Chapter 6

Domination Integrity - A measure of vulnerability


6.1 Introduction

The vulnerability of network have been studied in various contexts including road transportation system, information security, structural engineering and communication network. A graph structure is vulnerable if ‘any small damage produces large consequences’. In a communication network, the vulnerability measures the resistance of the network to disruption of operation after the failure of certain stations (junctions) or communication links (connections). In the theory of graphs, the vulnerability implies a lack of resistance(weakness) of graph network arising from deletion of vertices or edges or both. Communication networks must be so designed that they do not easily get disrupted under external attack and even if they get disturbed then they should be easily reconstructible. Many graph theoretic parameters have been introduced to describe the vulnerability of communication networks including binding number, rate of disruption, toughness, neighbor-connectivity, integrity, mean integrity, edge-connectivity and tenacity. This chapter is aimed to discuss new measure of vulnerability, domination integrity of graphs.

6.2 Integrity of graph network

In the analysis of the vulnerable communication network two quantities are playing vital role, namely (i) the number of elements that are not functioning (ii) the size of the largest remaining (survived) sub network within which mutual communication can still occur. In adverse relationship it is desirable that an opponent’s network would be such that the above referred two quantities can be made simultaneously small. Here the first parameter provides an information about nodes which can be targeted for more disruption while the later gives the impact of damage after disruption. To estimate these quantities Barefoot et al. [15] have introduced the concept of integrity.
Definition 6.2.1. The integrity of a graph $G$ is denoted by $I(G)$ and defined by $I(G) = \min\{|S| + m(G - S) : S \subseteq V(G)\}$ where $m(G - S)$ is the order of a maximum component of $G - S$.

Definition 6.2.2. A subset $S$ of $V(G)$ is said to be an $I$-set if $I(G) = |S| + m(G - S)$.

It is also observed that bigger the integrity of network, more reliable functionality of the network after any disruption caused by non-functional devices (elements). The connectivity is useful to identify local weaknesses in some respect while integrity gives brief account of vulnerability of the graph network.

Illustration 6.2.3. Consider the graph $G = P_{10}$ with vertices $v_1, v_2, \ldots, v_{10}$. Take $S = \{v_3, v_6, v_9\}$ then from Figure 6.1, $m(G - S) = 2$ so $I(P_{10}) = 5$.

\begin{figure}[h]
\centering
\begin{tikzpicture}
  \node[vertex] (v1) at (0,0) {}; \node[vertex] (v2) at (1,0) {}; \node[vertex] (v3) at (2,0) {}; \node[vertex] (v4) at (3,0) {}; \node[vertex] (v5) at (4,0) {}; \node[vertex] (v6) at (5,0) {}; \node[vertex] (v7) at (6,0) {}; \node[vertex] (v8) at (7,0) {}; \node[vertex] (v9) at (8,0) {}; \node[vertex] (v10) at (9,0) {};

  \draw (v1) -- (v2) -- (v3) -- (v4) -- (v5) -- (v6) -- (v7) -- (v8) -- (v9) -- (v10);
\end{tikzpicture}
\caption{$G = P_{10}$ and $G - S$}
\end{figure}

6.2.1 Some known results on integrity

Barefoot et al. [14, 15] have investigated many results on integrity, following theorem give the integrity of various family of graphs.

Theorem 6.2.4. [14, 15]

1. $I(K_n) = n$,
2. $I(K_{n,n}) = 1$,
3. $I(K_{1,n}) = 2$,
4. $I(P_n) = \lceil 2\sqrt{n+1} \rceil - 2$,

5. $I(C_n) = \lceil 2\sqrt{n} \rceil - 1$,

6. $I(K_{m,n}) = 1 + \min\{m,n\}$.

- Integrity of power of cycles have been discussed by Barefoot et al. [14, 15].

- Many results on the integrity of graphs in the context of union, join, composition and product of two graph have been reported by Goddard and Swart [37].

- Some general results on the interrelations between integrity and other graph parameters are investigated by Goddard and Swart [38].

- Many results are reported in a survey article on integrity by Bagga et al. [11].

- Mamut and Vumar [54] have determined the integrity of middle graph of some graphs.

- Dündar and Aytaç [31] discussed the integrity of total graphs via certain graph parameters.

**Proposition 6.2.5.** [31]

\[
\begin{align*}
(i) \quad \gamma(T(P_n)) = \begin{cases} \\
\frac{|V(T(P_n))|}{5} & \text{if } |V(T(P_n))| \equiv 0 \pmod{5} \\
\left\lfloor \frac{|V(T(C_n))|}{5} \right\rfloor + 1 & \text{otherwise}
\end{cases}
\end{align*}
\]

\[
\begin{align*}
(ii) \quad \gamma(T(C_n)) = \begin{cases} \\
\frac{|V(T(C_n))|}{5} & \text{if } |V(T(C_n))| \equiv 0 \pmod{5} \\
\left\lfloor \frac{|V(T(C_n))|}{5} \right\rfloor + 1 & \text{otherwise}
\end{cases}
\end{align*}
\]

6.3 Domination Integrity

**Definition 6.3.1.** A subset $S$ of $V(G)$ is called dominating set if for every $v \in V(G) - S$, there exists $u \in S$ such that $v$ is adjacent to $u$. 
Chapter 6. Domination Integrity - A measure of vulnerability

**Definition 6.3.2.** The minimum cardinality of a minimal dominating set in $G$ is called the domination number of $G$, denoted as $\gamma(G)$ and the corresponding minimal dominating set is called a $\gamma$-set of $G$.

The theory of domination plays vital role in determining decision making bodies of minimum strength or weakness of a network when certain part of it is paralysed. In the case of disruption of a network, the damage will be more when vital nodes are under siege. This motivated the study of domination integrity when the sets of non functioning nodes are dominating sets. The concept of domination integrity of a graph was introduced by Sundareswaran and Swaminathan [70] as a new measure of vulnerability which is defined as follows.

**Definition 6.3.3.** The domination integrity of a connected graph $G$ denoted by $DI(G)$ and defined as $DI(G) = \min \{ |X| + m(G - X) : X \text{ is a dominating set} \}$ where $m(G - X)$ is the order of a maximum connected component of $G - X$.

**Observation 6.3.4.** [70]

1. $1 \leq DI(G) \leq n$, where $G$ is a graph of order $n$.

2. $DI(G) = 1$ if and only if $G = K_1$.

3. $DI(G) = n$ if and only if $G$ is either $K_n$ or $\overline{K}_n$.

4. If $H$ is a subgraph of $G$ then $DI(G) \leq DI(H)$.

5. $I(G) \leq DI(G)$. Moreover difference between $I(G)$ and $DI(G)$ can be made arbitrarily large.

6. For all graph $G$, $DI(G) \geq \chi(G)$, since $I(G) \geq \chi(G)$.

**Illustration 6.3.5.** Consider the graph $G = C_{12}$ with vertices $v_1, v_2, \ldots, v_{12}$. Take $S = \{v_1, v_4, v_7, v_{10}\}$ then $S$ is a dominating set of $G = C_{12}$ and from the Figure 6.2, $m(G - S) = 2$ so $DI(C_{12}) = 4 + 2 = 6$. 
6.3.1 Some known results on domination integrity

Sundareswaran and Swaminathan [70–73] have investigated following results on domination integrity.

- They have investigated domination integrity of some standard graphs.

**Proposition 6.3.6. [70]**

(i) $DI(P_n) = \begin{cases} \left\lceil \frac{n}{2} \right\rceil + 1; & n = 2, 3, 4, 5 \\ \left\lceil \frac{n}{3} \right\rceil + 2; & n \geq 6 \end{cases}$

(ii) $DI(C_n) = \begin{cases} 3; & n = 3, 4 \\ \left\lceil \frac{n}{3} \right\rceil + 2; & n \geq 5 \end{cases}$

(iii) $DI(K_{m,n}) = \min\{m,n\} + 1$

- Investigated domination integrity of Binomial trees and Complete k-ary trees.

- They have investigated domination integrity of middle graph of some standard graphs.

- Investigated the domination integrity of powers of cycles.

- Discussed domination integrity of trees.
Vaidya and Kothari [88, 89] have discussed

- Domination integrity in the context of some graph operations.
- Domination integrity of splitting graph of path $P_n$ and cycle $C_n$.

### 6.4 Domination integrity of shadow graphs of some graphs

In this section we report some results on domination integrity of shadow graphs of some graphs.

**Theorem 6.4.1.** $\text{DI}(D_2(P_n)) = \begin{cases} 
3; & n = 2, 3 \\
5; & n = 4, 5 \\
7; & n = 6, 7 \\
9; & n = 8, 9 \\
11; & n = 10, 11 
\end{cases}$

**Proof.** Consider two copies of $P_n$. Let $v_1, v_2, \ldots, v_n$ be the vertices of the first copy of $P_n$ and $u_1, u_2, \ldots, u_n$ be the vertices of the second copy of $P_n$. Let $G$ be the graph $D_2(P_n)$. Then $|V(G)| = 2n$ and $|E(G)| = 4(n-1)$.

