CHAPTER V

ON THE FINSLER SPACE WITH METRIC

\[ ds = (g_{ij}(y) y^i y^j)^{1/2} + X_i(x,y) y^i \]

1. INTRODUCTION

Let \( F^n = (M^n, L) \) be an n-dimensional Finsler space, that is an n-dimensional differentiable manifold \( M^n \) equipped with a fundamental function \( L(x, y) \). In 1974 Matsumoto ([5]) introduced the transformation of the Finsler metric:

\[(1.1) \quad L^*(x, y) = L(x, y) + b_i(x) y^i\]

and obtained the relation between the Cartan's connection coefficients of \( (M^n, L) \) and \( (M^n, L^*) \). In 1985, [9] he also developed systematically the theory of induced Finsler connections and dealt, in particular, with the four Finsler connections namely Cartan connection, Rund Connection, Berwald connection and Hashiguchi connection. It has been assumed that the functions \( b_i(x) \) in (1.1) are functions of coordinates only. If \( L(x, y) \) is a metric function of Riemannian space then \( L^*(x, y) \) reduces to the metric function of Randers space. Such a Finsler metric was first introduced by G. Randers ([7]) from the viewpoint of general theory of relativity and applied to the theory of electron microscope by
R.S. Ingarden ([1]), who first named it a Randers space. In the papers ([4]), ([5]) and ([8]) this space has been studied from a geometrical view point. In 1978, Numata ([6]) has obtained the torsion tensors $R_{hjk}$ and $P_{hjk}$ of $(M^n,L^*)$ which is obtained from Minkowskian space $(M^n,L)$ by the transformation (1.1). In all these works the functions $b_i$ are assumed to be a function of coordinates only.

Izumi [2], while studying the conformal transformation of Finsler spaces, introduced the $h$-vector $X_i$ which is $v$-covariantly constant with respect to Cartan's connection $C\Gamma$ and satisfies the relation $L^h\xi_{ij} X_h = \rho h_{ij}$. Thus the $h$-vector $X_i$ is not only a function of coordinates but is also a function of directional arguments satisfying $L^h \dot{\xi}_j X_i = \rho h_{ij}$. In the second section of this chapter we shall find out the relation between Cartan's connection $C\Gamma$ of $(M^n, L)$ and $(M^n, L^*)$ where $L^* (x,y)$ is obtained from $L(x,y)$ by the transformation:

\begin{equation}
L^* (x,y) = L(x,y) + X_i (x,y) y^i
\end{equation}

under the assumption that $X_i (x,y)$ is an $h$-vector in $(M^n, L)$. 

In the third section of this chapter, we shall introduce a Finsler space with a metric $ds = \mu + \beta$, where $\mu = \sqrt{g_{ij}(dx)^i dx^j}$ is a Minkowskian metric and $\beta = X_i(x,y) y^j$, where $X_i$ is an h-vector in $(M^n, L)$. We shall find out the torsion tensor $R^*_{hijk}$ of $F^*_n$ and consider the case that this space is of scalar curvature. The fourth section is devoted to find the torsion tensor $P^*_{hijk}$ and to consider the case that this space is a Landsberg space.

2. **CARTAN’S CONNECTION OF THE SPACE $F^*_n$**

Let $X_i$ be a vector field in the Finsler space $(M^n, L)$. If $X_i$ satisfies the conditions

\[(2.1) \quad (i) \ X_{ij} = 0, \quad (ii) \ L C^h_{\ j} = \rho \ h_{ij} \]

then the vector field $X_i$ is called an h-vector ([2]). Here in the above equation (2.1) $\h{j}$ denotes the v-covariant derivative with respect to Cartan’s connection $C\Gamma$, $C^h_{\ ij}$ is the Cartan’s C-tensor, $h_{ij}$ is the the angular metric tensor and $\rho$ is a function given by
\[
(2.2) \quad \rho = \{1/(n-1)\} L C^i X_i, \quad C^i = C^i_{jk} g^{jk}
\]

from (2.1) we get

\[
(2.3) \quad \dot{\hat{X}}^i = L^{-1} \rho h_{ij}
\]

If we denote \(X_i, y^i\) as \(\beta\) then indicatory property

of \(h_{ij}\) yields \(\hat{\beta} = X_j\)

Thus differentiation of (1.2) with respect to \(y^i\)

gives

\[
(2.4) \quad L^*_i = L_i + X_i
\]

Throughout the chapter, we shall use the notations

\[
L_i = \hat{\partial}_i L, \quad L^*_{ij} = \hat{\partial}_j \hat{\partial}_i L \text{ etc.}
\]

