Chapter 4
COMPACT MATRIX OPERATORS ON $BK$ SPACES
In the present chapter, we establish some identities or estimates for operator norms and Hausdorff measures of noncompactness of certain operators given by infinite matrices that map an arbitrary BK space into the sequence spaces $c_0$, $c$, $\ell_\infty$ and $\ell_1$, and into the matrix domains of triangles in these spaces. Further, by using the Hausdorff measure of noncompactness, we apply our results to characterize some classes of compact matrix operators on BK spaces. The materials of this chapter can be found in [83].

4.1 Preliminary results

In this section, we prove some elementary results which will be needed in proving our main results in the subsequent section.

Lemma 4.1.1 Let $X$ be a BK space with AK or $X = \ell_\infty$. If $A \in (X, c)$, then the following hold:

\[
\alpha_k = \lim_{n \to \infty} a_{nk} \text{ exists for every } k \in \mathbb{N}, \quad (4.1.1)
\]

\[
\alpha = (\alpha_k) \in X^\beta, \quad (4.1.2)
\]

\[
\sup_n \| A_n - \alpha \|^* < \infty, \quad (4.1.3)
\]

\[
\lim_{n \to \infty} A_n(x) = \sum_{k=0}^{\infty} \alpha_k x_k \text{ for all } x = (x_k) \in X. \quad (4.1.4)
\]

Proof. Let us begin with the first case when $X$ is a BK space with AK.

We write $\| \cdot \| = \| \cdot \|^*_X$, for short. Since $A \in (X, c)$, we obtain by Lemma 1.2.7 that

\[
\| L_A \| = \sup_n \| A_n \|^*_X < \infty. \quad (4.1.5)
\]

Further, we have $e^{(k)} \in X$ and hence $A e^{(k)} \in c$ for all $k \in \mathbb{N}$. Consequently, the limits $\alpha_k$ in (4.1.1) exist for all $k \in \mathbb{N}$.

Now, let $x \in X$ be given. Since $X$ has AK, there is a positive constant $K$ such that $\| x^{[m]} \| \leq K \| x \|$ for all $m \in \mathbb{N}$, where $x^{[m]} = \sum_{k=0}^{m} x_k e^{(k)}$. We thus derive that

\[
\left| \sum_{k=0}^{m} a_{nk} x_k \right| = \left| A_n(x^{[m]}) \right| \leq \| A x^{[m]} \|_{\ell_\infty} = \| L_A(x^{[m]}) \|_{\ell_\infty} \leq K \| L_A \| \| x \|
\]

for all $m, n \in \mathbb{N}$. Therefore, we obtain from (4.1.1) that

\[
\left| \sum_{k=0}^{m} \alpha_k x_k \right| = \lim_{n \to \infty} \left| \sum_{k=0}^{m} a_{nk} x_k \right| \leq K \| L_A \| \| x \|; \quad (m \in \mathbb{N}).
\]
This implies that $\alpha x = (\alpha_k x_k) \in bs$, and since $x \in X$ was arbitrary, we deduce that $\alpha \in X^\gamma$. But $X$ has $AK$ which yields that $X^\gamma = X^\beta$ [117, Theorem 7.2.7] and hence (4.1.2) holds. Moreover, since $X \supset \phi$ is a $BK$ space, (4.1.2) implies $\|\alpha\|_X^* < \infty$ by [117, Theorem 7.2.9]. Therefore, we get (4.1.3) from (4.1.5) by using (1.2.1).

Now, define the matrix $B = (b_{nk})$ by $b_{nk} = a_{nk} - \alpha_k$ for all $n, k \in \mathbb{N}$. Then, it is obvious that $B_n \in X^\beta$ for all $n \in \mathbb{N}$. Also, it follows by (4.1.3) that

$$\sup_n \|B_n\|_X^* = \sup_n \|A_n - \alpha\|_X^* < \infty.$$  

Furthermore, we have from (4.1.1) that

$$\lim_{n \to \infty} b_{nk} = \lim_{n \to \infty} (a_{nk} - \alpha_k) = 0; \quad (k \in \mathbb{N}).$$

This leads us with Lemma 1.2.6 (c) to the consequence that $B \in (X, c_0)$. Hence $\lim_{n \to \infty} B_n(x) = 0$ for all $x \in X$ which yields (4.1.4).

Now, let us turn to the second case $X = \ell_\infty$, i.e., $A \in (\ell_\infty, c)$. In this case, it follows by Lemma 1.1.3 that (4.1.1) holds and

$$\sup_n \left( \sum_{k=0}^{\infty} |a_{nk}| \right) < \infty, \quad (4.1.6)$$

$$\lim_{n \to \infty} \left( \sum_{k=0}^{\infty} |a_{nk} - \alpha_k| \right) = 0. \quad (4.1.7)$$

From (4.1.1) and (4.1.6), we have for every $k \in \mathbb{N}$ that

$$\sum_{j=0}^{k} |\alpha_j| \leq \sup_n \left( \sum_{j=0}^{\infty} |a_{nj}| \right) < \infty$$

which implies that $\alpha = (\alpha_k) \in \ell_1$. Thus $\alpha \in \ell_\infty^\beta$ by Lemma 1.2.1, and so (4.1.2) holds. Moreover, it is immediate by (4.1.7) that

$$\sup_n \|A_n - \alpha\|_{\ell_1} = \sup_n \left( \sum_{k=0}^{\infty} |a_{nk} - \alpha_k| \right) < \infty$$

which is (4.1.3) for $X = \ell_\infty$ by Lemma 1.2.1.

Finally, let $x \in \ell_\infty$ be given. Then, we have for every $n \in \mathbb{N}$ that

$$\left| A_n(x) - \sum_{k=0}^{\infty} \alpha_k x_k \right| \leq \sum_{k=0}^{\infty} |a_{nk} - \alpha_k| |x_k| \leq \|x\|_{\ell_\infty} \left( \sum_{k=0}^{\infty} |a_{nk} - \alpha_k| \right).$$

Hence, from (4.1.7) we get (4.1.4) since $x \in \ell_\infty$ was arbitrary. This completes the proof. \[\square\]

Throughout, let $\mathcal{F}_r \ (r \in \mathbb{N})$ be the subcollection of $\mathcal{F}$ consisting of all nonempty and finite subsets of $\mathbb{N}$ with elements that are greater than $r$, that is

$$\mathcal{F}_r = \{ N \in \mathcal{F} : n > r \text{ for all } n \in N \}; \quad (r \in \mathbb{N}). \quad (4.1.8)$$
Then, we have the following result:

**Lemma 4.1.2** Let \( x = (x_n) \in \ell_1 \). Then, the inequalities

\[
\sup_{N \in \mathcal{F}_r} \left| \sum_{n \in N} x_n \right| \leq \sum_{n=r+1}^{\infty} |x_n| \leq 4 \cdot \sup_{N \in \mathcal{F}_r} \left| \sum_{n \in N} x_n \right|
\]

hold for every \( r \in \mathbb{N} \).

**Proof.** Let \( r \in \mathbb{N} \) be given. The first inequality on the left is trivial, since

\[
\left| \sum_{n \in N} x_n \right| \leq \sum_{n \in N} |x_n| \leq \sum_{n=r+1}^{\infty} |x_n| < \infty
\]

for all \( N \in \mathcal{F}_r \). To prove the other inequality, we have for every \( m > r \) that (see [68, Lemma 3.9] and [107, p.75])

\[
\sum_{n=r+1}^{m} |x_n| \leq 4 \cdot \max_{N \subset \{r+1, \ldots, m\}} \left| \sum_{n \in N} x_n \right|
\]

\[
\leq 4 \cdot \sup_{N \in \mathcal{F}_r} \left| \sum_{n \in N} x_n \right| < \infty.
\]

Hence, we obtain that

\[
\sum_{n=r+1}^{\infty} |x_n| = \lim_{m \to \infty} \left( \sum_{n=r+1}^{m} |x_n| \right) \leq 4 \cdot \sup_{N \in \mathcal{F}_r} \left| \sum_{n \in N} x_n \right|.
\]

This concludes the proof, since \( r \in \mathbb{N} \) was arbitrary. \( \square \)

**Remark 4.1.3** We should note that the characterizations of matrix classes which will be studied in the following sections are immediate by those results given in Chapter 1. Thus, we shall omit these characterizations here and only deal with the operator norms and the Hausdorff measures of noncompactness of some matrix operators which are given by infinite matrices in such classes.