For $n = 2i$ and $n = 2i+1$ where $1 \leq i \leq 5$ consider $S = \{u_{2k}, v_{2k}/1 \leq k \leq i\}$. $S$ is a dominating set of $G$ as $u_{2i-1}, v_{2i-1} \in N(u_{2i})$ and $u_{2i+1}, v_{2i+1} \in N(v_{2i})$ for $i = 1, 2, 3, 4$.

Moreover $|S| = 2i$ and $m(G-S) = 1$.

Now we discuss the minimality of $|S| + m(G-S)$.

There does not exist any dominating set $S_1$ of $G$ such that $|S_1| < |S|$ and $m(G-S_1) = 1$. It can be checked that for any dominating set $S_2$ of $G$ with $m(G-S_2) \geq 2$ then $|S| + m(G-S) \leq |S_2| + m(G-S_2)$.

Therefore, $\min\{|X| + m(G-X) : X \text{ is a dominating set}\} = |S| + m(G-S)$

$= 2i + 1$. 
Hence, \( DI(D_2(P_n)) = \begin{cases} 
3; & n = 2, 3 \\
5; & n = 4, 5 \\
7; & n = 6, 7 \\
9; & n = 8, 9 \\
11; & n = 10, 11 
\end{cases} \)

Theorem 6.4.2. For \( n \geq 12 \),
\[
DI(D_2(P_n)) = \begin{cases} 
\frac{2n}{3} + 4; & \text{if } n \equiv 0 \pmod{3} \\
\frac{2(n-1)}{3} + 5; & \text{if } n \equiv 1 \pmod{3} \\
\frac{2(n-2)}{3} + 6; & \text{if } n \equiv 2 \pmod{3} 
\end{cases}
\]

Proof. Consider two copies of \( P_n \). Let \( v_1, v_2, \ldots, v_n \) be the vertices of the first copy of \( P_n \) and \( u_1, u_2, \ldots, u_n \) be the vertices of the second copy of \( P_n \). Let \( G \) be the graph \( D_2(P_n) \).

- If \( n \equiv 0 \pmod{3} \) (i.e. \( n = 3k \)), consider \( S = \{u_{2+3i}, v_{2+3i} / 0 \leq i \leq k-1\} \) and \( |S| = 2k = \frac{2n}{3} \).
- If \( n \equiv 1 \pmod{3} \) (i.e. \( n = 3k+1 \)), consider \( S = \{u_{2+3i}, v_{2+3i} / 0 \leq i \leq k-1\} \cup \{v_{n-1}\} \) and \( |S| = 2k+1 = \frac{2(n-1)}{3} + 1 \).
- If \( n \equiv 2 \pmod{3} \) (i.e. \( n = 3k+2 \)), consider \( S = \{u_{2+3i}, v_{2+3i} / 0 \leq i \leq k\} \) and \( |S| = 2k+2 = \frac{2(n-2)}{3} + 2 \).

In all the above cases \( S \) is a dominating set for \( G \) as \( u_{1+3t}, u_{3+3t} \in N(u_{2+3t}) \) and \( v_{1+3t}, v_{3+3t} \in N(v_{2+3t}) \) for \( t \in \mathbb{N} \cup \{0\} \) moreover \( m(G - S) = 4 \).

Now we discuss minimality of \( |S| + m(G - S) \).

If we consider any dominating set \( S_1 \) of \( G \) with \( m(G - S_1) > 4 \) then due to construction of \( G = D_2(P_n) \) (i.e. to convert \( G - S_1 \) into disconnected graph the set \( S_1 \) must contain \( u_i \) and \( v_i \) both) and as \( S_1 \) is dominating set, \( |S_1| > |S| \). Hence, \( |S| + m(G - S) < |S_1| + m(G - S_1) \).
Chapter 6. Domination Integrity - A measure of vulnerability

If $S_2$ is any dominating set of $G$ with $m(G - S_2) < 4$ then $|S_2| > |S|$. Hence, $|S| + m(G - S) < |S_2| + m(G - S_2)$.

Therefore,

$$|S| + m(G - S) = \min\{|X| + m(G - X) : X \text{ is a dominating set}\} = DI(D_2(P_n)).$$

Hence, for $n \geq 12$

$$DI(D_2(P_n)) = \begin{cases} \frac{2n}{3} + 4; & \text{if } n \equiv 0 \pmod{3} \\ \frac{2(n-1)}{3} + 5; & \text{if } n \equiv 1 \pmod{3} \\ \frac{2(n-2)}{3} + 6; & \text{if } n \equiv 2 \pmod{3} \end{cases}$$

Corollary 6.4.3. From Proposition 6.3.6, Theorem 6.4.1 and Theorem 6.4.2 we can calculate the difference between domination integrity of $P_n$ and $D_2(P_n)$ as follows:

$$DI(D_2(P_n)) - DI(P_n) = \begin{cases} 0; & n = 3 \\ 1; & n = 2, 5 \\ 2; & n = 4 \\ 3; & n = 6 \\ 4; & n = 8, 9 \\ 5; & n = 10, 11 \\ \left\lceil \frac{n}{3} \right\rceil + 2; & n \geq 12 \& n \equiv 0 \pmod{3} \text{ or } n \equiv 2 \pmod{3} \\ \left\lceil \frac{n}{3} \right\rceil + 1; & n \geq 12 \& n \equiv 1 \pmod{3} \end{cases}$$

Theorem 6.4.4. $DI(D_2(C_n)) = \begin{cases} 5; & n = 3, 4 \\ 7; & n = 5, 6 \\ 9; & n = 7, 8 \\ 10; & n = 9 \\ 11; & n = 10 \end{cases}$
Proof. Consider two copies of $C_n$. Let $v_1, v_2, \ldots, v_n$ be the vertices of the first copy of $C_n$ and $u_1, u_2, \ldots, u_n$ be the vertices of the second copy of $C_n$. Let $G$ be the graph $D_2(C_n)$.

To prove this result we consider following three cases.

Case 1: $n = 3$ to 8

For $n = 2i + 1$ and $n = 2i + 2$ where $i = 1$ to 3, consider $S = \{u_{1+2k}, v_{1+2k} / 0 \leq k \leq i\}$ and $|S| = 2(i + 1)$. $S$ is dominating set for $G$ as $u_{2+2k}, v_{2+2k} \in N(u_{1+2k})$ and $u_n, v_n \in N(u_1)$ with $m(G - S) = 1$. If $S_1$ is any dominating set of $G$ with $m(G - S_1) > 1$ then due to construction of $D_2(C_n)$ and as $S_1$ is dominating set, $|S_1| + m(G - S_1) > |S| + m(G - S)$. Moreover there does not exist any dominating set $S_2$ with $m(G - S_2) = 1$ and $|S_2| < |S|$. Hence, $|S| + m(G - S) = 2i + 2 + 1$

$$= \min \{|X| + m(G - X) : X \text{ is a dominating set}|$$

Therefore, $DI(D_2(C_n)) = \begin{cases} 5; & n = 3, 4 \\ 7; & n = 5, 6 \\ 9; & n = 7, 8 \end{cases}$

Case 2: $n = 9$

Consider $S = \{u_1, u_4, u_7, v_1, v_4, v_7\}$ then $|S| = 6$ and $m(G - S) = 4$. Clearly $S$ is a dominating set of $D_2(C_9)$. Moreover for any other dominating set $S_1$ of $(D_2(C_9))$ we have $|S_1| + m(G - S_1) > |S| + m(G - S) = 10$.

Hence, $DI(D_2(C_9)) = 10$.

Case 3: $n = 10$

Consider $S = \{u_1, u_3, u_5, u_7, u_9, v_1, v_3, v_5, v_7, v_9\}$ then $|S| = 10$ and $m(G - S) = 1$. Clearly $S$ is a dominating set of $D_2(C_{10})$. Moreover for any other dominating set $S_1$ of $D_2(C_{10})$ we have $|S_1| + m(G - S_1) > |S| + m(G - S) = 11$.

Hence, $DI(D_2(C_{10})) = 11$. 


Therefore from above cases, \( DI(D_2(C_n)) = \begin{cases} 5; & n = 3, 4 \\ 7; & n = 5, 6 \\ 9; & n = 7, 8 \\ 10; & n = 9 \\ 11; & n = 10 \end{cases} \)

**Theorem 6.4.5.** For \( n \geq 11 \),

\[
DI(D_2(C_n)) = \begin{cases} \frac{2n}{3} + 4; & \text{if } n \equiv 0 \pmod{3} \\ \frac{2(n-1)}{3} + 6; & \text{if } n \equiv 1 \pmod{3} \\ \frac{2(n-2)}{3} + 6; & \text{if } n \equiv 2 \pmod{3} \end{cases}
\]

**Proof.** Consider two copies of \( C_n \). Let \( v_1, v_2, \ldots, v_n \) be the vertices of the first copy of \( C_n \) and \( u_1, u_2, \ldots, u_n \) be the vertices of the second copy of \( C_n \). Let \( G \) be the graph \( D_2(C_n) \).

- If \( n \equiv 0 \pmod{3} \) (i.e. \( n = 3k \)), consider \( S = \{u_{2+3i}, v_{2+3i}/0 \leq i \leq k-1\} \) and \( |S| = 2k = \frac{2n}{3} \).