The quantities and operations referring to \(F^*\) are marked with asterisks. Thus from (1.2) we get

\[
(2.4) \quad (a) \quad l_i^* = l_i + X_i
\]

where \(l_i\) is the normalized element of support. Again

from (2.3) and (2.4) we get
\[(2.5) \quad L_{ij}^* = (1+p) L_{ij}\]

If \(g_{ij} = \frac{1}{2} \partial_j \partial_i\), \(L^2\) denotes the metric tensor of \(F^a\) then the angular metric tensor \(h_{ij}\) of \(F^a\) is given by

\[(2.6) \quad h_{ij} = g_{ij} - l_i l_j = L^{-1} L_{ij}\]

Thus \((2.5)\) may be rewritten as

\[(2.5)(a) \quad h_{ij}^* = \tau(1+p) h_{ij},\]

where \(\tau = L^{-1} L^*\).

By virtue of \((2.6)\), \((2.5)\) (a) and \((2.4)(a)\) the relation between fundamental tensors is given by

\[(2.7) \quad g_{ij}^* = \tau(1+p) g_{ij} + \{1-\tau (1+p)\} l_i l_j + (l_i X_j + l_j X_i) + X_\lambda X_j\]

from \((2.7)\) the relation between contravariant components of the fundamental tensors will be derived as follows:

\[(2.8) \quad g^{*ij} = [\tau(1+p)]^{-1} g^{ij} - \tau^3 (1+p)^{-1} \{1-\tau (1+p)\} l^i l^j - \tau^2 (1+p)^{-1} (l^i x^j + l^j x^i),\]

Where \(X\) is the magnitude of the vector \(X^i = g^{ij} x_j\).

By virtue of lemma \((2.1)\) of chapter IV and \((2.5)\) it follows that all the successive derivatives of \(L_{ij}^*\) with respect to \(y^k\) are
proportional to the corresponding successive derivatives of \( L_{ij} \) with factor of proportionality \((1+\rho)\), i.e.

\[
(2.9) \quad \begin{align*}
\text{(a) } L^*_{ijk} &= (1+\rho) L_{ijk}, \\
\text{(b) } L^*_{ijkh} &= (1+\rho) L_{ijkh} \text{ and so on}.
\end{align*}
\]

From the equations

\[
(2.10) \quad L_{ijk} = 2L^{-1}C_{ijk} - L^{-2}(h_{ik} l_k + h_{jk} l_i + h_{ki} l_j), \quad C_{ijk} = 1/2 \dot{\delta}_{kl} g_{ij},
\]

\[(2.4)(a), \ (2.5)(a) \text{ and } (2.9)(a), \text{ we obtain the following relation between } C_{ijk} \text{ and } C^*_{ijk}:
\]

\[
(2.11) \quad C^*_{ijk} = \tau (1+\rho) C_{ijk} + \{(1+\rho) / 2L\} (h_{ij} u_k + h_{jk} u_i + h_{ki} u_j),
\]

where we put

\[
(2.12) \quad u_i = X_i - \beta L^{-1} l_i.
\]

From (2.1), (2.8) and (2.11) we get

\[
(2.13) \quad C^*_{ij} = C^h_{ij} + (2L^*)^{-1} (h_{ij} u^h + h_{ij} u_i + h^h_{ij} u_j) - L^*^{-1} \{ \rho + (2L^*)^{-1} L(X^2 - \beta^2 L^{-2}) \} h_{ij} + LL^*^{-1} u_i u_j \}
\]

Now we shall be concerned with Cartan's connection of \( F^0 \) and \( F^* \). This connection is denoted by \( \mathcal{C} = (F^i_{jk}, N^i_k, C^i_{jk}) \).
Here \( N'_{ik} = N_{0k} = \gamma^j F_{jk}^i \) and \( C^{h}_{ij} = g^{hk} C_{ikj} \).

Since for a Cartan's connection
\[
0 = D_{ijk} = \partial_k L_{ij} - L_{ijr} N^r_{ik} - L_{rj} F^r_{ik} - L_{ir} F^r_{jk},
\]

We obtain
\[
(2.14) \quad \partial_k L_{ij} = L_{ijr} N^r_{ik} + L_{rj} F^r_{ik} + L_{ir} F^r_{jk},
\]

Differentiation of (2.5) leads to
\[
(2.15) \quad \partial_k L^*_{ij} = (1 + \rho) \partial_k L_{ij} + \rho_k L_{ij},
\]

where \( \rho_k = \partial_k \rho \). If we put
\[
(2.16) \quad D^i_{jk} = F^*_{jk} - F^i_{jk}
\]

then the difference \( D^i_{jk} \) is obviously a tensor of (1,2) type. In view of (2.14) the equation (2.15) is written in the tensorial form

\[
(2.17) \quad (1 + \rho) (L_{ijr} D^r_{0k} + L_{rj} D^r_{ik} + L_{ir} D^r_{jk}) = \rho_k L_{ij}.
\]