### 4.2 Compact operators on \( BK \) spaces

In the present section, we establish some identities or estimates for the Hausdorff measures of noncompactness of certain operators given by infinite matrices that map an arbitrary \( BK \) space (with \( AK \), sometimes) into the spaces \( c_0, c, \ell_\infty \) and \( \ell_1 \). Further, by using the Hausdorff measure of noncompactness, we apply our results to characterize some classes of compact operators on \( BK \) spaces.

**Theorem 4.2.1** Let \( X \supset \phi \) be a \( BK \) space. Then, we have

(a) If \( A \in (X, c_0) \), then

\[
\|L_A\|_X = \lim_{r \to \infty} \left( \sup_{n > r} \|A_n\|_X \right).
\]  

(4.2.1)
(b) If $X$ has $AK$ or $X = \ell_\infty$ and $A \in (X,c)$, then

$$\frac{1}{2} \cdot \lim_{r \to \infty} \left( \sup_{n>r} \|A_n - \alpha\|_X^* \right) \leq \|L_A\|_X \leq \lim_{r \to \infty} \left( \sup_{n>r} \|A_n - \alpha\|_X^* \right),$$

(4.2.2)

where $\alpha = (\alpha_k)$ with $\alpha_k = \lim_{n \to \infty} a_{nk}$ for all $k \in \mathbb{N}$ (cf. [32, Theorem 3.4]).

(c) If $A \in (X,\ell_\infty)$, then

$$0 \leq \|L_A\|_X \leq \lim_{r \to \infty} \left( \sup_{n>r} \|A_n\|_X^* \right).$$

(4.2.3)

**Proof.** Let us remark that the limits in (4.2.1), (4.2.2) and (4.2.3) exist by Lemmas 1.2.7 and 4.1.1. For example, if $X$ is as in part (b) and $A \in (X,c)$, then the sequence $(\sup_{n>r} \|A_n - \alpha\|_X^*)_{r=0}^\infty$ of non-negative reals is nonincreasing and bounded by Lemma 4.1.1.

We write $S = S_X$, for short. Then, we have by (1.3.2) and Lemma 1.2.3 (a) that

$$\|L_A\|_X = \chi(AS).$$

(4.2.4)

For (a), we have $AS \in \mathcal{M}_{c_0}$. Thus, it follows by Lemma 1.3.2 that

$$\|L_A\|_X = \chi(AS) = \lim_{r \to \infty} \left( \sup_{x \in S} \|(I - P_r)(Ax)\|_\ell_\infty \right),$$

(4.2.5)

where $P_r : c_0 \to c_0$ ($r \in \mathbb{N}$) is the operator defined by $P_r(x) = (x_0, x_1, \ldots, x_r, 0, 0, \ldots)$ for all $x = (x_k) \in c_0$. This yields that $\|(I - P_r)(Ax)\|_\ell_\infty = \sup_{n>r} |A_n(x)|$ for all $x \in X$ and every $r \in \mathbb{N}$. Thus, by combining (1.1.1) and (1.2.1), we have for every $r \in \mathbb{N}$ that

$$\sup_{x \in S} \|(I - P_r)(Ax)\|_\ell_\infty = \sup_{n>r} \|A_n\|_X^*.$$ 

Hence, by (4.2.5) we get (4.2.1).

To prove (b), we have $AS \in \mathcal{M}_c$. Thus, we are going to apply Lemma 1.3.3 to get an estimate for the value of $\chi(AS)$ in (4.2.4). For this, we know that every $z = (z_n) \in c$ has a unique representation $z = \bar{z} e + \sum_{n=0}^{\infty} (z_n - \bar{z})e^{(n)}$, where $\bar{z} = \lim_{n \to \infty} z_n$. Also, let $P_r : c \to c$ ($r \in \mathbb{N}$) be the projectors defined by (1.3.1). Then, we have for every $r \in \mathbb{N}$ that $(I - P_r)(z) = \sum_{n=r+1}^{\infty} (z_n - \bar{z})e^{(n)}$ and hence

$$\|(I - P_r)(z)\|_\ell_\infty = \sup_{n>r} |z_n - \bar{z}|$$

(4.2.6)

for all $z \in c$ and every $r \in \mathbb{N}$, where $I$ is the identity operator on $c$.

Now, by using (4.2.4), we obtain by applying Lemma 1.3.3 that

$$\frac{1}{2} \cdot \mu(A) \leq \|L_A\|_X \leq \mu(A),$$

(4.2.7)

where

$$\mu(A) = \lim_{r \to \infty} \left( \sup_{x \in S} \|(I - P_r)(Ax)\|_\ell_\infty \right).$$

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On the other hand, it is given that $A \in (X, c)$, where $X$ is a $BK$ space with $AK$ or $X = \ell_\infty$. Thus, it follows from Lemma 4.1.1 that the limits $\alpha_k = \lim_{n \to \infty} a_{nk}$ exist for all $k$, $\alpha = (\alpha_k) \in X^\beta$ and

$$\lim_{n \to \infty} A_n(x) = \sum_{k=0}^{\infty} \alpha_k x_k$$

for all $x = (x_k) \in X$. Therefore, we derive from (4.2.6) that

$$\| (I - P_r)(Ax) \|_{\ell_\infty} = \sup_{n > r} \left| A_n(x) - \sum_{k=0}^{\infty} \alpha_k x_k \right|$$

$$= \sup_{n > r} \left| \sum_{k=0}^{\infty} (a_{nk} - \alpha_k) x_k \right|$$

for all $x = (x_k) \in X$ and every $r \in \mathbb{N}$. Consequently, we obtain by (1.2.1) that

$$\sup_{x \in S} \| (I - P_r)(Ax) \|_{\ell_\infty} = \sup_{n > r} \| A_n - \alpha \|_X^*; \quad (r \in \mathbb{N}).$$

Hence, we get (4.2.2) from (4.2.7).

Finally, to prove (c) we define $P_r : \ell_\infty \to \ell_\infty \ (r \in \mathbb{N})$ as in the proof of part (a) for all $x = (x_k) \in \ell_\infty$. Then, it is clear that

$$AS \subset P_r(AS) + (I - P_r)(AS); \quad (r \in \mathbb{N}).$$

Thus, it follows by the elementary properties of the function $\chi$ that

$$0 \leq \chi(AS) \leq \chi(P_r(AS)) + \chi((I - P_r)(AS))$$

$$= \chi((I - P_r)(AS))$$

$$\leq \sup_{x \in S} \| (I - P_r)(Ax) \|_{\ell_\infty}$$

$$= \sup_{n > r} \| A_n \|_X^*$$

for all $r \in \mathbb{N}$ which implies that

$$0 \leq \chi(AS) \leq \lim_{r \to \infty} \left( \sup_{n > r} \| A_n \|_X^* \right).$$

This and (4.2.4) together yield (4.2.3) and complete the proof. \hfill \Box

**Theorem 4.2.2** Let $X \supset \phi$ be a $BK$ space. Then, we have

(a) If $A \in (X, c_0)$, then

$L_A$ is compact if and only if $\lim_{r \to \infty} \left( \sup_{n > r} \| A_n \|_X^* \right) = 0.$

(b) If $X$ has $AK$ or $X = \ell_\infty$ and $A \in (X, c)$, then

$L_A$ is compact if and only if $\lim_{r \to \infty} \left( \sup_{n > r} \| A_n - \alpha \|_X^* \right) = 0,$
Proof. This result follows from Theorem 4.2.1 by using (1.3.3). \hfill \Box

It is worth mentioning that the condition in (4.2.8) is only a sufficient condition for the operator \( L_A \) to be compact, where \( A \subset (X, \ell_\infty) \) and \( X \ni \phi \) is a BK space. More precisely, we are going to show, in the following example, that it is possible for \( L_A \) to be compact while \( \lim_{r \to \infty} (\sup_{n>r} \| A_n \|_X^* ) \neq 0 \). Hence, in general, we have just ‘if’ in (4.2.8) of Theorem 4.2.2 (c).