- If \( n \equiv 1 \pmod{3} \) (i.e. \( n = 3k + 1 \)) or \( n \equiv 2 \pmod{3} \) (i.e. \( n = 3k + 2 \)) consider \( S = \{u_{2+3i}, v_{2+3i}/0 \leq i \leq k\} \cup \{v_n, u_n\} \). Then \( |S| = 2k + 2 = \frac{2(n-1)}{3} + 2 \) for \( n \equiv 1 \pmod{3} \) and \( |S| = 2k + 2 = \frac{2(n-2)}{3} + 2 \) for \( n \equiv 2 \pmod{3} \).

In all the above cases \( S \) is a dominating set for \( G \) as \( u_{2+3t}, v_{2+3t} \in N(u_{1+3t}) \), \( u_{3+3t}, v_{3+3t} \in N(u_{4+3t}) \) for \( t \in \mathbb{N} \cup \{0\} \) and \( u_n, v_n \in N(u_1) \) moreover \( m(G-S) = 4 \).

Now we discuss minimality of \( |S| + m(G-S) \).

If we consider any dominating set \( S_1 \) of \( G \) with \( m(G-S_1) > 4 \) then due to construction of \( G = D_2(C_n) \) (i.e. to convert \( G-S_1 \) into disconnected graph the set \( S_1 \) must contain \( u_i \) and \( v_i \) both) and as \( S_1 \) is dominating set, \( |S_1| > |S| \). Hence, \( |S| + m(G-S) < |S_1| + m(G-S_1) \).
If $S_2$ is any dominating set of $G$ with $m(G - S_2) < 4$ then $|S_2| > |S|$. Hence, $|S| + m(G - S) < |S_2| + m(G - S_2)$.

Therefore,

$$|S| + m(G - S) = \min \{ |X| + m(G - X) : X \text{ is a dominating set} \}$$

$$= DI(D_2(C_n)).$$

Hence, for $n \geq 11$

$$DI(D_2(C_n)) = \begin{cases} 
\frac{2n}{3} + 4; & \text{if } n \equiv 0 \pmod{3} \\
\frac{2(n-1)}{3} + 6; & \text{if } n \equiv 1 \pmod{3} \\
\frac{2(n-2)}{3} + 6; & \text{if } n \equiv 2 \pmod{3}
\end{cases}$$

Corollary 6.4.6. From Proposition 6.3.6, Theorem 6.4.4 and Theorem 6.4.5 we can calculate the difference between domination integrity of $C_n$ and $D_2(C_n)$ as follows:

$$DI(D_2(C_n)) - DI(C_n) = \begin{cases} 
2; & n = 3, 4 \\
3; & n = 5, 6 \\
4; & n = 7, 8 \\
5; & n = 10 \\
6; & n = 9 \\
\left\lceil \frac{n}{3} \right\rceil + 2; & n \geq 11
\end{cases}$$

Theorem 6.4.7. $DI(D_2(K_{m,n})) = 2m + 1$, where $m \leq n$.

Proof. Consider two copies of complete bipartite graph $K_{m,n}$ where $m \leq n$ with $A = \{v_1, v_2, \ldots, v_m\}$ and $B = \{u_1, u_2, \ldots, u_n\}$ are two partitions of first copy of $K_{m,n}$ where $C = \{v'_1, v'_2, \ldots, v'_m\}$ and $D = \{u'_1, u'_2, \ldots, u'_n\}$ are partitions of second copy of $K_{m,n}$. Let $G$ be the graph $D_2(K_{m,n})$.

Consider $S = \{v_1, v_2, \ldots, v_m, v'_1, v'_2, \ldots, v'_m\}$ then $|S| = 2m$ and $m(G - S) = 1$. $S$ is dominating set of $G$ as $u_i, u'_i \in N(v_1)$ for $1 \leq i \leq n$. Now we discuss the minimality of $|S| + m(G - S)$. 

Chapter 6. Domination Integrity - A measure of vulnerability

For above $S$, $m(G - S) = 1$ which is minimum. If we consider any dominating set $S_1$ of $G$ with $|S_1| = t < 2m = |S|$ then $m(G - S_1) = 2(m + n) - t$.

Therefore $|S_1| + m(G - S_1) = t + 2(m + n) - t$

$$= 2(m + n)$$

$$> 2m + 1$$

$$= |S| + m(G - S).$$

Hence, $|S| + m(G - S) = 2m + 1$

$$= min\{|X| + m(G - X) : X \text{ is a dominating set}\}$$

$$= DI(D_2(K_{m,n}))$$

Therefore $DI(D_2(K_{m,n})) = 2m + 1$, where $m \leq n$. ■

**Theorem 6.4.8.** $DI(B_{n,n}) = 3$.

**Proof.** Consider $B_{n,n}$ with vertex set $\{u, v, u_i, v_i, 1 \leq i \leq n\}$ where $u_i, v_i$ are pendant vertices.

Consider $S = \{u, v\}$ then $|S| = 2$ and $m(G - S) = 1$. Clearly $S$ is a dominating set of $B_{n,n}$ as $u_i \in N(u)$ and $v_i \in N(v)$ for $1 \leq i \leq n$. Moreover if $S_1$ is any dominating set of $B_{n,n}$ other than $S$ then $|S_1| > |S|$.

Hence, $|S| + m(G - S) = 2 + 3 = min\{|X| + m(G - X) : X \text{ is a dominating set}\}$.

Therefore $DI(B_{n,n}) = 3$. ■

**Theorem 6.4.9.** $DI(D_2(B_{n,n})) = 5$.

**Proof.** Consider two copies of $B_{n,n}$. Let $\{u, v, u_i, v_i, 1 \leq i \leq n\}$ and $\{u', v', u'_i, v'_i, 1 \leq i \leq n\}$ be the corresponding vertex sets of each copy of $B_{n,n}$ where $u_i, v_i, u'_i, u'_i$ are pendant vertices. Let $G$ be the graph $D_2(B_{n,n})$. 
Consider $S = \{u, v, u', v'\}$ then $|S| = 4$ and $m(G - S) = 1$. $S$ is dominating set of $G$ as $u_i, u'_i \in N(u)$ and $v_i, v'_i \in N(v)$ for $1 \leq i \leq n$. Now we discuss the minimality of $|S| + m(G - S)$. For above $S$, $m(G - S) = 1$ which is minimum.

We claim that there does not exist any dominating set $S_1$ of $G$ such that $|S_1| + m(G - S_1) < |S| + m(G - S)$. Due to construction of $D_2(B_{n,n})$ if $S_1$ is any dominating set then $S_1 \cap S \neq \emptyset$.

**Case 1:** $S_1 \subset S$

If we consider any dominating set $S_1 \subset S$ then clearly $m(G - S_1) = n > 1$ and $|S_1| + m(G - S_1) > |S| + m(G - S)$

**Case 2:** If $S_1$ is any one of following sets $\{u, v, u_i\}$, $\{u, v', v_i\}$, $\{u', v', u_i\}$, $\{u', v, v_i\}$ for any fixed $i$ then $m(G - S_1) = n > 1$.

Hence, $|S_1| + m(G - S_1) = n + 3 > |S| + m(G - S) = 4 + 1 = 5$.

Therefore $|S| + m(G - S) = \min\{|X| + m(G - X) : X \text{ is a dominating set}\}$

$$= DI(D_2(B_{n,n}))$$

Hence, $DI(D_2(B_{n,n})) = 5$  

---

**6.5 Domination integrity of total graphs of some graphs**

This section is devoted to discuss domination integrity of total graphs of $P_n, C_n$ and $K_{1,n}$.

**Theorem 6.5.1.** $DI(T(P_n)) = n + 1$ for $n = 2$ to $7$.

**Proof.** Let $v_1, v_2, \ldots, v_n$ be the vertices of path $P_n$ and $u_1, u_2, \ldots, u_{n-1}$ be the added vertices corresponding to edges $e_1, e_2, \ldots, e_{n-1}$ to obtain $T(P_n)$. Let $G$ be the graph $T(P_n)$. Then $|V(G)| = 2n - 1$ and $|E(T(G))| = 4n - 5$.

To prove this result we consider following six cases.
Case 1: \( n = 2 \)

Clearly \( T(P_2) \) is \( C_3 \) so \( DI(T(P_2)) = 3 \) from Proposition 6.3.6.

Case 2: \( n = 3 \)

Consider \( S = \{v_2, u_2\} \) then \( S \) is dominating set of \( T(P_3) \) and \( m(G - S) = 2 \), so \( |S| + m(G - S) = 4 \). There does not exist any dominating set \( S_1 \) of \( T(P_3) \) such that \( |S_1| + m(G - S_1) < |S| + m(G - S) \).

Hence, \( DI(T(P_3)) = 4 \).

Case 3: \( n = 4 \)

Consider \( S = \{v_2, u_2, u_4\} \) then \( S \) is dominating set of \( T(P_4) \) and \( m(G - S) = 2 \), so \( |S| + m(G - S) = 5 \). If \( S_1 \) is any dominating set of \( T(P_4) \) with \( |S_1| \leq 2 \) then \( m(G - S_1) = 5 \) so \( |S_1| + m(G - S_1) > 5 \). If we consider any dominating set \( S_2 \) of \( T(P_4) \) such that \( m(G - S_2) = 1 \) then \( |S_2| \geq 4 \) hence, \( |S_2| + m(G - S_2) \geq 5 \).

