Chapter 1

INTRODUCTION

In this thesis we attempt to make a probabilistic analysis of some physically realizable, though complex, storage and queueing models. It is essentially a mathematical study of the stochastic processes underlying these models. Our aim is to have an improved understanding of the behaviour of such models, that may widen their applicability. Different inventory systems with random lead times, vacation to the server, bulk demands, varying ordering levels, etc. are considered. Also we study some finite and infinite capacity queueing systems with bulk service and vacation to the server and obtain the transient solution in certain cases. Each chapter in the thesis is provided with self introduction and some important references. This chapter gives a brief general introduction to the subject matter and related topics.

1.1 INVENTORY THEORY

An inventory is an amount of material stored for the purpose of sale or production. The inventory models are usually characterized by the demand pattern and the
policy for replenishing the stock in the store. The two basic types of policy for replenishment are (i) the ordering cycle policy under which orders for replenishment are placed at regular intervals of time of length $T$, (ii) the $(s,S)$ policy under which orders are placed as and when the stock in the store plus the quantity already on order falls to some fixed level $s$. The replenishments ordered under any of these policies are assumed to arrive after a time lag $L$, which may be fixed or a random variable. This time lag $L$ is called 'lead time'. During a lead time the inventory level may fall to zero. The time duration for which the level of inventory continuously remains at zero is called a dry period.

A valuable review of the problems in the probability theory of storage systems is given by Gani [1957]. A systematic account of probabilistic treatment in the study of inventory systems using renewal theoretic arguments is given in Arrow, Karlin and Scarf [1958]. Hadley and Whitin [1963] deals with the applications of such models to practical situations. Tijms [1972] gives a detailed analysis of the inventory systems under $(s,S)$ policy.
The cost analysis of different inventory systems is given in Naddor [1966]. A practical treatment of the (s,S) lost sales model can be found in the recent books by Silver and Peterson [1984] and Tijms [1986].

Veinott [1966] gives a detailed review of the work carried out in (s,S) inventory systems up to 1966. We refer to the monograph by Ryshikov [1973] for inventory systems with random lead times. Gross and Harris [1971] and Gross, Harris and Lechner [1971] deal with one for one ordering inventory policies with state dependent lead times. Sivazlian [1974] considers an (s,S) inventory model in which unit demands of items occur with arbitrary interarrival times between demands, but lead time is assumed to be zero. His results are extended by Srinivasan [1979] to the case in which lead times are independent and identically distributed random variables having a general distribution. Sahin [1979] considers an (s,S) inventory system in which demand quantities are random but lead time is a constant. Again in 1983 Sahin discussed an inventory system in which the interarrival times between consecutive demands, quantities demanded and lead times are all independent and generally distributed sequences of independent and identically distributed random variables. He obtained the binomial
moments for the inventory deficit. Thangaraj and Ramanarayanan [1983] consider an inventory system with random lead times and having two ordering levels. Kalpakam and Arivirinanan [1985] deals with an inventory system having one exhibiting item subject to random failure. Daniel and Ramanarayanan [1987 a,b] consider inventory systems with vacation to the server during dry period.

1.2 QUEUEING THEORY

Queueing theory is a well developed branch of applied probability theory. Historically, the subject of queueing theory has been developed largely in the context of telephone traffic engineering. Over the past three decades, steady progress has been made towards solving increasingly difficult and realistic queueing models.

A queueing model is usually defined in terms of three characteristics— the input process, the service mechanism and the queue discipline. The input process describes the sequence of requests for service. Often the input process is specified in terms of the distribution of the lengths of time between consecutive customer arrival
instants. The service mechanism is the category that includes such characteristics as the number of servers and the lengths of time that customers hold the servers. The queue discipline deals with the rule by which customers are taken for service.

For the single server queue a busy period is the time interval during which the server is continuously busy. i.e. it is the length of time from the instant the (previously idle) server is seized until it next becomes idle. The time between the starting points of two consecutive busy periods is called a busy cycle. The actual waiting time in the queue of a customer is defined as the time between the moment of his arrival and the moment at which his service starts. The virtual waiting time at time t is the actual waiting time of a customer if he had arrived at time t.