Example 4.2.3 Let \( X \ni \phi \) be a BK space and choose a fixed \( m \in \mathbb{N} \) such that \( x'_m \neq 0 \) for some \( x' = (x'_k) \in S_X \). Now, we define the matrix \( A = (a_{nk}) \) by \( a_{nm} = 1 \) and \( a_{nk} = 0 \) for all \( k \neq m \) (\( n \in \mathbb{N} \)). Then, we have \( Ax = x_m e \) for all \( x = (x_k) \in X \) and hence \( A \subset (X, \ell_\infty) \). Further, it is obvious that \( L_A \) is of finite rank and so \( L_A \) is compact. On the other hand, we have \( A_n = e^{(m)} \) and hence \( \| A_n \|_X^* = \sup_{x \in S_X} |x_m| \) for all \( n \in \mathbb{N} \) by (1.2.1). This implies that
\[
\lim_{r \to \infty} \left( \sup_{n>r} \| A_n \|_X^* \right) = \sup_{x \in S_X} |x_m| \geq |x'_m| > 0.
\]

Now, it is obvious that Theorems 4.2.1 and 4.2.2 have several consequences, some of them are known results (cf. [102]). For instance, consider the particular case \( X = \ell_p \) (\( 1 \leq p < \infty \)). Then, we know by Lemma 1.2.3 (b) that every operator \( L \) in \( \mathcal{B}(\ell_p, c_0) \), \( \mathcal{B}(\ell_p, c) \) or \( \mathcal{B}(\ell_p, \ell_\infty) \) is given by an infinite matrix \( A \) belonging to the respective one of the classes \( (\ell_p, c_0) \), \( (\ell_p, c) \) or \( (\ell_p, \ell_\infty) \). Hence, by Theorem 4.2.1, we get an identity or estimate for \( \| L \|_X^* \). Further, by means of Theorem 4.2.2, we deduce the characterization of the classes \( \mathcal{C}(\ell_p, c_0) \) and \( \mathcal{C}(\ell_p, c) \) of compact operators, and the sufficient condition for an operator \( L \subset \mathcal{B}(\ell_p, \ell_\infty) \) to be compact. Moreover, we may note by Lemma 1.2.1 that if \( X = \ell_p \) (\( 1 \leq p < \infty \)) in Theorems 4.2.1 and 4.2.2, then \( \| \cdot \|_X^* \) is replaced by \( \| \cdot \|_{\ell_q} \), where \( q \) is the conjugate of \( p \) (see [49, Example 1.14] and [70, Theorem 4.6]).

On the other hand, let \( 1 < p < \infty \). Then, we write \( bv^p \) for the space of all sequences of \( p \)-bounded variation, that is
\[
bv^p = \left\{ x = (x_k) \in w : (x_k - x_{k-1}) \in \ell_p \right\}; \quad (1 < p < \infty).
\]

It is known that \( bv^p \) is a BK space with its natural norm [16, Theorem 2.1]. Further, we have for every \( a = (a_k) \in (bv^p)^\beta \) that
\[
\| a \|_{bv^p}^* = \left( \sum_{k=0}^{\infty} \left| \sum_{j=k}^{\infty} a_j \right|^q \right)^{1/q}, \quad (4.2.9)
\]
where \( 1 < p < \infty \) and \( q = p/(p-1) \) [71, Theorem 2.1].
Therefore, by using (4.2.9), we obtain the following consequence of Theorems 4.2.1 and 4.2.2, which can be found in [71, pp.42; 43].

**Corollary 4.2.4** Let $A$ be an infinite matrix, $1 < p < \infty$, $q = p/(p - 1)$ and

$$\|A\|^{(r)}_{(bv^p, \ell_\infty)} = \sup_{n > r} \left( \sum_{k=0}^{\infty} \left( \sum_{j=k}^{\infty} |a_{nj}|^q \right)^{1/q} \right); \quad (r \in \mathbb{N}).$$

Then, we have

(a) If $A \in (bv^p, c_0)$, then

$$\|L_A\|_{\chi} = \lim_{r \to \infty} \|A\|^{(r)}_{(bv^p, \ell_\infty)}$$

and

$L_A$ is compact if and only if $\lim_{r \to \infty} \|A\|^{(r)}_{(bv^p, \ell_\infty)} = 0$.

(b) If $A \in (bv^p, \ell_\infty)$, then

$$0 \leq \|L_A\|_{\chi} \leq \lim_{r \to \infty} \|A\|^{(r)}_{(bv^p, \ell_\infty)}$$

and

$L_A$ is compact if $\lim_{r \to \infty} \|A\|^{(r)}_{(bv^p, \ell_\infty)} = 0$.

We may note that part (b) of Theorem 4.2.1 can not be applicable for $X = bv^p$, since $bv^p$ does not have $AK$.

Now, let us recall that the *upper limit* (or *limit superior*) of a bounded real sequence $x = (x_n)$ can be defined by

$$\limsup_{n \to \infty} x_n = \lim_{r \to \infty} \left( \sup_{n > r} x_n \right). \quad (4.2.10)$$

Further, if $x_n \geq 0$ for all $n$ then we have

$$\limsup_{n \to \infty} x_n = 0 \quad \text{if and only if} \quad \lim_{n \to \infty} x_n = 0. \quad (4.2.11)$$

Then, by using the above notation [58, p.16], we have the following result:

**Theorem 4.2.5** Let $X \supset \phi$ be a BK space. Then, we have

(a) If $A \in (X, c_0)$, then

$$\|L_A\|_{\chi} = \limsup_{n \to \infty} \|A_n\|_{X}^*$$

and

$L_A$ is compact if and only if $\lim_{n \to \infty} \|A_n\|_{X}^* = 0$.

(b) If $X$ has AK or $X = \ell_\infty$ and $A \in (X, c)$, then

$$\frac{1}{2} \cdot \limsup_{n \to \infty} \|A_n - \alpha\|_{X}^* \leq \|L_A\|_{\chi} \leq \limsup_{n \to \infty} \|A_n - \alpha\|_{X}^*$$
and

\[ L_A \text{ is compact if and only if } \lim_{n \to \infty} \| A_n - \alpha \|_X^* = 0, \]

where \( \alpha = (\alpha_k) \) with \( \alpha_k = \lim_{n \to \infty} a_{nk} \) for all \( k \in \mathbb{N} \).

(c) If \( A \in (X, \ell_{\infty}) \), then

\[ 0 \leq \| L_A \|_X \leq \limsup_{n \to \infty} \| A_n \|_X^* \]

and

\[ L_A \text{ is compact if } \lim_{n \to \infty} \| A_n \|_X^* = 0. \]  \tag{4.2.12} \]

Proof. This is obtained from Theorems 4.2.1 and 4.2.2 by means of (4.2.10) and (4.2.11).

As we have seen in Example 4.2.3, the converse implication in (4.2.12) does not hold, in general.

Remark 4.2.6 By using Lemma 1.2.1, the special cases of Theorem 4.2.5 when \( A \in (c_0, c_0) \), \( A \in (c, c_0) \) or \( A \in (c_0, c) \) can be found in [62, Corollaries 5; 6] (see also [30, Corollary 3.7]). On the other hand, let \( X \) denote any of the spaces \( f \) or \( V_\sigma \) studied in [56, 100, 103]. Then, some applications of Theorem 4.2.5 can also be found in [94, Theorem 3.3].

Now, by using the notations of Lemma 1.2.3 (a), we can write the following (cf. [25, Corollary 4]):

Theorem 4.2.7 We have \((\ell_\infty, c_0) \subset C(\ell_\infty, c_0)\) and \((\ell_\infty, c) \subset C(\ell_\infty, c)\), that is, for every matrix \( A \in (\ell_\infty, c_0) \) or \( A \in (\ell_\infty, c) \), the operator \( L_A \) is compact.