Therefore, \( DI(T(P_4)) = 5 \).

Case 4: \( n = 5 \)

Consider \( S = \{v_2, u_2, v_4, u_4\} \) then \( S \) is dominating set of \( T(P_5) \) and \( m(G - S) = 2 \), so \( |S| + m(G - S) = 6 \). If \( S_1 \) is any dominating set of \( T(P_5) \) with \( |S_1| = 3 \) then \( m(G - S_1) \geq 4 \) so \( |S_1| + m(G - S_1) > 6 \). If \( S_2 \) is any dominating set of \( T(P_5) \) with \( |S_2| = 2 \) then \( m(G - S_2) = 7 \) so \( |S_2| + m(G - S_2) = 9 > 6 \). If we consider any dominating set \( S_3 \) of \( T(P_5) \) such that \( m(G - S_3) = 1 \) then \( |S_3| \geq 6 \) hence, \( |S_3| + m(G - S_3) \geq 7 \).

Therefore, \( DI(T(P_5)) = 6 \).

Case 5: \( n = 6 \)

Consider \( S = \{v_2, u_2, u_4, v_5\} \) then \( S \) is dominating set of \( T(P_6) \) and \( m(G - S) = 3 \), so \( |S| + m(G - S) = 7 \). If \( S_1 \) is any dominating set of \( T(P_6) \) with \( m(G - S_1) \geq 4 \) then \( |S_1| + m(G - S_1) = 8 > 7 \). If \( S_2 \) is any dominating set of \( T(P_6) \) with \( |S_2| = 2 \) then clearly \( |S_2| + m(G - S_2) > 7 \).
Therefore, \( DI(T(P_6)) = 7. \)

**Case 6: \( n = 7 \)**

Consider \( S = \{v_2, u_2, u_4, v_5, v_7\} \) then \( S \) is dominating set of \( T(P_7) \) and \( m(G - S) = 3 \), so \(|S| + m(G - S) = 8\). If \( S_1 \) is any dominating set of \( T(P_7) \) with \( m(G - S_1) \geq 4 \) then \(|S_1| + m(G - S_1) = 9 > 8\). If \( S_2 \) is any dominating set of \( T(P_7) \) with \(|S_2| = 2\) then clearly \(|S_2| + m(G - S_2) > 8\).

Therefore, \( DI(T(P_7)) = 8. \)

Hence \( DI(T(P_n)) = n + 1 \) for \( n = 2 \) to \( 7. \)

**Theorem 6.5.2.** For \( n \geq 8, \)

\[
DI(T(P_n)) = \begin{cases} 
\frac{2n}{3} + 4; & \text{if } n \equiv 0 \pmod{3} \\
\lceil \frac{2n}{3} \rceil + 4; & \text{if } n \equiv 1 \pmod{3} \\
\lfloor \frac{2n}{3} \rfloor + 4; & \text{if } n \equiv 2 \pmod{3}
\end{cases}
\]

**Proof.** Let \( v_1, v_2, \ldots, v_n \) be the vertices of path \( P_n \) and \( u_1, u_2, \ldots, u_{n-1} \) be the added vertices corresponding to edges \( e_1, e_2, \ldots, e_{n-1} \) to obtain \( T(P_n) \). Let \( G \) be the graph \( T(P_n) \).

**Proposition 6.2.5** gives the value of \( \gamma(T(P_n)) \), here we provide \( D(\gamma - \text{set}) \) for \( T(P_n) \) for different possibilities of \( n \) as below:

- If \( n \equiv 0 \pmod{5} \) (i.e. \( n = 5k \)), consider \( D = \{v_{2+5i}, u_{4+5i} | 0 \leq i < k\} \).

- If \( n \equiv 1 \pmod{5} \) (i.e. \( n = 5k + 1 \)) or \( n \equiv 2 \pmod{5} \) (i.e. \( n = 5k + 2 \)), consider \( D = \{v_{2+5i}, u_{4+5i}, v_n | 0 \leq i < k\} \).

- If \( n \equiv 3 \pmod{5} \) (i.e. \( n = 5k + 3 \)), consider \( D = \{v_{2+5i}, u_{4+5j} | 0 \leq i \leq k, 0 \leq j < k\} \).
• If \( n \equiv 4 \pmod{5} \) (i.e. \( n = 5k + 4 \)), consider \( D = \{v_{2+5i}, u_{4+5j} | 0 \leq i \leq k, 0 \leq j < k \} \).

Hence, \( \gamma(T(P_n)) = \begin{cases} \frac{2n-1}{5}; & \text{if } 2n-1 \equiv 0 \pmod{5} \\ \left\lfloor \frac{2n-1}{5} \right\rfloor + 1; & \text{otherwise} \end{cases} \)

Clearly, \( DI(T(P_n)) \leq |D| + m(G - D) \) .........................................................(i)

Now we define another subset \( S \) of \( V(T(P_n)) \) as below:

• If \( n \equiv 0 \pmod{3} \) (i.e. \( n = 3k \)), consider \( S = \{v_{2+3i}, u_{2+3i} | 0 \leq i < k \} \) and \( |S| = 2k \).

• If \( n \equiv 1 \pmod{3} \) (i.e. \( n = 3k + 1 \)) or \( n \equiv 2 \pmod{3} \) (i.e. \( n = 3k + 2 \)), consider \( S = \{v_{2+3i}, u_{2+3i}, v_n | 0 \leq i < k \} \) and \( |S| = 2k + 1 \).

In all the above cases \( S \) is a dominating set for \( G \) as \( u_{1+3t}, u_{3+3t} \in N(u_{2+3t}) \) and \( v_{1+3t}, v_{3+3t} \in N(v_{2+3t}) \) for \( t \in \mathbb{N} \cup \{0\} \) moreover \( m(G - S) = 4 \).

In order to compare the values of parameters \( |D| + m(G - D) \) and \( |S| + m(G - S) \) and to check the minimality of \( |S| + m(G - S) \), we prepare the Table 6.1 for random values of \( n \) between 8 to 20.

From columns 5 and 8 of Table 6.1, we can observe that for \( D \) (\( \gamma \)-set) and dominating set \( S \),

\( |S| + m(G - S) < |D| + m(G - D) \) .........................................................(ii)

We have verified that the above relation (ii) is valid even for larger values of \( n \).

From (i) and (ii), we have,

\( DI(T(P_n)) \leq |S| + m(G - S) < |D| + m(G - D) \).

Hence, \( DI(T(P_n)) \leq |S| + m(G - S) \) .........................................................(iii)
We claim that $DI(T(P_n)) = |S| + m(G - S)$.

If we consider any dominating set $S_1$ of $G$ such that, $|D| \leq |S_1| < |S|$ then due to construction of $T(P_n)$, it generates large value of $m(G - S_1)$ such that,

$|S| + m(G - S) < |S_1| + m(G - S_1)$.

Let $S_2$ be dominating set of $G$ with minimal cardinality such that $m(G - S_2) = 3$ then,

$|S| + m(G - S) \leq |S_2| + m(G - S_2)$, for $8 \leq n \leq 13$ and

$|S| + m(G - S) < |S_2| + m(G - S_2)$, for $n \geq 14$.

Moreover if $S_3$ is any dominating set of $G$ with $m(G - S_3) = 2$ or $m(G - S_3) = 1$ then clearly,

$|S| + m(G - S) < |S_3| + m(G - S_3)$

From above discussion we can say that among all dominating sets of $G$, $S$ is such that

$|S| + m(G - S)$ is minimum.

Therefore, $|S| + m(G - S) = \min\{|X| + m(G - X)|X$ is a dominating set}$

$= DI(T(P_n))$. 

---

**Table 6.1**

| n  | 2n - 1 | |D| | m(G - D) | |D| + m(G - D) | |S| | m(G - S) | |S| + m(G - S) |
|----|--------|---|---|--------|---------------|---|--------|---------------|
| 8  | 15     | 3 | 12| 15     | 5              | 4 | 9      |               |
| 9  | 17     | 4 | 13| 17     | 6              | 4 | 10     |               |
| 10 | 19     | 4 | 15| 19     | 7              | 4 | 11     |               |
| 11 | 21     | 5 | 16| 21     | 7              | 4 | 11     |               |
| 16 | 31     | 7 | 24| 31     | 11             | 4 | 15     |               |
| 20 | 39     | 8 | 31| 39     | 13             | 4 | 17     |               |
Chapter 6. Domination Integrity - A measure of vulnerability

Hence, for $n \geq 8$

$$DI(T(P_n)) = \begin{cases} 
\frac{2n}{3} + 4; & \text{if } n \equiv 0 \pmod{3} \\
\left\lceil \frac{2n}{3} \right\rceil + 4; & \text{if } n \equiv 1 \pmod{3} \\
\left\lfloor \frac{2n}{3} \right\rfloor + 4; & \text{if } n \equiv 2 \pmod{3} 
\end{cases}$$

\[\blacksquare\]

Corollary 6.5.3. $DI(T(P_n)) - DI(P_n) = \begin{cases} 
1; & n = 2, 3 \\
2; & n = 4, 5 \\
3; & n = 6, 7 
\end{cases}$

**Proof.** In view of Proposition 6.3.6, Theorem 6.5.1 and Theorem 6.5.2 the result is obvious.  \[\blacksquare\]

Theorem 6.5.4. $DI(T(C_n)) = \begin{cases} 
6; & n = 3, 4 \\
7; & n = 5 \\
8; & n = 6 \\
9; & n = 7 \\
10; & n = 8, 9 \\
11; & n = 10 
\end{cases}$

**Proof.** Let $v_1, v_2, \ldots, v_n$ be the vertices of cycle $C_n$ and $u_1, u_2, \ldots, u_n$ be the added vertices corresponding to edges $e_1, e_2, \ldots, e_n$ to obtain $T(C_n)$. Let $G$ be the graph $T(C_n)$. Then $|V(G)| = 2n$ and $|E(T(G))| = 4n$.