In order to find the difference \( D^i_{jk} \) we construct supplementary equations to (2.17). From (2.4) we obtain
\[
(2.18) \quad \partial_j L^*_{i} = \partial_j L_i + \partial_j X_i
\]

From \( L_{ij} = 0 \), the equation (2.18) is written in the form
\[ L^*_{ir} N^r_{ij} + L^*_{r} F^r_{ij} = (1+\rho) L_{ir} N^r_{ij} + (L_r + X_r) F^r_{ij} + X_{ij} \]

By means of (2.4), (2.5) and (2.16) this equation may be written in the tensorial form

(2.18)(a) \[ (1+\rho) L_{ir} D^r_{0j} + (L_r + X_r) D^r_{ij} = X_{ij} \]

To find the difference tensor \( D^r_{jk} \) we have the following ([4]):

**LEMMA (2.1)** The system of algebraic equations

(i) \( L^r_{ir} A^r = B_i \), (ii) \( (L_r + X_r) A^r = B \)

has a unique solution \( A^r \) for given \( B \) and \( B_i \) such that \( B_i l^i = 0 \). The solution is given by \( A^i = LB^i + \tau^{-1} (B - L B_\beta) l^i \), where the subscript \( \beta \) denotes the contraction by \( X^i \).

Now we establish the following:

**THEOREM (2.1):** the Cartan's connection of \( F^n \) is completely determined by the equations (2.17) and (2.18) (a) in terms of the one of \( F^n \).

**PROOF:** It is obvious that (2.18) (a) is equivalent to the two equations

(2.19) \[ (1+\rho) (L_{ir} D^r_{0j} + L_{ji} D^r_{0i}) + 2 (L_r + X_r) D^r_{ij} = 2 E_{ij} \]
\[(2.20) \quad (1+p) \left( L_{ir} D'_{ij} - L_{jr} D'_{0i} \right) = 2F_{ij} \]

where we put

\[(2.21) \quad 2 E_{ij} + X_{ij} + X_{ijk}, \quad 2 F_{ij} = X_{ij} - X_{ji} \]

On the other hand (2.17) is equivalent to

\[(2.22) \quad 2 (1+p) L_{jr} D'_{ik} + (1+p) (L_{ijr} D'_{0k} + L_{jkr} D'_{0i} - L_{kir} D'_{ij}) \]

\[= \rho_k L_{ij} + \rho_i L_{jk} - \rho_j L_{ki} \]

Contracting (2.19) with \(y^i\), we get

\[(2.23) \quad (1+p) L_{ir} D'_{0i} = 2 (I_i + X_i) D'_{0i} = 2E_{i0} \]

Similarly from (2.20) and (2.22) we obtain

\[(2.24) \quad (1+p) L_{ir} D'_{00} = 2F_{i0} \]

\[(2.25) \quad (1+p) (L_{ir} D'_{0j} + L_{jr} D'_{0i} + L_{ijr} D'_{00}) = \rho_0 L_{ij} \]

Contraction of (2.23) with \(y^i\) gives

\[(2.26) \quad (I_i + X_i) D'_{00} = E_{00} \]

Now first consider (2.24) and (2.26) and apply lemma (2.1) to obtain

\[(2.27) \quad D'_{00} = (1+p)^{-1} \left[ 2 L F'_0 + \tau^{-1} (E_{00} - 2L (1+p)^{-1} F_{j0}) I^i \right] \]

where we put \(F'_0 = g^{ij} F_{j0} \)

Secondly we add (2.20) and (2.25) to obtain
(2.28) \[ L_{ij} D'_{ij} = G_{ij} \]

where we put

(2.29) \[ G_{ij} = (2 (1+\rho))^{-1} (2 F_{ij} + \rho_0 L_{ij} - (1+\rho) L_{ijr} D'_{ij} \]

The equation (2.23) is written in the form

(2.23)(a) \[ (I_r + X_r) D'_{ij} = G_j \]

where we put

(2.30) \[ G_j = E_{j0} - 2^{-1} (1+\rho) L_{jr} D'_{00} \]

Substitution from (2.27) in (2.29) yields

(2.29)(a) \[ G_j = (1+\rho)^{-1} \left[ F_{ij} - LL_{ijr} F'_{00} + L_{ij} \{ (1+\rho) E_{00} \right. - 2LF_{\rho 0} + L^* \rho_0 \} (2L^*)^{-1} \]