Proof. Let \( A \in (\ell_\infty, c_0) \). Then, we have by Lemma 1.1.4 (c) that

\[ \lim_{n \to \infty} \left( \sum_{k=0}^{\infty} | a_{nk} | \right) = 0, \]

that is, \( \lim_{n \to \infty} \| A_n \|_{\ell_1} = 0 \). This implies that \( \lim_{n \to \infty} \| A_n \|_{\ell_\infty} = 0 \) by Lemma 1.2.1, since \( A_n \in \ell_\infty = \ell_1 \) for all \( n \in \mathbb{N} \). This leads us with part (a) of Theorem 4.2.5 to the consequence that \( L_A \) is compact.

Similarly, if \( A \in (\ell_\infty, c) \) then we obtain by (4.1.8) that \( \lim_{n \to \infty} \| A_n - \alpha \|_{\ell_1} = 0 \) and hence \( \lim_{n \to \infty} \| A_n - \alpha \|_{\ell_\infty} = 0 \). Hence, we deduce from Theorem 4.2.5 (b) that \( L_A \in C(\ell_\infty, c) \). This concludes the proof. \( \square \)

Moreover, we have the following result:

Theorem 4.2.8 Let \( X \supset \phi \) be a BK space. If \( A \in (X, \ell_1) \), then

\[ \lim_{r \to \infty} \left( \sup_{N \in \mathcal{F}_r} \left\| \sum_{n \in N} A_n \right\|_X^* \right) \leq \| L_A \|_X \leq 4 \cdot \lim_{r \to \infty} \left( \sup_{N \in \mathcal{F}_r} \left\| \sum_{n \in N} A_n \right\|_X^* \right) \] \tag{4.2.13} \]

and

\[ L_A \text{ is compact if and only if } \lim_{r \to \infty} \left( \sup_{N \in \mathcal{F}_r} \left\| \sum_{n \in N} A_n \right\|_X^* \right) = 0. \] \tag{4.2.14}
**Proof.** For simplicity in notation, we put

$$
\|A\|_{(X, \ell_1)}^{(r)} = \sup_{N \in F_r} \left\| \sum_{n \in N} A_n \right\|_{X}^{*}; \quad (r \in \mathbb{N}).
$$

Then, it is obvious by (4.1.8) that $F \supset F_0 \supset F_1 \supset \cdots$ which shows that the sequence $(\|A\|_{(X, \ell_1)}^{(r)})_{r=0}^{\infty}$ of non-negative reals is nonincreasing and bounded by Lemma 1.2.8. Therefore, the limit in (4.2.13) exists.

Now, let $S = S_X$. Then, we have by Lemma 1.2.3 (a) that $L_A(S) = AS \in \mathcal{M}_{\ell_1}$. Thus, it follows from (1.3.2) and Lemma 1.3.2 that

$$
\|L_A\|_X = \chi(AS) = \lim_{r \to \infty} \left( \sup_{x \in S} \| (I - P_r)(Ax) \|_{\ell_1} \right),
$$

where $P_r : \ell_1 \to \ell_1 \quad (r \in \mathbb{N})$ is defined by $P_r(x) = (x_0, x_1, \ldots, x_r, 0, 0, \ldots)$ for all $x = (x_k) \in \ell_1$. This yields that

$$
\|L_A\|_X = \lim_{r \to \infty} \left( \sup_{x \in S} \left( \sum_{n=r+1}^{\infty} |A_n(x)| \right) \right) \quad \text{ (4.2.15)}
$$

Since $A \in (X, \ell_1)$, we obtain by Lemma 4.1.2 that

$$
\sup_{N \in F_r} \left| \sum_{n \in N} A_n(x) \right| \leq \sum_{n=r+1}^{\infty} |A_n(x)| \leq 4 \cdot \sup_{N \in F_r} \left| \sum_{n \in N} A_n(x) \right| \quad \text{ (4.2.16)}
$$

for all $x \in X$ and every $r \in \mathbb{N}$.

On the other hand, since $A_n \in X^\beta$ for all $n \in \mathbb{N}$, we have from (1.2.1) that

$$
\left\| \sum_{n \in N} A_n \right\|_{X}^{*} = \sup_{x \in S} \left( \sum_{k=0}^{\infty} \left( \sum_{n \in N} a_{nk} \right) x_k \right) = \sup_{x \in S} \left| \sum_{n \in N} A_n(x) \right|
$$

for all $N \in F_r \quad (r \in \mathbb{N})$. This, together with (4.2.16), implies that

$$
\sup_{N \in F_r} \left\| \sum_{n \in N} A_n \right\|_{X}^{*} \leq \sup_{x \in S} \left( \sum_{n=r+1}^{\infty} |A_n(x)| \right) \leq 4 \cdot \sup_{N \in F_r} \left\| \sum_{n \in N} A_n \right\|_{X}^{*} \quad \text{ (4.2.17)}
$$

for every $r \in \mathbb{N}$. Hence, we get (4.2.13) by passing to the limits in (4.2.17) as $r \to \infty$ and using (4.2.15).

Finally, it is obvious that (4.2.14) is immediate by (1.3.3) and (4.2.13). This completes the proof. \( \square \)

Now, it is clear that Theorem 4.2.8 has many consequences with any particular case of the space $X$. For example, let $1 \leq p < \infty$. Then, it follows from Lemma 1.2.3 (b) that every operator $L \in B(\ell_p, \ell_1)$ is given by an infinite matrix $A \in (\ell_p, \ell_1)$. Therefore, by means of Theorem 4.2.8, we get estimates for the Hausdorff measures of noncompactness of operators in $B(\ell_p, \ell_1)$, and deduce the necessary and sufficient condition for an operator $L \in B(\ell_p, \ell_1)$ to be compact.
On the other hand, if \( L \in \mathcal{B}(\ell_1, \ell_1) \) is given by the matrix \( A \in (\ell_1, \ell_1) \) then we know by [30, Theorem 3.5] that

\[
\|L\|_\chi = \lim_{r \to \infty} \left( \sup_k \left( \sum_{n=r+1}^{\infty} |a_{nk}| \right) \right)
\]  

(4.2.18)

and

\[
L \text{ is compact if and only if } \lim_{r \to \infty} \left( \sup_k \left( \sum_{n=r+1}^{\infty} |a_{nk}| \right) \right) = 0
\]

which are compatible with Theorem 4.2.8 by Lemma 4.1.2, but (4.2.18) is more exact.

Remark 4.2.9 It is not true that every operator \( L \in \mathcal{B}(\ell_1, \ell_1) \) is compact (cf. [65, Remark 6.10]; [66, Remark 19]). For example, let \( I \in \mathcal{B}(\ell_1, \ell_1) \) be the identity operator on \( \ell_1 \). Then, it is obvious that \( I \) cannot be compact. To see this, we know that \( \chi(\bar{B}_\ell) = 1 \) [101, Theorem 2.3]. Hence, it follows from (1.3.2) that \( \|I\|_\chi = \chi(I(\bar{B}_\ell)) = \chi(\bar{B}_\ell) = 1 \).

Finally, by using (4.2.9) in Theorem 4.2.8, we end this section with the following result which can be found in [71, p.44].

Corollary 4.2.10 Let \( 1 < p < \infty \) and \( q = p/(p-1) \). If \( A \in (bv^p, p, \ell_1) \), then

\[
\lim_{r \to \infty} \|A\|^{(r)}_{(bv^p, \ell_1)} \leq \|L_A\|_\chi \leq 4 \cdot \lim_{r \to \infty} \|A\|^{(r)}_{(bv^p, \ell_1)}
\]

and

\[
L_A \text{ is compact if and only if } \lim_{r \to \infty} \|A\|^{(r)}_{(bv^p, \ell_1)} = 0,
\]

where

\[
\|A\|^{(r)}_{(bv^p, \ell_1)} = \sup_{N \in \mathcal{F}_r} \left( \sum_{k=0}^{\infty} \left( \sum_{n \in N} \left( \sum_{j=k}^{\infty} a_{nj} \right) \right)^q \right)^{1/q} ; \quad (r \in \mathbb{N}).
\]

4.3 Applications to matrix domains of triangles

In the present section, we apply our previous results to establish some identities or estimates for the operator norms and the Hausdorff measures of noncompactness of certain operators given by infinite matrices that map an arbitrary \( BK \) space (with \( AK \), sometimes) into the matrix domains of triangles in the spaces \( c_0, c, \ell_\infty \) and \( \ell_1 \). Further, we deduce the necessary and sufficient (or only sufficient) conditions for such operators to be compact.