For a complete reference on the earlier works of queueing theory we refer to the bibliographies given in the books by Syski [1960], Saaty [1961], Takacs [1962], Prabhu [1965], Cooper [1972], Gross and Harris [1974], Neuts [1981] and Medhi [1984].
M/G/1 queueing models where the server is not available over occasional intervals of time has been considered by many authors. The times, when the server is not available are called vacations (also referred to as rest). A queueing model in which the server goes for vacation whenever the system becomes empty is an 'exhaustive service system'. This model has been studied by Miller [1964], Cooper [1970], Levy and Yechiali [1975], Shantikumar [1980], Scholl and Kleinrock [1983], Lee [1984] and Fuhrmann [1984]. M/G/1 queueing systems without exhaustive service is studied by Neuts and Ramalhoto [1984], Ali and Neuts [1984] and Fuhrmann and Cooper [1985]. Daniel [1985] discusses several interesting models with vacation to the server. Doshi [1985] considers the G/G/1 exhaustive service system and proves that the 'decomposition property' holds. G/G/1 vacation system with Bernoulli schedules is considered by Keilson and Servi [1986]. For a complete survey of the queueing systems with vacations, we refer to Doshi [1986].

1.3 NOTATIONS

In this section we introduce the following notations, that may be frequently used in the thesis.
* denotes the convolution operator.

\[ f^{*n}(x) \] is the n-fold convolution of \( f(x) \) with itself.

For a distribution function \( F(x) \), \( \bar{F}(x) = 1 - F(x) \)

\( \delta_{ij} \) is the Kronecker delta function given by \( \delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases} \)

\([x]\) denotes the integral part of \( x \).

\( \gamma_{\alpha,\beta}(\cdot) \) is the Gamma density function with parameters \( \alpha \) and \( \beta \).

\( F_{\alpha,\beta}(\cdot) \) is the Gamma distribution function with parameters \( \alpha \) and \( \beta \).

\( E(X) \) is the expectation of the random variable \( X \).

We define the convolution of two matrices \( A \) and \( B \) as follows. If \( A(t) = [a_{ij}(t)] \) is a matrix of order \( m \times p \) and \( B(t) = [b_{ij}(t)] \) is a matrix of order \( p \times n \), then

\( A \ast B(t) = [c_{ij}(t)] \) is a matrix of order \( m \times n \) whose elements are given by

\[ c_{ij}(t) = \sum_{k=1}^{p} a_{ik} \ast b_{kj}(t). \]

1.4 **RENEWAL THEORY**

Let \( \{X_n, n=1,2, \ldots\} \) be a sequence of nonnegative independent random variables with a common distribution function \( F(x) \). Let \( S_0 = 0 \) and for \( n > 1 \), \( S_n = \sum_{i=1}^{n} X_i \). 
Define \( N(t) = \sup \{ n \mid S_n \leq t \} \)

If \( \mu = \int_0^\infty x \, dF(x) \), which we assume to exist, by the strong law of large numbers we have \( \frac{S_n}{n} \to \mu \) as \( n \to \infty \) with probability 1. Hence, for finite \( t \), \( S_n \leq t \) only finitely often and so \( N(t) < \infty \) with probability 1. The process \( \{N(t), t \geq 0\} \) is a Renewal process.

It is easy to note that \( N(t) \geq n \iff S_n \leq t \).

Using this one may obtain, \( P\{N(t)=n\} = F^*(n)(t)-F^*(n+1)(t) \).

Let \( M(t) = E(N(t)) \); \( M(t) \) is called the renewal function and it can be shown that \( M(t) = \sum_{n=1}^{\infty} F^*(n)(t) \).

Let \( m(t) = M'(t) \); \( m(t) \) is called the renewal density function and \( m(t) = \sum_{n=1}^{\infty} f^*(n)(t) \) if the density function \( f(x)=F'(x) \) exists.

Suppose \( \{X_n, n = 1,2, \ldots \} \) is a sequence of independent nonnegative random variables with \( X_1 \) having distribution function \( G(x) \) and \( X_n \) for \( n > 1 \) having distribution function \( F(x) \).