By virtue of (2.24), \( G_j \) are written as

(2.30)(a) \[ G_j = E_{j0} - F_{j0} \]

Thus we have obtained the system of equations (2.28) and (2.23)(a). Applying Lemma (2.1) to these equations we obtain

(2.31) \[ D'_{ij} = LG'_{ij} + \tau^{-1} (G_j - LG_{j0}) I' \]

where we put \( G'_{ij} = g^{ir} G_{ij} \)
Finally from (2.19) and (2.22), we obtain

\[ L_i \cdot D_{jk}^r = H_{ijk}, \quad (l_r + X_r) \cdot D_{jk}^r = H_{jk} \]

where we put

\[ H_{ijk} = \{2(1+p)\}^{-1} (\rho_k L_{ij} + \rho_j L_{ik} - \rho_i L_{kj}) \]

\[ \cdot \frac{1}{2} \left[ L_{ijr} D_{0kr}^r + L_{ikr} D_{0jr}^r - L_{kjr} D_{0ir}^r \right], \]

\[ H_{jk} = E_{jk} - \{(1+p)/2 \} \left( L_{jr} D_{0kr}^r + L_{kr} D_{0jr}^r \right) \]

Now applying Lemma (2.1) to (A), we get

\[ D_{jk}^i = L H_{jk}^i + \tau^{-1} (H_{jk} - LH_{ijk}) L^i \]

where we put \( H_{jk}^i = g^{hi} H_{hjk} \). By virtue of

\[ (2.31), H_{ijk} \text{ and } H_{jk} \text{ are written in terms of known quantities:} \]

\[ H_{ijk} = \left(1/2\right) L \left[ L_{kjr} G_i^r - L_{ijr} G_k^r - L_{ikr} G_j^r \right] \]

\[ + L_{ij} A_k + L_{ik} A_j - L_{jk} A_i, \]

\[ H_{jk} = E_{jk} - (1+p) L \cdot 2^{-1} (L_{jr} G_k^i + L_{kr} G_j^i) \]

where

\[ A_i = \{2(1+p)\}^{-1} \rho_i + (2\tau)^{-1} (G_i - L G_{Bi}) \]

3. THE (v) h - TORSION TENSOR \( R_{hijk}^* \) OF \( F^n \)

Let \( F^n \) be a locally Minkowskian space, where

fundamental function is expressed by \( L(y) = \{g_{ij}(y) y^i y^j\}^{1/2} \) in terms
of an adaptable coordinate system \((x^i)\). For a given \(h\)-vector \(X_i\) in \(F^n\), we obtain another Finsler space \(F^*^n\) with the fundamental function

\[
L^*(x, y) = L(y) + \beta(x, y),
\]

where \(\beta(x, y) = X_i (x,y) y^i\). With reference to the adaptable coordinate system \((x^i)\) the connection parameters \((F_{jk}^i, N_{jk}^i, C_{jk}^i)\) of the Cartan connection of \(F^n\) are given by

\[
F_{jk}^i = 0, \quad N_{jk}^i = F_{ij}^i = 0, \quad C_{jk}^i = g^{ir} C_{rjk} = (1/2) g^{ir} \partial_k g_{ij}.
\]

Thus the \(h\)-covariant differentiation \(X_{ij}\) of a covariant vector field \(X_i\) may be written as \(\partial_j X_i\) and the \(v\)-covariant differentiation of \(X_i\) as \(X_{ij} = \partial_j X_i - X_i C_{ij}^r\). In view of (2.16), (2.13) and (3.2) the connection parameter \(N^*_{ij}\) of \(F^*^n\) may be written as

\[
N^*_{ij} = LG_{ij}^i + \tau^{-1} (G_j - LG_{ij}) l^i.
\]

In view of (2.10) and (2.29) (a) the value of \(G_{ij}\) may be written as

\[
G_{ij} = (1+p)^{-1} [A_{ij} + L^{-1} (F_{jol} + F_{iol}) + G_{ij}]
\]

where

\[
G = (2LL^*)^{-1} \{(1+p) E_{00} - 2L F_{00} + L^* \rho_0\}
\]

and
\[ \text{(3.6)} \quad A_{ij} = F_{ij} - 2C_{ij} F'_{0} \]

The (v) h-torsion tensor \( R_{hjk}^{*} \) of \((M^{*}, L^{*})\) is defined by

\[ \text{(3.7)} \quad R_{hjk}^{*} = g_{hi}^{*} R_{jk}^{*} = h_{hi}^{*} R_{jk}^{*} \]

\[ = (\zeta_{(i,k)}^{(h)} \{ h_{hi}^{*} (\partial_{k} N_{j}^{*} - N_{k}^{*} \partial_{r} N_{j}^{*}) \}) \]