To prove this result we consider following two cases.
Chapter 6. Domination Integrity - A measure of vulnerability

Case 1: $n = 3, 4$

For $n = 3$, consider $S = \{v_2, v_3, u_3\}$ then $m(G - S) = 3$. There does not exist any dominating set $S_1$ of $T(C_3)$ such that $|S_1| + m(G - S_1) < |S| + m(G - S)$. Hence, $DI(T(C_3)) = 6$.

For $n = 4$, consider $S = \{v_2, v_4, u_2, u_4\}$ then $m(G - S) = 2$. There does not exist any dominating set $S_1$ of $T(C_4)$ such that $|S_1| + m(G - S_1) < |S| + m(G - S)$. Hence, $DI(T(C_4)) = 6$.

Case 2: $n = 5$ to 10

To explain this case we prepare the following Table 6.2.

| $n$ | $S$ | $|S|$ | $m(G - S)$ | $|S| + m(G - S)$ |
|-----|-----|------|------------|-----------------|
| 5   | $\{v_2, u_2, u_4, v_5\}$ | 4    | 3          | 7               |
| 6   | $\{v_2, u_2, v_5, u_5\}$ | 4    | 4          | 8               |
| 7   | $\{v_2, u_2, u_4, v_5, u_7, v_7\}$ | 6    | 3          | 9               |
| 8   | $\{v_2, u_2, u_4, v_5, u_7, v_7, v_8\}$ | 7    | 3          | 10              |
|     | $\{v_2, u_2, v_5, u_5, v_8, u_8\}$ | 6    | 4          | 10              |
| 9   | $\{v_2, u_2, v_5, u_5, v_8, u_8\}$ | 6    | 4          | 10              |
| 10  | $\{v_2, u_2, u_4, v_5, u_7, v_7, u_9, v_{10}\}$ | 8    | 3          | 11              |

The Table 6.2 gives dominating set $S$ and corresponding values of $m(G - S)$ for $n = 5$ to 10. It can be observed that among all dominating sets of $G$, above given $S$ gives the minimum value of $|S| + m(G - S)$.
Chapter 6. Domination Integrity - A measure of vulnerability

Hence, for \( n = 3 \) to \( 10 \)

\[
\text{DI}(T(C_n)) = \begin{cases} 
6; & n = 3, 4 \\
7; & n = 5 \\
8; & n = 6 \\
9; & n = 7 \\
10; & n = 8, 9 \\
11; & n = 10 
\end{cases}
\]

\[\text{Theorem 6.5.5.}\] For \( n \geq 11 \)

\[
\text{DI}(T(C_n)) = \begin{cases} 
\frac{2n}{3} + 4; & \text{if } n \geq 11 \text{ and } n \equiv 0 \pmod{3} \\
\frac{2(n+1)}{3} + 4; & \text{if } n \geq 11 \text{ and } n \equiv 1 \pmod{3} \\
\frac{2(n+1)}{3} + 4; & \text{if } n \geq 11 \text{ and } n \equiv 2 \pmod{3} 
\end{cases}
\]

**Proof.** Let \( v_1, v_2, \ldots, v_n \) be the vertices of cycle \( C_n \) and \( u_1, u_2, \ldots, u_n \) be the added vertices corresponding to edges \( e_1, e_2, \ldots, e_n \) to obtain \( T(C_n) \). Let \( G \) be the graph \( T(C_n) \).

Then \( |V(G)| = 2n \) and \( |E(T(G))| = 4n \).

Proposition 6.2.5 gives the value of \( \gamma(T(C_n)) \), here we provide \( D \) (\( \gamma \)-set) for \( T(C_n) \) for different possibilities of \( n \) as below:

- If \( n \equiv 0 \pmod{5} \) (i.e. \( n = 5k \)), consider \( D = \{v_{2+5i}, u_{4+5i} | 0 \leq i < k\} \).
- If \( n \equiv 1 \pmod{5} \) (i.e. \( n = 5k + 1 \)) or \( n \equiv 2 \pmod{5} \) (i.e. \( n = 5k + 2 \)), consider 
\[ D = \{v_{2+5i}, u_{4+5i}, v_n | 0 \leq i < k\} \]
- If \( n \equiv 3 \pmod{5} \) (i.e. \( n = 5k + 3 \)), consider 
\[ D = \{v_{2+5i}, u_{4+5j}, v_n | 0 \leq i \leq k, 0 \leq j < k\} \]
- If \( n \equiv 4 \pmod{5} \) (i.e. \( n = 5k + 4 \)), consider 
\[ D = \{v_{2+5i}, u_{4+5j}, v_n | 0 \leq i \leq k, 0 \leq j < k\} \]
Chapter 6. Domination Integrity - A measure of vulnerability

Hence, \( \gamma(T(C_n)) = \begin{cases} 
\frac{2n}{5}; & \text{if } 2n \equiv 0 \pmod{5} \\
\left\lfloor \frac{2n}{5} \right\rfloor + 1; & \text{otherwise}
\end{cases} \)

Clearly, \( DI(T(C_n)) \leq |D| + m(G - D) \) ............................................ (iv)

Now we define another subset \( S \) of \( V(T(C_n)) \) as below:

- If \( n \equiv 0 \pmod{3} \) (i.e. \( n = 3k \)) and \( n \equiv 2 \pmod{3} \) (i.e. \( n = 3k - 1 \)), consider \( S = \{v_{2+3i}, u_{2+3i} | 0 \leq i < k\} \) and \( |S| = 2k \).

- If \( n \equiv 1 \pmod{3} \) (i.e. \( n = 3k + 1 \)), consider \( S = \{v_{2+3i}, u_{2+3i}, v_n | 0 \leq i < k\} \cup \{v_n, u_n\} \) and \( |S| = 2(k + 1) = \frac{2(n+2)}{3} \).

In all the above cases \( S \) is a dominating set for \( G \) as \( u_{1+3t}, u_{3+3t} \in N(u_{2+3t}) \) and \( v_{1+3t}, v_{3+3t} \in N(v_{2+3t}) \) for \( t \in \mathbb{N} \cup \{0\} \) moreover \( m(G - S) = 4 \).

In order to compare the values of parameters \( |D| + m(G - D) \) and \( |S| + m(G - S) \) as well as to check the minimality of \( |S| + m(G - S) \), we prepare the Table 6.3 for random values of \( n \) between 11 to 25.

| n  | 2n | |D| | m(G - D) | |D| + m(G - D) | |S| | m(G - S) | |S| + m(G - S) |
|----|----|----------|----------|----------|----------------|--------|----------|----------------|
| 11 | 22 | 5        | 17       | 22       | 8              | 4      | 12       |
| 12 | 24 | 5        | 19       | 24       | 8              | 4      | 12       |
| 13 | 26 | 6        | 20       | 26       | 10             | 4      | 14       |
| 14 | 28 | 6        | 22       | 28       | 10             | 4      | 14       |
| 16 | 32 | 7        | 25       | 32       | 12             | 4      | 16       |
| 25 | 50 | 10       | 40       | 50       | 18             | 4      | 22       |
From columns 5 and 8 of Table 6.3, we can observe that for $D (\gamma - set)$ and dominating set $S$,

$$|S| + m(G - S) < |D| + m(G - D)$$

...................................................(v)

We have verified that the above relation (v) is valid even for larger values of $n$.

From (iv) and (v), we have,

$$DI(T(C_n)) \leq |S| + m(G - S) < |D| + m(G - D).$$

Hence, $DI(T(C_n)) \leq |S| + m(G - S)$............................................(vi)

We claim that $DI(T(C_n)) = |S| + m(G - S)$.

If we consider any dominating set $S_1$ of $G$ such that, $|D| \leq |S_1| < |S|$ then due to construction of $T(C_n)$, it generates large value of $m(G - S_1)$ such that,

$$|S| + m(G - S) < |S_1| + m(G - S_1).$$

Let $S_2$ be dominating set of $G$ with minimal cardinality such that $m(G - S_2) = 3$ then,

$$|S| + m(G - S) \leq |S_2| + m(G - S_2), \text{ for } n = 13 \text{ and}$$

$$|S| + m(G - S) < |S_2| + m(G - S_2), \text{ for } n = 11, 12 \text{ and } n \geq 14.$$

Moreover if $S_3$ is any dominating set of $G$ with $m(G - S_3) = 2$ or $m(G - S_3) = 1$ then clearly,

$$|S| + m(G - S) < |S_3| + m(G - S_3)$$

From above discussion we can say that among all dominating sets of $G$, $S$ is such that $|S| + m(G - S)$ is minimum.

