left( \sum_{k=0}^{\infty} \left( \sum_{n \in N} |\tilde{a}_{nk}| \right)^q \right) < \infty.
\]

**Corollary 2.4.5** Let \( \tilde{\ell}_1 = \ell_1(r, s, t) \). Then, we have

(a) \( A \in (\tilde{\ell}_1, \ell_\infty) \) if and only if (2.4.1) holds and

\[
sup_{n,k} |\tilde{a}_{nk}| < \infty,
\]  

(2.4.18)

\[
sup_{m} \left( \max_{0 \leq k \leq m} |a_{mk}| \right) < \infty \quad \text{for every } n \in \mathbb{N}.
\]  

(2.4.19)
(b) $A \in (\ell_1, c)$ if and only if (2.4.1), (2.4.13), (2.4.18) and (2.4.19) hold. Furthermore, if $A \in (\ell_1, c)$ then (2.4.14) holds for every $x \in \ell_1$ with $\bar{x} = (\bar{A}(r, s, t))x$.

(c) $A \in (\ell_1, c_0)$ if and only if (2.4.1), (2.4.15), (2.4.18) and (2.4.19) hold.

(d) Let $1 \leq p < \infty$. Then $A \in (\ell_1, \ell_p)$ if and only if (2.4.1) and (2.4.19) hold and

$$
\sup_k \left( \sum_{n=0}^{\infty} |\tilde{a}_{nk}|^p \right) < \infty.
$$

**Corollary 2.4.6** Let $\bar{c}_0 = c_0(r, s, t)$. Then, we have

(a) $A \in (\bar{c}_0, \ell_\infty)$ if and only if (2.4.1) and (2.4.5) hold and

$$
\sup_n \left( \sum_{k=0}^{\infty} |\tilde{a}_{nk}| \right) < \infty.
$$

(b) $A \in (\bar{c}_0, c)$ if and only if (2.4.1), (2.4.5), (2.4.13) and (2.4.20) hold. Furthermore, if $A \in (\bar{c}_0, c)$ then (2.4.14) holds for every $x \in \bar{c}_0$ with $\bar{x} = (\bar{A}(r, s, t))x$.

(c) $A \in (\bar{c}_0, c_0)$ if and only if (2.4.1), (2.4.5), (2.4.15) and (2.4.20) hold.

(d) Let $1 \leq p < \infty$. Then $A \in (\bar{c}_0, \ell_p)$ if and only if (2.4.1) and (2.4.5) hold and

$$
\sup_{K \in \mathbb{F}} \left( \sum_{n=0}^{\infty} \left| \sum_{k \in K} \tilde{a}_{nk} \right|^p \right) < \infty.
$$

**Corollary 2.4.7** Let $\bar{\ell}_\infty = \ell_\infty(r, s, t)$. Then, we have

(a) $A \in (\bar{\ell}_\infty, \ell_\infty)$ if and only if (2.4.1) and (2.4.20) hold and

$$
\lim_{m \to \infty} \left( \sum_{k=0}^{m} |a_{mk}^{(n)}| \right) = 0 \text{ for all } n \in \mathbb{N}.
$$

(b) $A \in (\bar{\ell}_\infty, c)$ if and only if (2.4.1), (2.4.13), (2.4.20) and (2.4.22) hold and

$$
\lim_{n \to \infty} \left( \sum_{k=0}^{\infty} |\tilde{a}_{nk} - \bar{a}_k| \right) = 0.
$$

Moreover, if $A \in (\bar{\ell}_\infty, c)$ then (2.4.14) holds for every $x \in \bar{\ell}_\infty$ with $\bar{x} = (\bar{A}(r, s, t))x$.

(c) $A \in (\bar{\ell}_\infty, c_0)$ if and only if (2.4.1) and (2.4.22) hold and

$$
\lim_{n \to \infty} \left( \sum_{k=0}^{\infty} |\tilde{a}_{nk}| \right) = 0.
$$

(d) Let $1 \leq p < \infty$. Then $A \in (\bar{\ell}_\infty, \ell_p)$ if and only if (2.4.1), (2.4.21) and (2.4.22) hold.