In view of (2.5) (a) and (2.6), we have

\[ \text{(3.8)} \quad R_{hjk}^{*} = (\zeta_{(i,k)}^{(h)} \{ (1+p) L^{*} L_{hi}^{*} (\partial_{k} N_{j}^{*}) \}

\[ - N_{k}^{*} \partial_{r} N_{j}^{*} \}) \]

By virtue of (3.2) and (2.16) the equation (2.28) may be written as \( L_{hi} N_{j}^{*} = G_{hj} \) from which we get

\[ L_{hi} \partial_{k} N_{j}^{*} = G_{hjk} \]

and

\[ (\zeta_{(i,k)}^{(h)} \{ L_{hi} N_{j}^{*} \partial_{r} N_{j}^{*} \} = (\zeta_{(i,k)}^{(h)} \{ L \dot{G}_{k}^{*} \partial_{r} G_{hj} \}) \]

Thus (3.8) may be written as

\[ \text{(3.9)} \quad R_{hjk}^{*} = [(1+p) (\zeta_{(i,k)}^{(h)} \{ L^{*}(G_{hjk} + L \dot{G}_{k}^{*} \partial_{r} G_{hj}) \})] \]

By virtue of (3.4) we have

\[ \text{(3.10)} \quad G_{hjk} = (1+p)^{-1} [A_{hjk} + L^{-1} (I_{h} F_{i(jk) + I_{h}} + F_{hok}) + G_{jk} h_{hj}] \]
\[ -(1+p)^2 \rho_k \{ A_{hj} + L^{-1} (l_h F_{j0} + l_j F_{ho}) + \mathcal{G}_{hij} \}, \]

\[ (3.11) \]

\[ \dot{\mathcal{G}}_{hij} = (1+p)^{-1} \left[ -2(F_{uo} \dot{\mathcal{G}}_{hij} + C_{hj}^{\mu} F_{ur} \right. \]

\[ + \dot{\mathcal{G}}_{hj} + \{ G - \rho_0 (2L)^{-1} \} \{ 2C_{hjr} - L^{-1} (l_h h_{jr} + l_j h_{hr}) \} \]

\[ + L^{-2} \{ (h_{hr} - l_h l_r) F_{j0} + (h_{jr} - l_j l_r) F_{ho} \} \]

\[ + L^{-1} \{ l_h F_{jr} + l_j F_{hr} + 2^{-1} (\rho_j h_{hr} - \rho_h h_{jr}) \} \] .

From (3.4) and (3.11) we get

\[ (3.12) \]

\[ (1+p)^2 \mathcal{L}_{(g,k)} \{ G_{k}^{\prime} \dot{\mathcal{G}}_{hij} \}

\[ = \mathcal{L}_{(g,k)} \{ -[A_{j}^{\prime} \dot{\mathcal{G}}_{hij} \}

\[ + G (F_{j0} \dot{F}_{ho} + C_{hj}^{\mu} F_{sr} + (2L)^{-1} \rho_0 C_{hkr}) + 2GF_{s0}

\[ l_j + 2L \rho_j) h_{jk} + 2A_{j}^{\prime} \rho_0 \dot{F}_{hj} + C_{hj}^{\mu} F_{sr} + (2L)^{-1} \rho_0 C_{hkr} + 2GF_{s0}

\[ (\dot{\mathcal{G}}_{j} C_{hj}^{\mu} + 2C_{j}^{\mu} C_{hj}^{\nu})

\[ - L^{-2} (A_{hj} F_{k0} - F_{h0} F_{jk} - F_{0} F_{jk} l_h l_k)

\[ - L^{-1} [A_{j}^{\prime} F_{hr} l_k + 2F_{0} (F_{s0} \dot{A}_{j} C_{hj}^{\mu} + C_{hj}^{\mu} F_{sr} l_k + 2F_{0} C_{hj}^{\nu} F_{sk} l_k)]

\[ - L^{-2} \rho_0 C_{hj} F_{0} l_k + 2^{-1} L^{-2} \rho_0 (l_h A_{jk} + l_j A_{hk} + L^{-1} l_j l_r A_{k0})

\[ + 2^{-1} L^{-1} (\rho_j A_{hk} - \rho_h A_{jk}) + 2^{-1} L^{-2} \rho_j (l_h A_{k0} + l_k F_{h0}) \} .