Throughout this section, we assume that \( A = (a_{nk}) \) is an infinite matrix and \( T = (t_{nk}) \) is a triangle, and we define the matrix \( B = (b_{nk}) \) by

\[
b_{nk} = \sum_{m=0}^{n} t_{nm} a_{mk} ; \quad (n, k \in \mathbb{N}), \quad (4.3.1)
\]
that is, \( B = TA \) and hence
\[
B_n = \sum_{m=0}^{n} t_{nm} A_m = \left( \sum_{m=0}^{n} t_{nm} a_{mk} \right)_{k=0}^{\infty}; \quad (n \in \mathbb{N}). \tag{4.3.2}
\]

Then, by using (4.3.1) and (4.3.2), we have the following result on operator norms.

**Theorem 4.3.1** Let \( X \supset \emptyset \) be a BK space and \( T \) a triangle. Then, we have
(a) Let \( Y \) be any of the spaces \( c_0, c \) or \( \ell_\infty \). If \( A \in (X, Y_T) \), then
\[
\| L_A \| = \| A \|_{(X, (\ell_\infty)_T)} = \sup_{n} \left\| \sum_{m=0}^{n} t_{nm} A_m \right\|_{X}^* < \infty.
\]
(b) If \( A \in (X, (\ell_1)_T) \), then
\[
\| A \|_{(X, (\ell_1)_T)} \leq \| L_A \| \leq 4 \cdot \| A \|_{(X, (\ell_1)_T)},
\]
where
\[
\| A \|_{(X, (\ell_1)_T)} = \sup_{N \in \mathcal{F}} \left\| \sum_{n \in N} \left( \sum_{m=0}^{n} t_{nm} A_m \right) \right\|_{X}^* < \infty.
\]

**Proof.** This is obtained from Lemmas 1.2.7 and 1.2.8 by using Lemma 1.2.12 (b). \( \square \)

With the notations of (4.3.1) and (4.3.2), we define the sequence \( a = (a_k)_{k=0}^{\infty} \) by
\[
a_k = \lim_{n \to \infty} \left( \sum_{m=0}^{n} t_{nm} a_{mk} \right); \quad (k \in \mathbb{N}) \tag{4.3.3}
\]
provided the limits in (4.3.3) exist for all \( k \in \mathbb{N} \) which is the case whenever \( A \in (X, c_T) \) by Lemmas 1.2.12 (a) and 4.1.1, where \( X \) is a BK space with \( AK \) or \( X = \ell_\infty \).

Now, by means of Remark 1.3.6, we have the following result on the Hausdorff measure of noncompactness.

**Theorem 4.3.2** Let \( X \supset \emptyset \) be a BK space and \( T \) a triangle. Then, we have
(a) If \( A \in (X, (c_0)_T) \), then
\[
\| L_A \|_{X} = \limsup_{n \to \infty} \left\| \sum_{m=0}^{n} t_{nm} A_m \right\|_{X}^* \quad \text{and} \quad L_A \text{ is compact if and only if } \lim_{n \to \infty} \left\| \sum_{m=0}^{n} t_{nm} A_m \right\|_{X}^* = 0.
\]
(b) If \( X \) has \( AK \) and \( A \in (X, c_T) \), then
\[
\frac{1}{2} \cdot \limsup_{n \to \infty} \left\| \sum_{m=0}^{n} t_{nm} A_m - a \right\|_{X}^* \leq \| L_A \|_{X} \leq \limsup_{n \to \infty} \left\| \sum_{m=0}^{n} t_{nm} A_m - a \right\|_{X}^*
\]

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\[ L_A \text{ is compact if and only if } \lim_{n \to \infty} \left\| \sum_{m=0}^{n} t_{nm} A_m - a \right\|_X^* = 0. \]

(c) If \( A \in (X, (\ell_\infty)_T) \), then
\[
0 \leq \| L_A \|_X \leq \limsup_{n \to \infty} \left\| \sum_{m=0}^{n} t_{nm} A_m \right\|_X^*
\]
and
\[ L_A \text{ is compact if } \lim_{n \to \infty} \left\| \sum_{m=0}^{n} t_{nm} A_m \right\|_X^* = 0. \quad (4.3.4) \]

**Proof.** This result follows from Theorem 4.2.5 by means of Lemmas 1.2.12 (a) and 1.3.4. \( \square \)

As we have seen in Example 4.2.3, it can easily be seen that the equivalence in (4.3.4) does not hold.

Moreover, by using Lemmas 1.2.12 (a) and 1.3.4, the following result is immediate by Theorem 4.2.7.

**Theorem 4.3.3** Let \( T \) be a triangle. If either \( A \in (\ell_\infty, (c_0)_T) \) or \( A \in (\ell_\infty, c_T) \), then \( L_A \) is compact.

**Theorem 4.3.4** Let \( X \supset \phi \) be a BK space and \( T \) a triangle. If \( A \in (X, (\ell_1)_T) \), then
\[
\lim_{r \to \infty} \| A \|_{(X, (\ell_1)_T)}(r) \leq \| L_A \|_X \leq 4 \cdot \lim_{r \to \infty} \| A \|_{(X, (\ell_1)_T)}(r)
\]
and
\[ L_A \text{ is compact if and only if } \lim_{r \to \infty} \| A \|_{(X, (\ell_1)_T)}(r) = 0, \]
where
\[
\| A \|_{(X, (\ell_1)_T)}(r) = \sup_{n \in \mathbb{N}_r} \left\| \sum_{m=0}^{n} t_{nm} A_m \right\|_X^*; \quad (r \in \mathbb{N}).
\]

**Proof.** This is obtained from Theorem 4.2.8 by using Lemmas 1.2.12 (a) and 1.3.4. \( \square \)

It is worth mentioning that Theorems 4.3.1–4.3.4 have several consequences with any particular case of the triangle \( T \) or any particular BK space \( X \) (with \( AK \) in part (b) of Theorem 4.3.2). For instance, we have the following results:

**Corollary 4.3.5** Let \( X \supset \phi \) be a BK space and \( A \) an infinite matrix. If \( A \) is in any of the classes \( (X, cs_0) \), \( (X, cs) \) or \( (X, bs) \), then
\[
\| L_A \| = \sup_{n} \| A \|_{(X, bs)}^{(n)} < \infty,
\]
where
\[
\| A \|_{(X, bs)}^{(n)} = \left\| \sum_{m=0}^{n} A_m \right\|_X^*; \quad (n \in \mathbb{N}).
\]
Furthermore, we have the following:
(a) If \( A \in (X, cs_0) \), then
\[
\| L_A \|_\chi = \limsup_{n \to \infty} \| A \|^{(n)}_{(X, bs)}
\]
and
\[ L_A \text{ is compact if and only if } \lim_{n \to \infty} \| A \|^{(n)}_{(X, bs)} = 0. \]
(b) If \( A \in (X, bs) \), then
\[
0 \leq \| L_A \|_\chi \leq \limsup_{n \to \infty} \| A \|^{(n)}_{(X, bs)}
\]
and
\[ L_A \text{ is compact if } \lim_{n \to \infty} \| A \|^{(n)}_{(X, bs)} = 0. \]
(c) If \( X \) has \( AK \) and \( A \in (X, cs) \), then
\[
\frac{1}{2} \cdot \limsup_{n \to \infty} \left\| \sum_{m=0}^{n} A_m - b \right\|_X^* \leq \| L_A \|_\chi \leq \limsup_{n \to \infty} \left\| \sum_{m=0}^{n} A_m - b \right\|_X^*
\]
and
\[ L_A \text{ is compact if and only if } \lim_{n \to \infty} \left\| \sum_{m=0}^{n} A_m - b \right\|_X^* = 0, \]
where \( b = (b_k) \) with \( b_k = \lim_{n \to \infty} (\sum_{m=0}^{n} a_{mk}) \) for all \( k \in \mathbb{N} \).