Therefore,

$$|S| + m(G - S) = \min\{|X| + m(G - X)|X \text{ is a dominating set}\}$$

$$= DI(T(C_n)).$$
Hence, for $n \geq 11$

$$DI(T(C_n)) = \begin{cases} 
\frac{2n}{3} + 4; & \text{if } n \geq 11 \text{ and } n \equiv 0 \pmod{3} \\
\frac{2(n+2)}{3} + 4; & \text{if } n \geq 11 \text{ and } n \equiv 1 \pmod{3} \\
\frac{2(n+1)}{3} + 4; & \text{if } n \geq 11 \text{ and } n \equiv 2 \pmod{3}
\end{cases}$$

Corollary 6.5.6. $DI(T(C_n)) - DI(C_n) = \begin{cases} 
\frac{n}{3} + 2; & n \geq 11 \text{ and } n \equiv 0 \pmod{3} \\
\left\lceil \frac{n}{3} \right\rceil + 2; & n \geq 11 \text{ and } n \equiv 1 \pmod{3} \text{ or } n \equiv 2 \pmod{3}
\end{cases}$

Proof. In view of Proposition 6.3.6, Theorem 6.5.4 and Theorem 6.5.5 the proof is obvious.

Theorem 6.5.7. $DI(T(K_{1,n})) = n + 2$.

Proof. Let $v$ be the apex vertex of $K_{1,n}$ and $v_1, v_2, \ldots, v_n$ be the pendant vertices of $K_{1,n}$ and $u_1, u_2, \ldots, u_n$ be the added vertices corresponding to edges $e_1, e_2, \ldots, e_n$ to obtain $T(K_{1,n})$. Let $G$ be the graph $T(K_{1,n})$.

Consider $S = \{v, u_1, u_2, \ldots, u_n\}$ then $|S| = n + 1$ and $m(G - S) = 1$. Clearly $S$ is a dominating set of $G$ and $|S| + m(G - S) = n + 2$.

For $S_1 = \{v, u_1, u_2, \ldots, u_{n-1}\}$ then $|S_1| = n$ and $m(G - S_1) = 2$ and $|S_1| + m(G - S_1) = n + 2$.

For $S_2 = \{v, u_1, u_2, \ldots, u_{n-2}\}$ then $|S_2| = n - 1$ and $m(G - S_2) = 4$ and $|S_2| + m(G - S_2) = n + 3$.

Similarly for any other dominating set $S_3$ of $G$, $|S| + m(G - S) \leq |S_3| + m(G - S_3)$.
\[ |S| + m(G - S) = \min \{ |X| + m(G - X) | X \text{ is a dominating set} \} = DI(T(K_{1,n})). \]

Hence, \( DI(T(K_{1,n})) = n + 2. \) ■

### 6.6 Domination integrity of some path related graphs

In this section we report some results of domination integrity of path related graphs.

**Theorem 6.6.1.** \( DI(P_n^2) = \begin{cases} 
2; & n = 2 \\
3; & n = 3, 4 \\
4; & n = 5, 6 
\end{cases} \)

**Proof.** Let \( V(P_n) = \{v_1, v_2, \ldots, v_n\} \) and \( P_n^2 \) be the square graph of \( P_n \). Then \( |V(P_n^2)| = n \) and \( |E(P_n^2)| = 2n - 3 \). The proof is divided into following three cases.

**Case 1:** \( n = 2 \)

\( P_2^2 \) is \( P_2 \) itself. Consider \( S = \{v_1\} \) which is a dominating set of \( P_2^2 \) then \( m(G - S) = 1 \). Thus \( |S| + m(G - S) = 2 \). If we choose \( S = \{v_2\} \) then also \( |S| + m(G - S) = 2 \).

Hence, \( DI(P_2^2) = 2 \)

**Case 2:** \( n = 3, 4 \)

For \( n = 3 \) consider \( S = \{v_2, v_3\} \) which is a dominating set for \( P_3^2 \) and \( m(G - S) = 1. \) Therefore \( |S| + m(G - S) = 3 \). For \( S = \{v_1, v_3\} \) also \( |S| + m(G - S) = 3 \). If \( S = \{v_1\} \) or \( S = \{v_2\} \) or \( S = \{v_3\} \) then \( m(G - S) = 2 \) so \( |S| + m(G - S) = 3 \).

Hence, \( DI(P_3^2) = 3 \).

For \( n = 4 \) consider \( S = \{v_2, v_3\} \) which is a dominating set of \( P_4^2 \) and \( m(G - S) = 1 \). Therefore \( |S| + m(G - S) = 3 \). For \( S = \{v_1, v_3\} \) or \( S = \{v_2, v_4\} \) or \( S = \{v_1, v_2\} \) or...
Chapter 6. Domination Integrity - A measure of vulnerability

\[ S = \{v_3, v_4\}, \ m(G - S) = 2 \] so for these choices of \( S \) we get \(|S| + m(G - S) = 4\). If \( S = \{v_i\}, i = 1, 2, 3, 4 \) then \( m(G - S) = 3 \) so \(|S| + m(G - S) = 4\).

Hence, \( DI(P^2_4) = 3 \).

**Case 3: \( n = 5, 6 \)**

\( S = \{v_3, v_4\} \) is a dominating set for \( P^2_5 \) and \( P^2_6 \). Then \( m(G - S) = 2 \) and \(|S| + m(G - S) = 4\). It is easy to observe that there does not exist dominating set \( S \) for which \(|S| + m(G - S) \leq 3\).

Therefore \( DI(P^2_n) = 4 \) for \( n = 5, 6 \).

Hence from above three cases, \( DI(P^2_n) = \begin{cases} 2; & n = 2 \\ 3; & n = 3, 4 \\ 4; & n = 5, 6 \end{cases} \)

**Theorem 6.6.2.** For \( n = 7 \) to 15,

\[
DI(P^2_n) = \begin{cases} 5 + 2i \quad \text{if } n = 7 + 4i \\ \quad \quad \text{where } i = 0, 1, 2 \\ 6 + 2i \quad \text{if } n = 7 + 4i + k, \\ \quad \quad \text{where } k = 1, 2, 3 \text{ and } i = 0, 1 \end{cases}
\]

**Proof.** If \( S = \{v_{3+4j}, v_{4+4j}/j = 0 \text{ to } i\} \cup \{v_n\} \) when \( n = 7 + 4i, i = 0, 1, 2 \) (i.e. \( n = 7, 11, 15 \)) then \(|S| = 2i + 3 \) and \( m(G - S) = 2 \). If \( S = \{v_{3+4j}, v_{4+4j}/j = 0 \text{ to } i + 1\} \) when \( n = 7 + 4i + k, k = 1, 2, 3, i = 0, 1 \) (i.e. \( n = 8, 9, 10, 12, 13, 14 \)) then \(|S| = 2i + 4 \) and \( m(G - S) = 2 \). In both the cases \( S \) is a dominating set of \( P^2_n \) as \( v_1, v_2 \in N(v_3) \) and \( v_{5+4t}, v_{6+4t} \in N(v_{4+4t}) \), for \( t = 0, 1, 2, \ldots, i \) or \( i - 1 \).

Now we claim that there does not exist any dominating set \( S_1 \) such that \(|S_1| = |S|\) and \( m(G - S_1) < m(G - S) \). If \( S_1 \) is a dominating set and \( m(G - S_1) < m(G - S) = 2 \) then all the components will be \( K_1 \), consequently \(|S_1| > |S|\).

We also claim that there does not exist any dominating set \( S_2 \) such that \(|S_2| < |S|\) and \( m(G - S_2) = m(G - S) = 2 \). But if \( S_2 \) is a dominating set and \(|S_2| < |S|\). Then due to
construction of $P_n^2$, $G - S_2$ will give rise to at least one component with the number of vertices more then two because each vertex of $P_n^2$ is adjacent to the vertices which are at the distance two apart. This implies that there does not exist any dominating set $S_2$ such that $|S_2| < |S|$ and consequently $m(G - S_2) = m(G - S) = 2$.

Moreover if we consider any dominating set $S_3$ of $P_n^2$ such that $m(G - S_3) > 2$ then $|S| + m(G - S) \leq |S_3| + m(G - S_3)$.

Therefore,

$$|S| + m(G - S) = \min\{|X| + m(G - X) : X \text{ is a dominating set}\}$$

$$= DI(P_n^2).$$

Hence, for $n = 7$ to $15$ $DI(P_n^2) = \begin{cases} 2i + 3 + 2 = 5 + 2i & \text{if } n = 7 + 4i \text{ where } i = 0, 1, 2 \\ 2i + 4 + 2 = 6 + 2i & \text{if } n = 7 + 4i + k, \text{ where } k = 1, 2, 3 \text{ and } i = 0, 1 \end{cases}$

**Theorem 6.6.3.** $DI(P_n^2) = \begin{cases} 9; & n = 16 \\ 10; & n = 17, 18 \end{cases}$  

**Proof.** To prove this result we consider following two cases.

**Case 1:** $n = 16$

If $S = \{v_3, v_4, v_8, v_9, v_{13}, v_{14}\}$ then $|S| = 6$ and $m(G - S) = 3$. Moreover $S$ is a dominating set as $v_1, v_2 \in N(v_3), v_5, v_6 \in N(v_4), v_7 \in N(v_8), v_{10}, v_{11} \in N(v_9)$ and $v_{12} \in N(v_{13}), v_{15}, v_{16} \in N(v_{14})$. If for some dominating set $S_1$ of $P_{16}^2, m(G - S_1) = 2$ then clearly $|S_1| > |S|$ so $|S_1| + m(G - S_1) > |S| + m(G - S)$. It can be verified that for any other dominating set $S_2$ of $P_{16}^2$ for which $m(G - S_2) = 4$ then $|S_2| + m(G - S_2) \geq |S| + m(G - S)$. Thus among all dominating set, $|S| + m(G - S) = 6 + 3 = 9$ is minimum.