Substituting from (3.10) and (3.12) in (3.9),

we obtain

**THEOREM (3.1)**: The (v)h-torsion tensor \( R_{hjk}^{\ast} \) of the space \( F^{n} \) is written in the form
\begin{equation}
(3.13) \quad R^*_{hkj} = (1+\rho)^{-1} \left( G^*_{ik} \right) \{ L^* \ L \ G^*_{ij} h_{ik} + L^2 \ K^*_{ijk} + L (l_j K_{ik}) \}
+ L j K_{hk}\}
\end{equation}

where
\begin{align*}
G^*_{ij} &= A^*_{ij} \dot{\sigma} \ G + G \dot{\sigma} \ G - L^{-1} \{ G_{ij} (1+\rho) - (F_0 \dot{\sigma} \ G + G^2) l_j \}
- L^{-2} G F_{ij} + 2^{-1} L^{-2} (L \rho_j - \rho_0 l_j),
K^*_{ijk} &= \tau \left[ L^{-1} (1+\rho) A^*_{ijk} - 2 A^*_{ij} (F_{0j} \dot{\sigma} \ C^*_{hk} + F_{sr}) \right.
- 2 G F_{so} \left( \dot{\sigma} \ C^*_{hk} + 2 C^*_{jr} C^*_{hk} \right) + L^{-2} (A_{kj} F_{ko} - F_{he} F_{jk})
+ (2 L^{-1} (A_{hk} + \rho_j A_{jk}))
- \tau (A^*_{ik} F_{ji} + 2 G C^*_{jk} F_{so} + L^{-1} (2 F_{jo} F_{ko})
+ \rho_k A_{jk} + \rho_0 C_{jk} F^*_{ij}) + (2 L^{-1} \rho_j F_{ko}) \}
\end{align*}

Now we shall be concerned with the contracted tensor \( R^*_{ijk} \) of \( R^*_{ikj} \). The space \( F^n \) is of scalar curvature \( R^* \) if the equation \( R^*_{ijk} = R^* \ L^{-2} h^*_{ij} \) holds good ([3]). If the scalar \( R^* = \) constant, then \( F^n \) is said to be of constant curvature.

From (3.13) the contracted (v) h-torion tensor \( R^*_{ij} \) of \( F^n \) is given by
\begin{equation}
(3.14) \quad R^*_{ij} = (1+\rho)^{-1} \{ L^* \ L \ G^*_{ij} h_{ij} + L^2 \ W_{ij} - L (l_i W_{jo}) \}
+ L j W_{jo} + W_{00} l_i l_j \}
\end{equation}

where we put
\[ W_{ij} = K_{ii} - K_{ij} + K_{ij} \]

and \( W_{ij} \) is symmetric in the indices \( i \) and \( j \). It is to be noted that \( R^*_{i0j} = R^* L^* L^* h_{ij} \) is rewritten as \( R^*_{i0j} = \tau (1+\rho) R^* L^* L^* h_{ij} \).

Therefore we obtain easily from (3.14) the following:

**THEOREM (3.2)**: Let \( F^n \) be a Finsler space with a metric

\[ L^* = L + \beta \]

Where \( L = \{g_{ij}(y) y^i y^j\}^{1/2} \), \( \beta = X_t(x,y) y^i \) and \( X_t \)

is an \( h \)-vector in \( F^n \). If \( F^n \) is of scalar curvature \( R^* \) then the matrix \( \lambda = h_{ij} - W_{ij} \) is of rank less than three, where we put

\[ \lambda = \tau \{(1+\rho)^2 \tau^2 R^* - G_0'\} \]

Now we consider the case \( F_{ij} = 0 \). In this case

\[ A_{ij} = 0, K_{ijk} = 0, K_{ij} = 0 \] and \( W_{ij} = 0 \) hold good. Therefore the tensor

\[ R^*_{i0j} = \{1/(1+\rho)\} L^* L^* G_0' h_{ij} \]. Consequently, we obtain

**THEOREM (3.3)**: Let \( F^n \) be an above mentioned Finsler space if the condition \( F_{ij} = 0 \) is satisfied, then \( F^n \) is of scalar curvature

\[ R^* = \{(1+\rho)\tau\}^2 G_0'. \]

Though the concept of a Finsler space of scalar curvature was introduced by L. Berwald in 1947, we have no
concrete example of non-Minkowski space of scalar curvature.

Moreover we can show the following:

**THEOREM (3.4):** In Theorem (3.3) if the scalar curvature $R^*$ is constant, then $R^* = 0$ and the space $F_0^*$ is a locally Minkowskian space.

