Chapter 1

INTRODUCTION

1.1 Mathematical Programming Problems

The history of mathematical programming can be traced back to the period of World War II. Complexity of the situation during World War II and the urgency in assigning scarce resources to different military operations in the best possible way led to collective efforts of scientists toward developing optimal strategies and solving optimization problems. Mathematical scientists were first gathered in England and then in other countries as part of these efforts. Because the goal of their work in the beginning was to optimize military operations, this subject became known as “operations research (OR)”. At the end of World War II, those involved in industrial development used the operations research methods for other purposes. Also, many of them were interested in continuing their work of developing methods for other similar optimization problems. Some of these are mentioned in the following sections and chapters.

Mathematical programming, alternatively known as constrained optimization, is a broad subject area that covers a great variety of theoretical and applied problem areas such as operations research [46, 32], network analysis [55, 44], game theory and economics [47, 46], and production engineering [52, 36]. Mathematical
programming is a class of methods for solving problems which have an objective function to be optimized (that is, to be minimized or maximized) in order to find the best solution under specified constraints expressed as inequalities or equations involving the variables that form the objective function. Every optimization problem essentially consists of an objective function and a set of constraints, which may include requirements like non-negativity of the variables or finite bounds for some or all of the variables. Depending on the nature of the problem, the objective function is to be either maximized or minimized. For example, the target of a problem can be any of the following:

(a) Maximize the profit.

(b) Maximize the reliability of an equipment.

(c) Minimize the cost.

(d) Minimize the weight of an engineering component or structure, etc.

Mathematical problems impose certain constraints on the variables that define the objective function, and therefore the problem is referred to as a constrained optimization problem. A problem without any constraints is an unconstrained optimization problem [59]. In this thesis, we consider only constrained optimization problems. As mentioned above, mathematical programming problems (that is, optimization problems subject to constraints) are divided into different parts like, for instance, linear programming problems, quadratic programming problems, nonlinear programming problems, dynamic programming problems and stochastic programming problems. In this thesis, we discuss only linear programming problems, quadratic programming problems and nonlinear programming problems, though
the methods developed in this thesis can be useful in other forms of mathematical programming problems as well.

1.2 Linear Programming Problems

Linear programming is one of the basic problems in constrained optimization and has received considerable attention during the last six decades. Linear programming was developed by the Russian mathematician Leonid V. Kantorovich in 1939 [38, 62] and was extended by the American mathematician George B. Dantzig, who published his work in 1947. Its acceptance and usefulness have been greatly enhanced by the availability of powerful digital computers, since the technique often requires a vast amount of computation [1]. Many algorithms have been proposed for solving linear programming problems, starting with the simplex algorithm of Dantzig and its relatives [19], continuing via the polynomial-time algorithms of Khachiyan [45] and Karmarkar [34], through to several more recent techniques reviewed in [43]. While some of the proposed algorithms have proven to be extremely efficient in practice, analysis of their execution time has not been fully satisfactory so far. For example, the simplex algorithm was shown to be exponential in the worst case by Klee and Minty [37]. Klee and Minty (1972) gave a linear programming problem in which the polytope $P$ is a distortion of an $n$-dimensional cube. They showed that the simplex method as formulated by Dantzig visits all the $2^n$ vertices before arriving at the optimal vertex. This shows that the worst-case complexity of the simplex algorithm is exponential time. Also, the number of arithmetic operations in Simplex algorithm grows exponentially as the number of variables increases. Khachiyan [35] proposed the first polynomial-time algorithm, the ellipsoid method. Karmarkar [34] suggested the second polynomial-time algo-
rithm that appears to be more efficient than the simplex method, especially when the problem size increases beyond some thousands of variables [16].

1.2.1 Simplex method

The simplex method is the most popular exterior point algorithm for solving linear programming problems since the 1940's. In short, the simplex method moves from vertex to vertex on the bounding of the feasible solution set which is a polyhedron, repeatedly improving the objective function until either an optimal solution is found or it is established that no finite optimal solution exists. As it can be understood easily, the time required might be an exponential function of the number of variables, and this can happen in some special cases [50].