Thus $DI(P_{16}^2) = 9$. 

---

**Chapter 6. Domination Integrity - A measure of vulnerability**

214
Case 2: \( n = 17, 18 \)

If \( S = \{v_3, v_4, v_9, v_{13}, v_{14}, v_{17}\} \) then \(|S| = 7\) and \(m(G - S) = 3\). Moreover \( S \) is dominating set for \( P^2_{17} \) and \( P^2_{18} \). If for some dominating set \( S_1 \) of \( P^2_n \), \( m(G - S_1) = 2 \) then clearly \(|S_1| > |S|\) so \(|S_1| + m(G - S_1) > |S| + m(G - S)|. It can be verified that for any other dominating set \( S_2 \) of \( P^2_n \) for which \( m(G - S_2) = 4 \) then \(|S_2| + m(G - S_2) \geq |S| + m(G - S)|.

Thus in all the cases \(|S| + m(G - S)| = 7 + 3 = 10\) is minimum.

Thus \( DI(P^2_n) = 10 \) for \( n = 17, 18 \).

Hence, from above two cases, \( DI(P^2_n) = \begin{cases} 9; & n = 16 \\ 10; & n = 17, 18 \end{cases} \)

**Theorem 6.6.4.** For \( n \geq 19 \),

\[
DI(P^2_n) = \begin{cases} 11 & \text{if } n = 19, 20 \\ 11 + 2i & \text{if } n = 21 + 6i \text{ where } i = 0, 1, 2, \ldots \\ 12 + 2i & \text{if } n = 21 + 6i + k \\ & \text{where } k = 1, 2, 3 \text{ and } i = 0, 1, 2, \ldots \\ 13 + 2i & \text{if } n = 21 + 6i + k \\ & \text{where } k = 4, 5 \text{ and } i = 0, 1, 2, \ldots \\ \end{cases}
\]

**Proof.** To prove this result we consider following two cases.

**Case 1:** \( n = 19, 20 \)

Consider \( S = \{v_3, v_4, v_9, v_{15}, v_{16}, v_{19}\} \) then \(|S| = 7\) and \(m(G - S) = 4\). Clearly \( S \) is a dominating set of \( P^2_n \), for \( n = 19, 20 \).

**Case 2:** \( n \geq 21 \)

Let \( S_1 = \{v_3, v_4, v_9, v_{10}\} \).

- If \( n = 21 + 6i \text{ where } i = 0, 1, 2, \ldots \) (i.e. for \( n = 21, 27, 33, \ldots \)), consider \( S = S_1 \cup \{v_{15+6j}, v_{16+6j} / j = 0 \text{ to } i\} \cup \{v_n\} \) then \(|S| = 7 + 2i\).
Chapter 6. Domination Integrity - A measure of vulnerability

- If \( n = 21 + 6i + k \) where \( k = 1, 2, 3 \) and \( i = 0, 1, 2, \ldots \) (i.e. for \( n = 22, 23, 24, 28, 29, 30 \ldots \)), consider \( S = S_1 \cup \{v_{15+6j}, v_{16+6j}/j = 0 \text{ to } i + 1\} \) then \(|S| = 8 + 2i\).

- If \( n = 21 + 6i + k \) where \( k = 4, 5 \) and \( i = 0, 1, 2, \ldots \) (i.e. for \( n = 25, 26, 31, 32, \ldots \)), consider \( S = S_1 \cup \{v_{15+6j}, v_{16+6j}/j = 0 \text{ to } i + 1\} \cup \{v_n\} \) then \(|S| = 9 + 2i\).

In all the above cases \( S \) will be a dominating set for \( P_n^2 \) as \( v_1, v_2 \in N(v_3), v_{5+6t}, v_{6+6t} \in N(v_{4+6t}) \) and \( v_{7+6t}, v_{8+6t} \in N(v_{9+6t}) \) where \( t \in \mathbb{N} \cup \{0\} \). Moreover \( m(G - S) = 4 \).

Thus we have found dominating sets for \( P_n^2 \). Now we discuss about minimality of \(|S| + m(G - S)\).

If we consider any dominating set \( S_1 \) of \( G \) such that \(|S_1| < |S|\) then due to construction of \( P_n^2 \) (i.e. to convert \( G - S_1 \) into disconnect graph we must include at least two consecutive vertices in \( S_1 \)), it generates large value of \( m(G - S_1) \) such that,

\[
|S| + m(G - S) < |S_1| + m(G - S_1).
\]

Let \( S_2 \) be any dominating set of \( P_n^2 \) such that \( m(G - S_2) = 3 \) then,

\[
|S| + m(G - S) \leq |S_2| + m(G - S_2), \text{ for } n = 19, 20, 21, 22, 23, 28, 31 \text{ and }
\]

\[
|S| + m(G - S) < |S_2| + m(G - S_2), \text{ for } n = 24, 25, 26, 27, 29, 30, n \geq 32.
\]

Moreover if \( S_3 \) is any dominating set of \( P_n^2 \) with \( m(G - S_3) = 2 \) or \( m(G - S_3) = 1 \) then clearly,

\[
|S| + m(G - S) < |S_3| + m(G - S_3)
\]

Therefore in both the cases we have,

\[
|S| + m(G - S) = \min\{ |X| + m(G - X) : X \text{ is a dominating set} \}
\]

\[
= DI(P_n^2).
\]
Hence, for \( n \geq 19 \),
\[
DI(P_n^2) = \begin{cases} 
7 + 4 = 11 & \text{if } n = 19, 20 \\
7 + 2i + 4 = 11 + 2i & \text{if } n = 21 + 6i \text{ where } i = 0, 1, 2, \ldots \\
8 + 2i + 4 = 12 + 2i & \text{if } n = 21 + 6i + k \text{ where } k = 1, 2, 3 \text{ and } i = 0, 1, 2, \ldots \\
9 + 2i + 4 = 13 + 2i & \text{if } n = 21 + 6i + k \text{ where } k = 4, 5 \text{ and } i = 0, 1, 2, \ldots 
\end{cases}
\]

**Definition 6.6.5.** The composition of two graphs \( G \) and \( H \) is denoted as \( G[H] \) (also known as *lexicographic product*) whose vertex set is \( V(G) \times V(H) \) and two vertices \((u_1, v_1)\) and \((u_2, v_2)\) are adjacent if either \( u_1 \) is adjacent to \( u_2 \) in \( G \) or \( u_1 = u_2 \) and \( v_1 \) is adjacent to \( v_2 \) in \( H \).

It is worth to mention that, unlike the union, join, cartesian product, direct product and strong product of two graphs the composition of two graphs is not commutative.

**Theorem 6.6.6.** \( DI(P_2[P_n]) = n + \lceil 2 \sqrt{n+1} \rceil - 2 \).

**Proof.** Let \( P_2 \) be a path with vertices \( u_1, u_2 \) and \( P_n \) with \( v_1, v_2, \ldots, v_n \). Let \( G \) be the graph \( P_2[P_n] \). Then \( V(G) = \{(u_i, v_j) / 1 \leq i \leq 2, 1 \leq j \leq n \} \) and
\[
E(G) = \{(u_1, v_j)(u_2, v_k) / 1 \leq j \leq n, 1 \leq k \leq n \} \cup \{(u_1, v_j)(u_1, v_{j+1}), (u_2, v_j)(u_2, v_{j+1}) / 1 \leq j \leq n - 1 \}.
\]

For the sake of convenience we denote the vertices \((u_1, v_j) = w_{1j}, 1 \leq j \leq n \) and \((u_2, v_j) = w_{2j}, 1 \leq j \leq n \).

The graph of \( P_2[P_n] \) is shown in Figure 6.3 for the better understanding of the notations and arrangement of vertices.

Moreover, \( K_{n,n} \) is a subgraph of \( G \) and \( DI(K_{n,n}) = n + 1, DI(G) > n + 1 \).

Consider \( S_1 = \{w_{2j} / 1 \leq j \leq n \} \), \(|S_1| = n \). Then \( S_1 \) is a dominating set of \( G \) and \( G - S_1 = P_n \) so \( m(G - S_1) = n \).
Let $S_2 = \{w_{1k} = (u_1, v_k) / v_k \in I\text{-set of } P_n\}$. Take $V_1 = \{v_k / v_k \in I\text{-set of } P_n\}$ so $|S_2| = |V_1|$. Consider $S = S_1 \cup S_2$ then $S$ is also dominating set of $G$ as $S_1 \subset S$. Here $|S| = |S_1| + |S_2| = |S_1| + |V_1|$ and $G - S = P_n - V_1$ so $m(G - S) = m(P_n - V_1)$. Note that $I(P_n) = \lceil 2 \sqrt{n + 1} \rceil - 2$.