**PROOF:** From (2.3) and $F_y = 0$, we get

\[(3.15) \quad 2 \dot{\delta}_F \cdot F_y = L^{-1} (\rho_j h_{ij} - \rho_i h_{jj}) = 0.\]

which after contraction with $y^j$ gives $\rho_0 = 0$. Thus contracting (3.15) with $g^{ij}$ we get $\rho_j = 0$. Therefore the scalar $R^*$ is written in the form

\[(3.16) \quad R^* = \{ (1+p) \tau \}^2 \{ G^2 - L^{-1} (1+p) G_0 \} .\]

It follows from (3.16) and $G = \{(1+p) / 2L^* \} E_{00}$ that the condition $R^* = C (\text{= Constant})$ is written in the form

\[(3.17) \quad [2 \beta E_{0000} - 3 E^2 \cdot_{00} + 4 (L^4 + 6L^2 \beta^2 + \beta^4) C] + 2L[E_{000} + 8 \beta (L^2 + \beta^2) C] = 0.\]

The term in the first (resp. second) bracket of the left hand side of (3.17) is a polynomial of the fourth (resp. third) order with respect to $y^i$. Therefore (3.17) is equivalent to
(3.18) \[ 2 \beta E_{000} - 3E_{00}^2 + 4 (L^4 + 6L^2 \beta^2 + \beta^4) C = 0, \]

(3.19) \[ E_{000} + 8 (L^2 - \beta^2) C = 0. \]

From (3.18) and (3.19) we obtain

(3.20) \[ 3E_{00}^2 = 4C (L^2 - \beta^2) (L^2 + 3\beta^2). \]

If C doesn’t vanish then in view of \( F_{ij} = 0 \) and \( \beta^2 = E_{00} \), the h-covariant differentiation of (3.20) yields

(3.21) \[ 3E_{00} = 8\beta C (L^2 - 3\beta^2). \]

Elimination of \( E_{000} \) from (3.19) and (3.21) gives \( L^2 \beta C = 0 \) from which we get \( \beta = 0 \) as \( L^2 C \neq 0 \). Since \( \partial_j \beta = X_j \), therefore \( \beta = 0 \) implies \( X_i = 0 \), \( E_{ij} = 0 \). Hence (3.20) gives \( C = 0 \) as \( L^2 - \beta^2 \neq 0 \), \( L^2 + 3\beta^2 \neq 0 \). This contradicts our assumption \( C \neq 0 \). Hence the scalar \( R^* = C = 0 \) and from (3.20) we get \( E_{00} = 0 \). Since the assumption \( F_{ij} = 0 \) implies \( \rho_i = 0 \), therefore \( E_{00} = 0 \) implies that \( F_{ij} = 0 \) so that \( X_{ij} = \partial_j X_i = 0 \). Thus \( X_i \) does not contain \( x^i \). Thus \( F^{*n} \) is locally Minkowskian space.
4. **THE (v)hv-TORSION TENSOR P*_{hjk} OF F*^n**

We shall continue to be concerned with the above mentioned Finsler space F*^n. The (v)hv - torsion tensor P*_{hjk} of F*^n is defined as

\[(4.1) \quad P^*_{hjk} = C^*_{hjk0} = y' \partial_r C^*_{hjk} - (\dot{\partial}_r C^*_{hjk}) N^*_{0}
- \sum_{(h,j,k)} \{ C_{hjr} F^{*r}_{k0} \}\]

Where and in the following the symbol \(\sum_{(h,j,k)}\)
denotes the cyclic permutation of h,j,k and summation.

In view of (2.11) and \(P_{hjk} = C_{hjk0} = 0\), we obtain

\[(4.2) \quad y' \partial_r C^*_{hjk} = C^*_{hjk0} = 2(L^* G + F_{r0}) C_{hjk}
+ \sum_{(h,j,k)} \{ (2L)^{-1} (\rho_0 u_k + (1+\rho) (X_{k0} - L^{-1} G_{01}) l_k) h_{ij} \}\]

\[(4.3) \quad \dot{\partial}_r C^*_{hjk} = \tau (1+\rho) \dot{\partial}_r C_{hjk} + L^{-1}(1+\rho)C_{hjk} u_r
+ \sum_{(h,j,k)} \{ (1+\rho)L^{-1} C_{hjr} u_k - (2L^2)^{-1}(1+\rho)h_{jr}(n_{kr})
+(\rho-\beta L^{-1})h_{kr} + (2L^2)^{-1}(1+\rho)h_{hr} n_{kj} \}\]

where we put \(n_j = l_i \ u_j + l_j \ u_i\). Therefore (3.3), (3.4) and (4.3) lead us to

\[(4.4) \quad \dot{\partial}_r C^*_{hjk} N^*_0 = 2 L^* \dot{\partial}_r C_{hjk} F^*_0 - (2L^* G - \tau \rho_0 - 2F_{r0}) C_{hjk}\]
\begin{align*}
+ \mathcal{L}_{(h,i,k)} \{ 2F_{\rho_0} C'_{hj} u_k - L^{-1} F_{\rho_0} n_{jk} - h_{ij} (L^{-1} F_{\rho_0} l_k - L^{-1} (\rho - \beta L^{-1}) F_{\rho_0} + (G - (2L)^{-1} \rho_0) u_k) \}.
\end{align*}