Corollary 4.3.6 Let \( X \supset \phi \) be a BK space. Then, we have
(a) If \( A \in (X, bv) \), then
\[
\| A \|^{(r)}_{(X, bv)} \leq \| L_A \| \leq 4 \cdot \| A \|^{(r)}_{(X, bv)},
\]
where
\[
\| A \|^{(r)}_{(X, bv)} = \sup_{N \in \mathcal{F}} \left\| \sum_{n \in N} (A_n - A_{n-1}) \right\|_X^* < \infty.
\]
(b) Furthermore, if \( A \in (X, bv) \) then
\[
\lim_{r \to \infty} \| A \|^{(r)}_{(X, bv)} \leq \| L_A \|_\chi \leq 4 \cdot \lim_{r \to \infty} \| A \|^{(r)}_{(X, bv)}
\]
and
\[ L_A \text{ is compact if and only if } \lim_{r \to \infty} \| A \|^{(r)}_{(X, bv)} = 0, \]
where
\[
\| A \|^{(r)}_{(X, bv)} = \sup_{N \in \mathcal{F}_r} \left\| \sum_{n \in N} (A_n - A_{n-1}) \right\|_X^* ; \quad (r \in \mathbb{N}).
\]

Similarly, some applications concerning the spaces of generalized means and \( \lambda \)-sequence spaces can be obtained by using (2.1.2) and (3.1.5) in Theorems 4.3.1–4.3.4.
On the other hand, there are many special cases of Corollaries 4.3.5 and 4.3.6. For example, by using (4.2.9) in Corollary 4.3.5, we have

**Corollary 4.3.7** Let \( A \) be an infinite matrix, \( 1 < p < \infty \) and \( q = p/(p - 1) \). If \( A \) is in any of the classes \((bv^p, cs_0)\), \((bv^p, cs)\) or \((bv^p, bs)\), then

\[
\|L_A\| = \sup_n \|A\|_{(bv^p, bs)}^{(n)} < \infty,
\]

where

\[
\|A\|_{(bv^p, bs)}^{(n)} = \left( \sum_{k=0}^{\infty} \left| \sum_{j=k}^{\infty} \left( \sum_{m=0}^{n} a_{mj} \right) \right|^q \right)^{1/q}; \quad (n \in \mathbb{N}).
\]

Furthermore, we have the following:

(a) If \( A \in (bv^p, cs_0) \), then

\[
\|L_A\|_\chi = \limsup_{n \to \infty} \|A\|_{(bv^p, bs)}^{(n)}
\]

and

\( L_A \) is compact if and only if \( \lim_{n \to \infty} \|A\|_{(bv^p, bs)}^{(n)} = 0 \).

(b) If \( A \in (bv^p, bs) \), then

\[
0 \leq \|L_A\|_\chi \leq \limsup_{n \to \infty} \|A\|_{(bv^p, bs)}^{(n)}
\]

and

\( L_A \) is compact if \( \lim_{n \to \infty} \|A\|_{(bv^p, bs)}^{(n)} = 0 \).

Finally, we conclude this section by the following result obtained similarly from Corollary 4.3.6 (cf. [71, pp.40; 44]).

**Corollary 4.3.8** Let \( 1 < p < \infty \) and \( q = p/(p - 1) \). If \( A \in (bv^p, bv) \), then

\[
\|A\|_{(bv^p, bv)} \leq \|L_A\| \leq 4 \cdot \|A\|_{(bv^p, bv)},
\]

where

\[
\|A\|_{(bv^p, bv)} = \sup_{N \in \mathcal{F}} \left( \sum_{k=0}^{\infty} \left| \sum_{j=k}^{\infty} \left( \sum_{n \in N} (a_{nj} - a_{n-1, j}) \right) \right|^q \right)^{1/q} < \infty.
\]

Moreover, if \( A \in (bv^p, bv) \) then

\[
\lim_{r \to \infty} \|A\|_{(bv^p, bv)}^{(r)} \leq \|L_A\|_\chi \leq 4 \cdot \lim_{r \to \infty} \|A\|_{(bv^p, bv)}^{(r)}
\]

and

\( L_A \) is compact if and only if \( \lim_{r \to \infty} \|A\|_{(bv^p, bv)}^{(r)} = 0 \),

where

\[
\|A\|_{(bv^p, bv)}^{(r)} = \sup_{N \in \mathcal{F}_r} \left( \sum_{k=0}^{\infty} \left| \sum_{j=k}^{\infty} \left( \sum_{n \in N} (a_{nj} - a_{n-1, j}) \right) \right|^q \right)^{1/q}; \quad (r \in \mathbb{N}).
\]
4.4 Some particular cases

In this final section, we shall confine ourselves to apply our main results to the spaces \( s_0^0, s_0^{(c)}, s_0^{(c)} \) and \( \ell_p \) of de Malafosse [59, 60, 63] which can be defined as follows:

Let \( \alpha = (\alpha_k)_{k=0}^\infty \) be a sequence of positive reals and define the diagonal matrix \( D_{1/\alpha} \) by \( d_{nn} = 1/\alpha_n \) for all \( n \in \mathbb{N} \). Then, the spaces \( s_0^0, s_0^{(c)}, s_0^{(c)} \) and \( \ell_p \) are defined as the matrix domains of \( D_{1/\alpha} \) in the spaces \( c_0, c, \ell_\infty \) and \( \ell_p \), respectively, where \( 1 \leq p < \infty \).

It is known that these spaces are BK spaces with their natural norms, and the spaces \( s_0^0 \) and \( \ell_p \) (\( 1 \leq p < \infty \)) have AK [59, Lemma 4 (i)].

Throughout this section, let \( \beta = (\beta_k)_{k=0}^\infty \) be a sequence of positive reals. If \( A = (a_{nk}) \) is an infinite matrix, then we have

\[
\frac{1}{\beta_n} A_n = \left( \frac{a_{nk}}{\beta_n} \right)_{k=0}^\infty; \quad (n \in \mathbb{N}) \tag{4.4.1}
\]

and we define the sequence \( \gamma = (\gamma_k)_{k=0}^\infty \) by

\[
\gamma_k = \lim_{n \to \infty} \left( \frac{a_{nk}}{\beta_n} \right); \quad (k \in \mathbb{N}) \tag{4.4.2}
\]

provided the limits in (4.4.2) exist for all \( k \in \mathbb{N} \).

Now, let us consider the special cases of Theorems 4.3.1—4.3.4 when \( T = D_{1/\beta} \). Then, by using (4.4.1) and (4.4.2), we have the following results:

**Corollary 4.4.1** Let \( X \ni \phi \) be a BK space and \( A \) an infinite matrix. Then, we have

(a) If \( A \) is in any of the classes \( (X, s_0^0), (X, s_0^{(c)}) \) or \( (X, s_0^{(c)}) \), then

\[
\| L_A \| = \sup_n \left\| \frac{1}{\beta_n} A_n \right\| = \sup_n \left( \frac{1}{\beta_n} \| A_n \|_X \right) < \infty.
\]

(b) If \( A \in (X, \ell_1) \), then

\[
\sup_{N \in \mathcal{F}} \left\| \sum_{n \in N} \frac{1}{\beta_n} A_n \right\|_X \leq \| L_A \| \leq 4 \cdot \sup_{N \in \mathcal{F}} \left\| \sum_{n \in N} \frac{1}{\beta_n} A_n \right\|_X.
\]