So, $|S| + m(G - S) = |S_1| + |V_1| + m(P_n - V_1)$

$$= |S_1| + I(P_n).$$

$$= n + \lceil 2 \sqrt{n + 1} \rceil - 2 > n + 1.$$

Now we discuss the minimality of $|S| + m(G - S)$.

If $S_3$ be any dominating set of $G$ which is not containing $S_1$ or $S_2$ as a proper subset and $|S_3| = k < 2n$. Then due to construction $G$ ($w_{1j}$ is adjacent to $w_{2k}$ for $1 \leq i, k \leq n$),

$$|S_3| + m(G - S_3) = k + 2n - k = 2n > |S| + m(G - S).$$

Let $S_5$ be another dominating set of $G$ such that $S_5 = S_4 \cup S_2$, where $S_4 \subset S_1$ with $|S_4| < n$. In $G$, $w_{1j}$ is adjacent to $w_{2k}$ for $1 \leq i, k \leq n$ therefore $m(G - S_5) = |S_2| + n - |S_4|.$

Therefore, $|S_5| + m(G - S_5) = |S_2| + |S_4| + |S_2| + n - |S_4|$

$$= 2|S_2| + n.$$

$$> |S| + m(G - S).$$
Therefore from above discussion, among all dominating sets of $G$, $S$ is such that $|S| + m(G - S)$ is minimum.

Hence, $DI(P_2 [P_n]) = \min \{|X| + m(G - X) : X \text{ is a dominating set}\}$

$$= |S| + m(G - S).$$

$$= n + \left\lfloor 2 \sqrt{n+1} \right\rfloor - 2.$$  

\[\begin{aligned}
4; & \quad \text{if } n = 2, 3 \\
6; & \quad \text{if } n = 4, 5 \\
\frac{2n}{3} + 4; & \quad \text{if } n \geq 6 \text{ and } n \equiv 0 \pmod{3} \\
\frac{2(n-1)}{3} + 4; & \quad \text{if } n \geq 6 \text{ and } n \equiv 1 \pmod{3} \\
\frac{2(n+1)}{3} + 4; & \quad \text{if } n \geq 6 \text{ and } n \equiv 2 \pmod{3}
\end{aligned}\]

\textbf{Theorem 6.6.7.} $DI(P_n [P_2]) = \begin{cases} 4; & \text{if } n = 2, 3 \\ 6; & \text{if } n = 4, 5 \\ \frac{2n}{3} + 4; & \text{if } n \geq 6 \text{ and } n \equiv 0 \pmod{3} \\ \frac{2(n-1)}{3} + 4; & \text{if } n \geq 6 \text{ and } n \equiv 1 \pmod{3} \\ \frac{2(n+1)}{3} + 4; & \text{if } n \geq 6 \text{ and } n \equiv 2 \pmod{3} \end{cases}$

\textbf{Proof.} Let $P_n$ be a path with vertices $u_1, u_2, \ldots, u_n$ and $P_2$ with $v_1, v_2$. Let $G$ be the graph $P_n [P_2]$. Then $V(G) = \{(u_i, v_j) / 1 \leq i \leq n, 1 \leq j \leq 2\}$ and

$E(G) = \{(u_i, v_j)(u_{i+1}, v_j) / 1 \leq i \leq n - 1, 1 \leq j \leq 2\} \cup \{(u_i, v_1)(u_{i+1}, v_2) / 1 \leq i \leq n - 1\} \cup \{(u_i, v_2)(u_{i+1}, v_1) / 1 \leq i \leq n - 1\}$. Without loss of generality we denote vertices $(u_i, v_1) = w_{i1}, 1 \leq i \leq n$ and $(u_i, v_2) = w_{i2}, 1 \leq i \leq n$.

The graph of $P_3 [P_2]$ is shown in Figure 6.4 for the better understanding of the notations and arrangement of vertices.

\[\begin{aligned}
\text{Figure 6.4: } P_3 [P_2]
\end{aligned}\]

To prove this result we consider following two cases.
Chapter 6. Domination Integrity - A measure of vulnerability

**Case 1:** \( n = 2 \) to \( 5 \).

For \( n = 2 \), \( P_2 \ [P_2] \) is isomorphic to complete graph \( K_4 \), hence \( DI (P_2 \ [P_2]) = 4 \).

For \( n = 3 \), consider \( S = \{ w_{21}, w_{22} \} \), which is a dominating set for \( P_3 \ [P_2] \) and \( m(G - S) = 2 \). There does not exist any dominating set \( S_1 \) of \( G \) such that \( |S_1| + m(G - S_1) < |S| + m(G - S) \).

Hence, \( DI (P_3 \ [P_2]) = 4 \).

For \( n = 4 \), consider \( S = \{ w_{21}, w_{22}, w_{42} \} \), which is a dominating set for \( P_4 \ [P_2] \) and \( m(G - S) = 3 \). Moreover for any dominating set \( S_1 \) of \( G \) we have, \( |S_1| + m(G - S_1) > |S| + m(G - S) \).

Hence, \( DI (P_4 \ [P_2]) = 6 \).

For \( n = 5 \), consider \( S = \{ w_{21}, w_{22}, w_{41}, w_{42} \} \), which is a dominating set for \( P_5 \ [P_2] \) and \( m(G - S) = 2 \). Moreover for any dominating set \( S_1 \) of \( G \) we have, \( |S_1| + m(G - S_1) > |S| + m(G - S) \).

Hence, \( DI (P_5 \ [P_2]) = 6 \).

**Case 2:** \( n \geq 6 \).

Now we consider subset \( S \) of \( G \) as below:

- If \( n \equiv 0 \ (\text{mod } 3) \) (i.e. \( n = 3k \)) and \( n \equiv 2 \ (\text{mod } 3) \) (i.e. \( n = 3k - 1 \)), consider
  \[ S = \{ w_{(2+3)j} \ (0 \leq j \leq k - 1) \} \cup \{ w_{(2+3)j} \ (0 \leq j \leq k - 1) \} \text{ and } |S| = 2k \]. So \( |S| = \frac{2n}{3} \) for \( n \equiv 0 \ (\text{mod } 3) \) and \( |S| = \frac{2(n+1)}{3} \) for \( n \equiv 2 \ (\text{mod } 3) \).

- If \( n \equiv 1 \ (\text{mod } 3) \) (i.e. \( n = 3k + 1 \)), consider \( S = \{ w_{(2+3)j} \ (0 \leq j \leq k - 1) \} \cup \{ w_{(2+3)j} \ (0 \leq j \leq k - 1) \} \cup \{ w_{n1} \} \text{ and } |S| = 2k + 1 = \frac{2(n-1)}{3} \).

In all the above cases \( S \) is a dominating set for \( G \) as \( w_{(i-1)1}, w_{(i+1)1} \in N(w_{i1}) \) and \( w_{(i-1)2}, w_{(i+1)2} \in N(w_{i2}) \) moreover \( m(G - S) = 4 \).
Now we discuss the minimality of $|S| + m(G - S)$.

If we consider any dominating set $S_1$ of $G$ such that, $|S_1| < |S|$ then due to construction of $G$ (i.e. to convert $G - S_1$ into disconnect graph we must include vertices $w_{i1}$ and $w_{i2}$ in $S_1$), it generates large value of $m(G - S_1)$ such that,

$$|S| + m(G - S) < |S_1| + m(G - S_1).$$

Let $S_2$ be any dominating set of $G$ such that $m(G - S_2) = 3$ then,

$$|S| + m(G - S) < |S_2| + m(G - S_2),$$

for $n \geq 6$.

Moreover if $S_3$ is any dominating set of $G$ with $m(G - S_3) = 2$ or $m(G - S_3) = 1$ then clearly,

$$|S| + m(G - S) < |S_3| + m(G - S_3)$$

Therefore in both the cases we have,

$$|S| + m(G - S) = \min\{ |X| + m(G - X) : X \text{ is a dominating set} \}$$

$$= DI(G).$$

Hence, $DI(P_n [P_2]) = \begin{cases} 4; & \text{if } n = 2, 3 \\ 6; & \text{if } n = 4, 5 \\ \frac{2n}{3} + 4; & \text{if } n \geq 6 \text{ and } n \equiv 0(\ mod \ 3) \\ \frac{2(n-1)}{3} + 4; & \text{if } n \geq 6 \text{ and } n \equiv 1(\ mod \ 3) \\ \frac{2(n+1)}{3} + 4; & \text{if } n \geq 6 \text{ and } n \equiv 2(\ mod \ 3) \end{cases}$

### 6.7 Conclusion and Scope of Further Research

The rapid growth of various modes of communication have emerged as a search for sustainable and secured network. The vulnerability of network is an important issue with special reference to defence objectives. We take up this problem in the context of expansion of graph network by means of various graph operations and investigate
domination integrity of resultant graphs. From the results reported in this chapter, we conclude that the domination integrity increases when the graph network is expanded.

To measure the graph vulnerability by various measures different from domination integrity and establish a better sustainable network by comparison of such measures as well as to investigate domination integrity for various graphs is an open area of research.