Let \( S_0 = 0 \) and \( S_n = \sum_{i=1}^{n} X_i \) for \( n \geq 1 \).
Define \( N_D(t) = \text{Sup}\{ n \mid S_n \leq t \} \), \( N_D(t) \) is called a Delayed renewal process or a Modified renewal process.

Here we have, \( P\{N_D(t)=n\} = G^{*}F^{*n-1}(t) - G^{*}F^{*n}(t) \).

The modified renewal function is \( M_D(t) = E(N_D(t)) \) and

\[
M_D(t) = \sum_{n=1}^{\infty} G^{*}F^{*n-1}(t).
\]

The modified renewal density function is \( m_D(t) = M'_D(t) \) and it is given by

\[
m_D(t) = \sum_{n=1}^{\infty} g^{*}f^{*n-1}(t),
\]

under the additional assumption that the density function \( g(x) = G'(x) \) and \( f(x) = F'(x) \) exist.

For more details of the renewal theory we refer to Cox [1962].

1.5. SUMMARY OF THE WORK INCLUDED IN THIS THESIS

In the second chapter we consider three models on \((s,S)\) inventory systems with finite backlog of demands and vacation to the server. In all the models the inter-arrival times of demands and lead times are independent sequences of independent and identically distributed random variables having general distributions. In the first two models, whenever the inventory becomes dry, the server goes for vacation. In the third model when
the inventory becomes dry, a local purchase is made according to the availability and the server goes for vacation only if the local purchase is impossible. The vacation period is also random with a general distribution. If the server returns from vacation before the realization of the order, he permits a finite number of demands to wait. All the demands arriving during the vacation period of the server are lost. In the first and third model, order size is a constant and in the second model the order size can vary according to the inventory level. Using renewal theory, the inventory level and queue size probabilities are presented explicitly.

In chapter 3, we derive expressions to find the correlation between lead time and dry period for \((s,S)\) inventory systems and finite capacity dam models. Also, assuming exponential distributions for interarrival times of demands and lead times, simple expressions for the joint moments are obtained.

Fourth chapter deals with an \((s,S)\) policy inventory system under the assumption that intervals of time between successive demand points, quantities demanded at these points and lead times are independent sequences of independent and identically distributed random variables. Interarrival times
of demands and lead times follow general distributions. The quantity demanded each time is a discrete random variable taking values between two integers \(a\) and \(b\) such that \(s < a \leq b < S - s\). Backlogging of demands are not allowed. Exact expressions for the system size probabilities are derived.

In chapter 5, we consider an inventory system in which an ordering level is decided according to the number of demands during the previous lead time. Interarrival times of demands and lead times are generally distributed random variables and each demand is for one unit. All the demands that occur during the inventory dry period are lost. Using renewal theoretic arguments we derive the inventory level probabilities. Also we discuss the correlation between the number of demands in a lead time and the next dry period.

\(G/M^{a,b}/1\) queueing system with vacation to the server is considered in chapter 6. The service time is exponentially distributed with parameter \(\mu_i\), if \(i\) is the size of the batch being served. The vacation periods are also exponentially distributed. Matrix-geometric method of Neuts is used to find the steady state probabilities of the system size. The structure of the matrix geometric equation is not simple
and is not yielding to any easy algorithmic approach for solution in the general setup. Probability distribution of waiting time is given explicitly.

In chapter 7, we consider a finite capacity $M/G/1$ queueing system with server going for vacation whenever there is no unit in the system. The vacation periods are independent and identically distributed random variables having a general probability distribution function. The capacity of the waiting room is finite and all the demands that arrive when the waiting room is full are lost. Using renewal theory, we derive the transient system size probabilities at arbitrary time points. Also we derive expressions for the probability distribution of virtual waiting time in the queue at any time $t$.

In the last chapter we consider an $M/G^{a,b}/1$ queueing system with a waiting room that allows only a maximum of $b$ customers to wait at any time. A minimum of $a$ customers are required to start a service and the server goes for vacation whenever he finds less than $a$ customers in the waiting room after a service. If the server returns from vacation to find less than $a$ customers waiting, he begins
another vacation immediately. Here also expressions for the time dependent system size probabilities at arbitrary time points are derived.

The expressions we derive are complicated and hence do not easily yield to give numerical solutions. Developing algorithms for these will be quite worthwhile work.