Further, we have the following corollary which is obtained similarly from Theorem 2.4.2.
Corollary 2.4.8 Let $\bar{c} = c(r, s, t)$. Then, we have

(a) $A \in (\bar{c}, \ell_\infty)$ if and only if (2.4.1), (2.4.5), (2.4.6) and (2.4.20) hold and

$$\sup_n \left| \sum_{k=0}^{\infty} \tilde{a}_{nk} - \alpha_n \right| < \infty.$$  

(b) $A \in (\bar{c}, c)$ if and only if (2.4.1), (2.4.5), (2.4.6), (2.4.13) and (2.4.20) hold and

$$\lim_{n \to \infty} \left( \sum_{k=0}^{\infty} \tilde{a}_{nk} - \alpha_n \right) = a.$$  

Furthermore, if $A \in (\bar{c}, c)$ then we have

$$\lim_{n \to \infty} A_n(x) = \sum_{k=0}^{\infty} \tilde{a}_k(x_k - l) + la$$

for all $x \in \bar{c}$ with $\bar{x} = (\bar{A}(r, s, t))x$ and $l = \lim_{k \to \infty} \bar{x}_k$.

(c) $A \in (\bar{c}, c_0)$ if and only if (2.4.1), (2.4.5), (2.4.6), (2.4.15) and (2.4.20) hold and

$$\lim_{n \to \infty} \left( \sum_{k=0}^{\infty} \tilde{a}_{nk} - \alpha_n \right) = 0.$$  

(d) Let $1 \leq p < \infty$. Then, we have $A \in (\bar{c}, \ell_p)$ if and only if (2.4.1), (2.4.5), (2.4.6) and (2.4.21) hold and

$$\sum_{n=0}^{\infty} \left( \sum_{k=0}^{\infty} \tilde{a}_{nk} - \alpha_n \right)^p < \infty.$$  

Remark 2.4.9 As we have seen in Remark 2.3.8, there exist some $s \in U_o$ such that the condition (2.4.1) is redundant in all our results of this section. On the other hand, let us consider the special case $p = 1$ of part (d) in Corollaries 2.4.6, 2.4.7 and 2.4.8. Then, it is obvious by Remark 1.1.8 that condition (2.4.21) can equivalently be replaced by

$$\sup_{N \in \mathcal{F}} \left( \sum_{k=0}^{\infty} \left| \sum_{n \in N} \tilde{a}_{nk} \right| \right) < \infty.$$  

Now, let $1 < p < \infty$ and $X, Y \in \{c_0, c, \ell_\infty, \ell_1, \ell_p\}$ except the case $X = Y = \ell_p$. Also, let us write $\bar{X} = X(r, s, t)$ and $\bar{Y} = Y(r', s', t')$, where $r, t, r', t' \in U$ and $s, s' \in U_o$. Then, we have the following corollary which is immediate by Lemma 1.2.12 (a).

Corollary 2.4.10 Let $A = (a_{nk})$ be an infinite matrix and define the matrix $B = (b_{nk})$ by

$$b_{nk} = \frac{1}{r'} \sum_{j=0}^{n} s_{n-j} t' j' a_{jk}; \quad (n, k \in \mathbb{N}).$$
Then, the necessary and sufficient conditions in order that $A \in (\bar{X}, \hat{Y})$ are obtained from the respective one in Corollaries 2.4.4–2.4.8 by replacing the entries of the matrix $A$ by those of the matrix $B$, where

$$
\tilde{b}_{nk} = \frac{1}{r'_n} \sum_{j=0}^{n} s'_{n-j} t'_j \tilde{a}_{jk}; \quad (n, k \in \mathbb{N})
$$

and

$$
b_{mk}^{(n)} = \frac{1}{r'_m} \sum_{j=0}^{n} s'_{n-j} t'_j a_{mk}^{(j)}; \quad (0 \leq k \leq m; \ m, n \in \mathbb{N}).
$$

Finally, let us conclude this chapter by noting that the conclusion of Corollary 2.4.10 can be extended, by means of Theorems 2.4.1 and 2.4.2, to the general case when $X$ and $Y$ are arbitrary $BK$ spaces such that $X$ has $AK$ or $X \in \{c, \ell_{\infty}\}$.

**Remark 2.4.11** We should note that Corollary 2.4.10 includes the characterizations of matrix mappings between the sequence spaces mentioned in Remark 2.2.1, and this can be achieved by replacing $r, s, t, r', s'$ and $t'$ in Corollary 2.4.10 by some suitable sequences.