The original simplex method calculates and updates all numbers in the tableau though many of them do not change. At every iteration of the simplex method, the only data required are the first row of the tableau representing the relative profit coefficients \((c_j - z_j)\) row, the pivotal column and the pivotal row of the tableau corresponding to the entering and leaving variables respectively, and the right-hand-side constants represented in the last column of the tableau. Revised simplex method, which is an improvement over the original simplex method, consists of almost the same steps as those in the simplex method. The only difference occurs in the details of identifying the entering variable and leaving variable. In fact, the simplex method updates the entire simplex tableau when only a small part of it changes. The revised simplex method is computationally more efficient and accurate. Benefits of revised simplex method are clearly understood in case of large LP problems. As a matter of fact, all the mathematical programming software routines for linear programming problems implement only the revised simplex method rather than the original simplex method.
1.2.2 Interior point methods

Linear programming problems are usually solved using the simplex method, which is an exterior point method. An alternative to the simplex method is called the Interior Point method. In 1979 L. Khachiyan presented the first polynomial-time algorithm which is known as ellipsoid method for solving linear programming problems [35]. Khachiyan proposed to construct the smallest ellipsoid that contains the polytope (or simplex) representing the solution set of the linear programming problem and then to search for the optimal solution in the interior of this ellipsoid, thus avoiding traversing the surface of the polytope from vertex to vertex. N. Karmarkar (1984) suggested the second polynomial-time algorithm, combining the admirable theoretical properties of the ellipsoid method and practical advantages of the simplex method. His method appeared to be more efficient than the simplex method. He also claimed that his algorithm is a polynomial-time algorithm and works better than Khachiyan’s algorithm [34]. When the problem size increases above some thousands of variables, Karmarkar’s algorithm is more efficient than the simplex method [16]. These interior point methods don’t move from vertex to vertex, but pass through the interior of the feasible solution set in order to reach the optimal solutions faster than the simplex method. In a way, the interior point algorithms avoid visiting some vertices by jumping through the interior, thus attempting to reduce the execution time when the number of iterations is expected to grow exponentially with the size of the problem usually measured in terms of the number of variables and the number of constraints.
1.2.3 Duality theory

Linear programming has an interesting theory of duality. The most important aspect of this theory is: corresponding to every primal linear programming problem, there is a (unique) dual linear programming problem. As a matter of fact, there are unique pairs of primal and dual linear programming problems, one of them is a maximization problem while the other is a minimization problem. The convention in naming these problems is that the maximization problem is called the primal and the minimization problem is called the dual, though there is no mathematical reason for this naming system. There are some relationships between primal and dual problems that are not only important to find optimality conditions, but also provide an understanding of the problem and provides some meaningful economic interpretations of the optimization model. More specifically, the post optimality analysis depends heavily on the understanding of duality.

John von Neumann played a key role in the development of linear programming based on duality. He developed the theory of duality. Far from being confined only to linear programming, duality theory has contributed considerably to the development of other mathematical optimization techniques. Intimately associated with Lagrangian functions and Lagrange multipliers, it has important applications in integer and nonlinear programming [23]. The origins of duality go back to John von Neumann (1947) and, independently, to Gale et al. [28]. For a historical perspective, see Dantzig [20].

Duality in linear programming problems is a useful property that makes the problem easier in some cases and leads to the dual simplex method. This is also helpful in sensitivity or post optimality analysis of decision variables and scarce resources. We will discuss the duality principle and relationships between the
primal and dual problems in Chapter 2.

1.3 Quadratic Programming Problems

Quadratic programming is a special class of optimization problem. In this problem, a quadratic objective function of several variables is optimized (that is, maximized or minimized) subject to linear constraints on the variables. While interior point algorithms are extensively studied for LP problems, several researchers have also paid attention to interior point algorithms for quadratic programming (QP) or linear complementarity problem (LCP), which is a special case of QP [51]. Megiddo [44] has studied the barrier path to the optimal solution for LCP.

Many efficient methods have been developed for solving QP problems in the last three decades. One of them is called the penalty method. In this class of methods, we replace the original constrained problem by a sequence of unconstrained sub-problems that minimize the penalty function. It is an important approach to solve constrained optimization problems. Kapoor and Vidya [33], Ye and Tse [68] have proposed an extension of Karmarkar’s algorithm for solving convex quadratic programs with a global convergence rate \(1 - O\left(\frac{1}{n}\right)\). Ye [67] developed an improved interior point algorithm for QP. His algorithm creates a sequence of primal and dual interior feasible points converging to the optimal solution. At each iteration, the gap between primal and dual objective function is reduced at a global rate \(1 - \frac{1}{\sqrt{n}}\). Özdemir and Evirgen [51] have proposed a dynamical system model approach to find a unique minimum of quadratic programming problems with equality constraints. Further information about some approaches to solve quadratic programming problems can be found in references [8, 13, 22, 60].
1.4 General Non-Linear Programming Problems

Nonlinear programming problems represent the general case of mathematical pro­
gramming problems such that both the objective function and constraints are non­
linear and are the most difficult cases of smooth optimization problems to solve.
In fact, there is no general agreement on the best approach to solve every nonlin­
ear programming problem in practice and there is no guarantee of convergence to
the optimal solution [25, 32, 54, 63]. Constrained nonlinear programming prob­
lems often arise in many Science, Engineering and Economics applications. We are
therefore going to consider this case of programming problem.