**Corollary 4.4.2** Let \( X \ni \phi \) be a BK space. Then, we have

(a) If \( A \in (X, s_0^0) \), then

\[
\| L_A \|_X = \limsup_{n \to \infty} \left( \frac{1}{\beta_n} \| A_n \|_X \right)
\]

and

\( L_A \) is compact if and only if \( \lim_{n \to \infty} \left( \frac{1}{\beta_n} \| A_n \|_X \right) = 0. \)

(b) If \( X \) has AK and \( A \in (X, s_0^{(c)}) \), then

\[
\frac{1}{2} \cdot \limsup_{n \to \infty} \left\| \frac{1}{\beta_n} A_n - \gamma \right\|_X \leq \| L_A \|_X \leq \limsup_{n \to \infty} \left\| \frac{1}{\beta_n} A_n - \gamma \right\|_X.
\]
and
\[ L_A \text{ is compact if and only if } \lim_{n \to \infty} \left\| \frac{1}{\beta_n} A_n - \gamma \right\|_X^* = 0. \]

(c) If \( A \in (X, s_\beta) \), then
\[
0 \leq \|L_A\|_X = \limsup_{n \to \infty} \left( \frac{1}{\beta_n} \|A_n\|_X^* \right)
\]
and
\[ L_A \text{ is compact if } \lim_{n \to \infty} \left( \frac{1}{\beta_n} \|A_n\|_X^* \right) = 0. \]

(d) If \( A \in (X, \ell^1_\beta) \), then
\[
\lim_{r \to \infty} \left( \sup_{N \in \mathcal{F}_r} \left| \sum_{n \in N} \frac{1}{\beta_n} A_n \right|^* \right) \leq \|L_A\|_X \leq 4 \lim_{r \to \infty} \left( \sup_{N \in \mathcal{F}_r} \left| \sum_{n \in N} \frac{1}{\beta_n} A_n \right|^* \right)
\]
and
\[ L_A \text{ is compact if and only if } \lim_{r \to \infty} \left( \sup_{N \in \mathcal{F}_r} \left| \sum_{n \in N} \frac{1}{\beta_n} A_n \right|^* \right) = 0. \]

**Corollary 4.4.3** If either \( A \in (\ell_\infty, s^0_\beta) \) or \( A \in (\ell_\infty, s^{(c)}_\beta) \), then \( L_A \) is compact.

**Remark 4.4.4** Many new results can be obtained from Corollaries 4.4.1 and 4.4.2 with any particular case of the space \( X \), e.g., we may use (4.2.9) for \( X = \mathbb{b}v^p \ (1 < p < \infty) \) in the above corollaries (except part (b) of Corollary 4.4.2).

Now, let us turn to the \( \beta \)-duals of the spaces \( s^0_\alpha \), \( s^{(c)}_\alpha \), \( s_\alpha \) and \( \ell^1_\alpha \). Then, it can be easily seen that \((s^0_\alpha)^\beta = (s^{(c)}_\alpha)^\beta = (s_\alpha)^\beta = \ell^1_{1/\alpha} \), \((\ell^1_\alpha)^\beta = s_{1/\alpha} \) and \((\ell^p_\alpha)^\beta = \ell^q_{1/\alpha} \), where \( 1 < p < \infty \), \( q = p/(p-1) \) and \( 1/\alpha = (1/\alpha_k)_{k=0}^\infty \). Furthermore, if \( a = (a_k) \) is in the \( \beta \)-dual of any of the spaces \( s_{1/\alpha} \), \( \ell^1_{1/\alpha} \) or \( \ell^p_{1/\alpha} \), then we have
\[
\|a\|_{s^0_\alpha}^* = \|a\|_{s^{(c)}_\alpha}^* = \|a\|_{s_\alpha}^* = \sum_{k=0}^{\infty} \alpha_k |a_k|, \tag{4.4.3}
\]
\[
\|a\|_{\ell^1_{1/\alpha}}^* = \sup_k (\alpha_k |a_k|) \tag{4.4.4}
\]
and
\[
\|a\|_{\ell^p_{1/\alpha}}^* = \left( \sum_{k=0}^{\infty} (\alpha_k |a_k|)^q \right)^{1/q}, \tag{4.4.5}
\]
respectively, where \( 1 < p < \infty \) and \( q = p/(p-1) \). Therefore, by using these notations in Theorems 4.2.5, 4.2.8, 4.3.1, 4.3.2 and 4.3.4, and Corollaries 4.3.5 and 4.3.6, we get some applications as special cases of these results when \( X \) is any of the spaces \( s^0_\alpha \), \( s^{(c)}_\alpha \), \( s_\alpha \), \( \ell^1_{1/\alpha} \) or \( \ell^p_{1/\alpha} \) (except the cases \( X = s^{(c)}_\alpha \) and \( X = s_\alpha \) when \( X \) has AK). For instance, by using Lemma 1.2.3 (b), we have the following consequence of Corollaries 4.3.5 and 4.3.6.
Corollary 4.4.5  Let $1 < p < \infty$ and $q = p/(p - 1)$. Then, we have
(a) If $L \in B(\ell^p_\alpha, cs_0)$ is given by the matrix $A \in (\ell^p_\alpha, cs_0)$, then
\[
\|L\|_\chi = \limsup_{n \to \infty} \|A\|^{(n)}_{(\ell^p_\alpha, bs)}
\]
and $L \in C(\ell^p_\alpha, cs_0)$ if and only if $\lim_{n \to \infty} \|A\|^{(n)}_{(\ell^p_\alpha, bs)} = 0$, where
\[
\|A\|^{(n)}_{(\ell^p_\alpha, bs)} = \left( \sum_{k=0}^{\infty} \left| \alpha_k \left( \sum_{m=0}^{n} a_{mk} \right) \right|^q \right)^{1/q}; \quad (n \in \mathbb{N}).
\]
(b) If $L \in B(\ell^p_\alpha, bs)$ is given by the matrix $A \in (\ell^p_\alpha, bs)$, then
\[
0 \leq \|L\|_\chi \leq \limsup_{n \to \infty} \|A\|^{(n)}_{(\ell^p_\alpha, bs)}
\]
and $L \in C(\ell^p_\alpha, bs)$ if $\lim_{n \to \infty} \|A\|^{(n)}_{(\ell^p_\alpha, bs)} = 0$.
(c) If $L \in B(\ell^p_\alpha, cs)$ is given by the matrix $A \in (\ell^p_\alpha, cs)$, then
\[
\frac{1}{2} \cdot \limsup_{n \to \infty} \|A\|^{(n)}_{(\ell^p_\alpha, cs)} \leq \|L\|_\chi \leq \limsup_{n \to \infty} \|A\|^{(n)}_{(\ell^p_\alpha, cs)}
\]
and $L \in C(\ell^p_\alpha, cs)$ if and only if $\lim_{n \to \infty} \|A\|^{(n)}_{(\ell^p_\alpha, cs)} = 0$, where
\[
\|A\|^{(n)}_{(\ell^p_\alpha, cs)} = \left( \sum_{k=0}^{\infty} \left| \alpha_k \left( \sum_{m=0}^{n} a_{mk} - a_k \right) \right|^q \right)^{1/q}; \quad (n \in \mathbb{N})
\]and $a_k = \lim_{n \to \infty} (\sum_{m=0}^{n} a_{mk})$ for all $k \in \mathbb{N}$.
(d) If $L \in B(\ell^p_\alpha, bv)$ is given by the matrix $A \in (\ell^p_\alpha, bv)$, then
\[
\lim_{r \to \infty} \|A\|^{(r)}_{(\ell^p_\alpha, bv)} \leq \|L\|_\chi \leq 4 \cdot \lim_{r \to \infty} \|A\|^{(r)}_{(\ell^p_\alpha, bv)}
\]
and $L \in C(\ell^p_\alpha, bv)$ if and only if $\lim_{r \to \infty} \|A\|^{(r)}_{(\ell^p_\alpha, bv)} = 0$, where
\[
\|A\|^{(r)}_{(\ell^p_\alpha, bv)} = \sup_{N \in F_r} \left( \sum_{k=0}^{\infty} \sum_{n \in N} \alpha_k (a_{nk} - a_{n-1,k}) \right)^{1/q}; \quad (r \in \mathbb{N}).
\]

Remark 4.4.6  The conclusions of Corollary 4.4.5 still hold for $s^0_\alpha$ instead of $\ell^p_\alpha$ with $q = 1$.