By virtue of (3.3), (3.4) and (2.11) we have

\begin{equation}
\mathcal{L}_{(h,i,k)} \{ C^*_{hj} F^*_{k0} \} = 3L^*G C_{hjk} + \mathcal{L}_{(h,i,k)} \{ L^* C'_{hj} \}
(A'_{i k} + L^{-1} F_{\rho_0} l_k) - 2 C'_{hj} F_{\rho_0} u_k + L^{-1} F_{\rho_0} n_{jk}
+ 2^* l_{ij} (A_{\rho k} + L^{-1} F_{\rho_0} l_k + L^{-2} \beta F_{k0} + 3G u_k) \}.
\end{equation}

Substituting from (4.2), (4.4) and (4.5) in (4.1) we obtain

**THEOREM (4.1)**: The \((v)\) hv – torsion tensor \(P^*_{hijk}\) of the space \(F^n\)

is written in the form

\[ P^*_{hijk} = -2\tau T_{hijk} F_{0} + (L^*G - \tau \rho_0) C_{hjk} \]
\[ + \mathcal{L}_{(h,i,k)} \{ \tau C'_{hj}(F_{\rho_0} l_k + L F_{kr}) + h_{ij} P_k \} \]

where we put

\[ T_{hijk} = L C_{hjk} r + C_{hjk} l_r + \mathcal{L}_{(h,i,k)} \{ C_{ijk} l_h \} \]

\[ 2 P_k = -A_{\rho k} + L^{-1} [(1+\rho) E_{k0} + (\tau-\rho) F_{k0} - (F_{\rho 0} + 2L^*G - \tau \rho_0) l_k] \]
\[ - G u_k. \]

If the condition \(F_{ij} = 0\) is satisfied then the \((v)\) hv – torsion tensor \(P^*_{hijk}\) of \(F^n\) is given by

\begin{equation}
P^*_{hijk} = (L^*G - \tau \rho_0) C_{hjk} + \mathcal{L}_{(h,i,k)} \{ h_{ij} P_k \}
\end{equation}
Where
\[ G = (2L L^*)^{-1} [(1 + \rho) E_{00} + L^* \rho_0] \]
\[ 2P_k = L^{-1} [(1 + \rho) F_{k0} - (2L^* G - \tau \rho_0) l_k] - G u_k \]

Now we shall treat a Landsberg space \( F^n \). Such a space is by definition a Finsler space with the (v) \( h \) - torsion tensor \( P^*_{hjk} = C^*_{hjk;0} = 0 \). On the other hand, a Finsler space \( F^n \) with \( C^*_{hijk} = 0 \) is called a Berwald space (or an affinely connected space).

**THEOREM (4.2)**: Let \( F^n \) \((n \geq 3)\) be a Finsler space with a metric \( L^* = L + \beta \), where \( L = \{g_{ij} (dx) dx^i dx^j\}^{1/2} \), \( \beta = X_i (x, y) y^i \) and \( X_i \) is an \( h \) - vector in \((M^n, L)\). In the case \( F_{ij} = 0 \), if \( F^n \) is a Landsberg space then \( F^n \) is reduced to a Berwald space.

**PROOF**: It is easy to see that the condition \((L^* G - \tau \rho_0) = 0\) means \( E_{00} = 0, i.e., E_{ij} = 0\). From \( E_{ij} = F_{ij} = 0\) we have \( X_i = \text{constant} \), so that \( F^n \) reduces to a locally Minkowskian space. In the case \( L^* G - \tau \rho_0 \neq 0 \), by virtue of (4.6), the equation \( P^*_{hjk} = 0 \) is equivalent to

\[ C_{hjk} = -(L^* G - \tau \rho_0)^1 \mathcal{J}_{(h,j,k)} \{h_{ij} P_k\}, \]
that is, $\tilde{F}$ is $C$ - reducible. By virtue of Theorem 1 of ([10]), the space $F^n$ 
($n \geq 3$) turns out to be a Berwald space. Consequently the proof of
Theorem (4.2) is complete.
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with a metric $ds = \{g_{ij} (dx^i dx^j)\}^{1/2} + b_i (x) dx^i$.

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