Nonlinear programming problems can be solved using a variety of methods
such as penalty and barrier methods, gradient projection methods and sequential
quadratic programming (SQP) methods [25]. All of these optimization algorithms
are iterative procedures which start from a given point and then generate a se­
quence of solutions which converges to the optimal solution.

1.4.1 Penalty and Barrier methods

The basic principle of these methods is to transform the constrained problem
to unconstrained problem by adding a penalty or barrier term to the objective
function. This penalty/barrier term increases the objective function value when
the constraints are violated or the bounding of feasible solution set is approached.

1.4.2 Generalized reduced gradient method

The generalized reduced gradient (GRG) algorithm was first developed by Abadie
and Carpentier [1] as an extension of the reduced gradient method which is also
known as Frank-Wolfe algorithm. GRG transforms inequality constraints into
international journals for publication.

1.6 Chapter Wise Summary

In preparing this thesis, a special effort has been made to develop a method which is different from classical methods to solve mathematical programming problem. This method is a stochastic search method. It is similar to Monte Carlo methods in the sense that it is a process of using random variables to solve deterministic problems. In this sense, Monte Carlo methods obtain “estimate” of natural constants treated as parameters of the model. A stochastic search will “approximate” the optimal solution by generating a monotone sequence converging to the optimal solution.

To illustrate our point of view, we begin with applying stochastic search algorithm to Linear, Quadratic and General nonlinear programming problems in the chapters 2, 3 and 4, respectively.

This thesis is divided into five chapters.

Chapter One: Chapter one is the current chapter and gives some background and history on the mathematical programming (Linear, Quadratic and General nonlinear programming) problem and also some related methods to solve the mathematical programming problems.

Chapter Two: This chapter is divided into seven sections. We give a brief introduction on linear programming and its background. Linear programming in canonical form with some of its features are given. Simplex method and revised simplex method is introduced in tableau form by giving a numerical example. Duality theory and its applications with economic interpretation of the dual problem is given and the concept of duality is introduced by
an example. Some computational problems in simplex method have been discussed, also complexity of the simplex method.

The proposed stochastic search method has been investigated in a great detail. We have begun with a uniform distribution between zero and the upper bound for every decision variable. This is like using the least informative prior because the distribution provides no information on the location of the optimal solution. We know that the optimal solution is away from the origin. We therefore try a normal distribution with the mean at the upper bound for a decision variable. Here, we know that at least 50% of points generated from such a normal distribution will be infeasible solutions. What is surprisingly is that in spite of such a high rejection rate, the near optimal solution is obtained very fast. We then experiment by moving the mean of the normal distribution in the interior of the feasible set, away from the upper bound. On one hand, this change reduces the rejection rate. On the other hand there is only a marginal effect on the convergence time. We then try some other more skew probability distributions in order to establish some benchmark for the new algorithm. Also, the stochastic algorithm uses multiple starts to find the best near-optimal solution among feasible solutions which are generated. It also uses the duality theory in linear programming.

The stochastic search algorithm is programmed in MATLAB for easy execution. The result of simulation studies by applying the stochastic search algorithm on four numerical examples along with conclusions has been provided.

**Chapter Three:** This chapter is divided into five sections. It deals with quadratic programming problems. The statement of quadratic programming problem
are given. We have given the necessary and sufficient conditions on optimality
the solution of quadratic programming problems. These conditions are known
as Karush-Kuhn-Tucker, also known as Kuhn-Tucker, (KKT) conditions. We
apply these KKT conditions to obtain optimal solution on one numerical
example.

The proposed stochastic search method has been investigated in a great
detail. This method is a reliable method of obtaining a sufficiency near-
optimal solution without any initial guess or starting point. The stochastic
search algorithm has been applied on three numerical examples. The results
of simulation studies and conclusions have been provided. The stochastic
search algorithm is programmed in MATLAB code.

**Chapter Four:** This chapter is divided into six sections. This chapter deals with
general class of nonlinear programming problem which can be considered
as a direct extension of linear programming where we replace linear model
functions by nonlinear ones. In general nonlinear programming problem
both the objective function and the constraints are nonlinear. Nonlinear
programming is “hard”, because does not exist an algorithm that can solve
every *NLP* problem efficiently in practice.

The statement of problem and a brief on some classical methods (Penalty
and Barrier method, Sequential quadratic programming method) which are
applied to solve constrained nonlinear programming problem have been in-
vestigated.

The proposed stochastic search method has been given in a great detail. The
stochastic search algorithm is applied on three numerical general nonlinear
programming problems to find the near-optimal solution of those problems. The stochastic search algorithm is programmed in MATLAB for easy execution. The results of these simulation studies and conclusion are mentioned.

**Chapter Five**: In this chapter the results of simulation studies from all numerical examples and further research are discussed.