Similarly, several results and consequences can be obtained from Corollaries 4.4.1 and 4.4.2 by using (4.4.3), (4.4.4) and (4.4.5). The most of these results can be found in [63]. For example, if $A \in (s^0_\alpha, s^0_\beta)$ then we have by Corollary 4.4.2 (b) that (see [63, Theorem 2.14 (iv)]
\[
\frac{1}{2} \cdot \limsup_{n \to \infty} \left( \sum_{k=0}^{\infty} \alpha_k \left| \frac{a_{nk}}{\beta_n} - \gamma_k \right| \right) \leq \|L_A\|_\chi \leq \limsup_{n \to \infty} \left( \sum_{k=0}^{\infty} \alpha_k \left| \frac{a_{nk}}{\beta_n} - \gamma_k \right| \right)
\]
and

\[ L_A \text{ is compact if and only if } \lim_{n \to \infty} \left( \sum_{k=0}^{\infty} \frac{\alpha_k}{\beta_n} |a_{nk}| \right) = 0, \]

where, here and in what follows, \( \alpha = (\alpha_k)_{k=0}^{\infty} \) and \( \beta = (\beta_k)_{k=0}^{\infty} \) are sequences of positive reals and \( \gamma = (\gamma_k)_{k=0}^{\infty} \) is given by (4.4.2). Thus, in the following consequences of Corollaries 4.4.1 and 4.4.2, we shall only consider those results which are not covered in [63] or which have other formulae.

**Corollary 4.4.7** If \( A \) is in any of the classes \( (s_\alpha, s_{\beta}^0) \), \( (s_\alpha, s_{\beta}^{(c)}) \) or \( (s_\alpha, s_{\beta}) \), then

\[ \| L_A \| = \sup_n \left( \frac{1}{\beta_n} \sum_{k=0}^{\infty} \alpha_k |a_{nk}| \right) < \infty. \]

**Remark 4.4.8** It is obvious by (4.4.3) that the conclusion of Corollary 4.4.7 still holds if we replace the space \( s_\alpha \) by any of the spaces \( s_\alpha^0 \) or \( s_\alpha^{(c)} \) (see [63, Theorem 1.5]).

**Corollary 4.4.9** Let \( 1 < p < \infty \) and \( q = p/(p - 1) \). Then, we have

(a) If \( A \) is in any of the classes \( (s_\alpha^0, \ell_\beta^1) \), \( (s_\alpha^{(c)}, \ell_\beta^1) \) or \( (s_\alpha, \ell_\beta^1) \), then

\[ \sup_{N \in \mathcal{F}} \left( \sum_{k=0}^{\infty} \frac{\sum_{n \in N} \alpha_k a_{nk}}{\beta_n} \right)^{1/q} \leq \| L_A \| \leq 4 \cdot \sup_{N \in \mathcal{F}} \left( \sum_{k=0}^{\infty} \frac{\sum_{n \in N} \alpha_k a_{nk}}{\beta_n} \right)^{1/q}. \]

(b) If \( L \in B(\ell_p^\alpha, \ell_\beta^1) \) is given by the matrix \( A \in (\ell_p^\alpha, \ell_\beta^1) \), then

\[ \sup_{N \in \mathcal{F}} \left( \sum_{k=0}^{\infty} \frac{\sum_{n \in N} \alpha_k a_{nk}}{\beta_n} \right)^{1/q} \leq \| L \| \leq 4 \cdot \sup_{N \in \mathcal{F}} \left( \sum_{k=0}^{\infty} \frac{\sum_{n \in N} \alpha_k a_{nk}}{\beta_n} \right)^{1/q}. \]

**Remark 4.4.10** We may note that the matrix classes in part (a) of Corollary 4.4.9 are equal, which is immediate by (4.4.3) and Lemmas 1.2.4 (b) and 1.2.12 (a).

**Corollary 4.4.11** If \( A \in (s_\alpha, s_{\beta}) = (s_\alpha^{(c)}, s_{\beta}) = (s_\alpha^0, s_{\beta}) \), then

\[ 0 \leq \| L_A \|_\chi \leq \limsup_{n \to \infty} \left( \frac{1}{\beta_n} \sum_{k=0}^{\infty} \alpha_k |a_{nk}| \right) \]

and

\[ L_A \text{ is compact if } \lim_{n \to \infty} \left( \frac{1}{\beta_n} \sum_{k=0}^{\infty} \alpha_k |a_{nk}| \right) = 0. \]

**Corollary 4.4.12** Let \( 1 < p < \infty \) and \( q = p/(p - 1) \). Then, we have

(a) If \( L \in B(\ell_p^\alpha, s_\beta^0) \) is given by the matrix \( A \in (\ell_p^\alpha, s_\beta^0) \), then

\[ \| L \|_\chi = \limsup_{n \to \infty} \left( \sum_{k=0}^{\infty} \frac{\alpha_k a_{nk}}{\beta_n} \right)^{1/q}. \]
and
\[ L \in C(\ell^p, s^0_\beta) \text{ if and only if } \lim_{n \to \infty} \left( \sum_{k=0}^{\infty} \left| \frac{\alpha_k a_{nk}}{\beta_n} \right|^q \right) = 0. \]

(b) If \( L \in B(\ell^p, s^{(c)}_\beta) \) is given by the matrix \( A \in (\ell^p, s^{(c)}_\beta) \), then
\[
\frac{1}{2} \cdot \limsup_{n \to \infty} \left( \sum_{k=0}^{\infty} \left| \frac{\alpha_k a_{nk}}{\beta_n} - \alpha_k \gamma_k \right|^q \right)^{1/q} \leq \|L\|_\chi \leq \limsup_{n \to \infty} \left( \sum_{k=0}^{\infty} \left| \frac{\alpha_k a_{nk}}{\beta_n} - \alpha_k \gamma_k \right|^q \right)^{1/q}
\]
and
\[
L \in C(\ell^p, s^{(c)}_\beta) \text{ if and only if } \lim_{n \to \infty} \left( \sum_{k=0}^{\infty} \left| \frac{\alpha_k a_{nk}}{\beta_n} - \alpha_k \gamma_k \right|^q \right) = 0.
\]

(c) If \( L \in B(\ell^p, s_\beta) \) is given by the matrix \( A \in (\ell^p, s_\beta) \), then
\[
0 \leq \|L\|_\chi \leq \limsup_{n \to \infty} \left( \sum_{k=0}^{\infty} \left| \frac{\alpha_k a_{nk}}{\beta_n} \right|^q \right)^{1/q}
\]
and
\[
L \in C(\ell^p, s_\beta) \text{ if } \lim_{n \to \infty} \left( \sum_{k=0}^{\infty} \left| \frac{\alpha_k a_{nk}}{\beta_n} \right|^q \right) = 0.
\]

(d) If \( L \in B(\ell^p, \ell^1_\beta) \) is given by the matrix \( A \in (\ell^p, \ell^1_\beta) \), then
\[
\lim_{r \to \infty} \|A\|_{(\ell^p, \ell^1_\beta)}^{(r)} \leq \|L\|_\chi \leq 4 \cdot \lim_{r \to \infty} \|A\|_{(\ell^p, \ell^1_\beta)}^{(r)}
\]
and
\[
L \in C(\ell^p, \ell^1_\beta) \text{ if and only if } \lim_{r \to \infty} \|A\|_{(\ell^p, \ell^1_\beta)}^{(r)} = 0,
\]
where
\[
\|A\|_{(\ell^p, \ell^1_\beta)}^{(r)} = \sup_{N \in \mathcal{F}_r} \left( \sum_{k=0}^{\infty} \left( \sum_{n \in N} \left| \frac{\alpha_k a_{nk}}{\beta_n} \right|^q \right)^{1/q} \right) \quad ; \quad (r \in \mathbb{N}).
\]