Chapter 4

Cordial and 3-equitable labelings
Chapter 4. Cordial and 3-equitable labelings

4.1 Introduction

The concept of cordial labeling was introduced by Cahit [24]. The partial motivation to define cordial labeling was an attempt to prove the existence (conjectured) of graceful and harmonious labelings for all trees. It turns out to be relatively easy to prove that every tree admits a cordial labeling. Unfortunately, cordial labeling is too weak to handle Ringel’s conjecture, even though vast amount of research papers are available and many graph or graph families are proved to be cordial. On the other side some variants of cordial labeling are also introduced. This chapter is aimed to give a brief account of cordial labeling and some new results investigated by us.

4.2 Cordial labeling of graphs

Definition 4.2.1. A mapping $f : V(G) \to \{0, 1\}$ is called binary vertex labeling of $G$ and $f(v)$ is called the label of the vertex $v$ of $G$ under $f$.

If for an edge $e = uv$, the induced edge labeling $f^* : E(G) \to \{0, 1\}$ is given by $f^*(e) = |f(u) - f(v)|$ then we introduce following notations,

$$
\begin{align*}
  v_f(i) &= \text{number of vertices of } G \text{ having label } i \text{ under } f \\
  e_f(i) &= \text{number of edges of } G \text{ having label } i \text{ under } f^*
\end{align*}
$$

where $i = 0$ or 1

Definition 4.2.2. A binary vertex labeling $f$ of a graph $G$ is called a cordial labeling if $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$. A graph $G$ is cordial if it admits cordial labeling.
Illustration 4.2.3. A cordial labeling of helm $H_3$ is shown in Figure 4.1.

![Figure 4.1: $H_3$ and its cordial labeling](image)

As reported in Rosa [24] graceful, harmonious and cordial trees are related as follows.

- Any graceful tree is cordial.
- Any harmonious tree is cordial.

But for general graphs the relation between the existence of graceful, harmonious and cordial labeling is more complicated.

- The cycle $C_4$ is graceful and cordial but not harmonious.
- The cycle $C_5$ is harmonious and cordial but not graceful.
- The complete graph $K_4$ is graceful and harmonious but not cordial.
- Complete graph $K_5$ is neither of the three.

### 4.2.1 Some known results on cordial labeling

- Cahit[25] has proved that,
  - Every tree is cordial.
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- Complete graphs $K_n$ are cordial if and only if $n \leq 3$.
- Complete bipartite graphs $K_{m,n}$ are cordial for all $m$ and $n$.
- The one point union of $t$ copies of $3$–cycles denoted as $C_3^{(t)}$ is cordial if and only if $n \neq 3 \pmod{4}$.
- Wheels $W_n = C_n + K_1$ are cordial if and only if $n \neq 3 \pmod{4}$.
- Maximal outer planar graphs are cordial.
- An Eulerian graph is not cordial if its number of edges $q \equiv 2 \pmod{4}$.
- All fans $f_n = P_n + K_1$ are cordial.

- Ho et al. [45] have proved that
  - Any unicyclic graph is cordial except $C_{4k+2}$.
  - Generalized Petersen graph $P(n,k)$ is cordial if and only if $n \neq 2 \pmod{4}$.

- Kirchherr [51] has given a characterization of cordial graphs in terms of their adjacency matrices.

- Cairnie and Edwards [28] have determined the computational complexity of cordial and $k$–cordial labelings. They also prove the conjecture of Kirchherr [51] that deciding whether a graph admits a cordial labeling is NP-complete.

- Andar et al. in [2–4] proved that
  - Helms, closed helms and generalized helms are cordial.
  - Flower graph (graphs obtained by joining the vertices of degree one of a helm to the central vertex)
  - Sunflower graph (graphs obtained by taking a wheel with the central vertex $v$ and $n$–cycle $v_1, v_2, \ldots, v_n$ and additional vertices $w_1, w_2, \ldots, w_n$ where $w_i$ is joined by edges to $v_i, v_{i+1}$, where $i + 1$ is taken modulo $n$)
  - Multiple shells are cordial.
The one point unions of helms, closed helms, flowers, gears, and sunflower graphs, where in each case the central vertex is the common vertex.

- Vaidya et al., [84, 86, 90] have discussed cordial labeling for some cycle related graphs arising from different graph operations.
- Vaidya et al., [85] have discussed some new cordial graphs.
- Vaidya et al., [80] have discussed some shell related cordial graphs.
- Vaidya and Dani, [81] and Vaidya et al., [79] have discussed the cordial labeling for some star related graphs.
- Vaidya and Vihol [93] prove that the middle graph \( M(G) \) of an Eulerian graph is Eulerian and \( |E(M(G))| = \sum_{i=1}^{n} \frac{d(v_i)^2 + 2e}{2} \). They also prove that the middle graphs of path, crown, \( K_{1,n} \), tadpol \( T(n, l+1) \) admit cordial labeling.
- Raj and Koilraj [61] have discussed cordial labeling for the splitting graph of some standard graphs.
- Motivated through the concept of cordial labeling Babujee and Shobana [9] have introduced the concepts of cordial languages and cordial numbers.

### 4.3 Cordial labeling of some star and bistar related graphs

In this section we report some result on cordial labeling of bistar related graph which are investigated by us.

**Theorem 4.3.1.** \( K_{1,n} \ast K_{1,n} \) is a cordial graph.

**Proof.** Consider two copies of \( K_{1,n} \) with vertex sets \( \{u, u_i : 1 \leq i \leq n\} \) and \( \{v, v_i : 1 \leq i \leq n\} \) respectively where \( u_i, v_i, 1 \leq i \leq n \) are the pendant vertices. Let \( G \) be the graph \( K_{1,n} \ast K_{1,n} \) with \( V(G) = \{u, v, u_i, v_i : 1 \leq i \leq n\} \) and \( E(G) = \{uv, uu_i, uv_i, vv_i, vu_i : 1 \leq i \leq n\} \) then \( |V(G)| = 2n + 2 \) and \( |E(G)| = 4n + 1 \).
We define vertex labeling \( f : V(G) \rightarrow \{0, 1\} \) as follows:

\[
\begin{align*}
  f(u) &= 1, \\
  f(v) &= 0, \\
  f(u_i) &= 1; \quad 1 \leq i \leq n \\
  f(v_i) &= 0; \quad 1 \leq i \leq n
\end{align*}
\]

In view of the above defined labeling pattern we have, 
\( v_f(0) = n + 1 = v_f(1) \)

\( e_f(0) = 2n, e_f(1) = 2n + 1 \)

Thus we proved that \( |v_f(0) - v_f(1)| \leq 1 \) and \( |e_f(0) - e_f(1)| \leq 1. \)

Hence, \( K_{1,n} \ast K_{1,n} \) is a cordial graph.

\[\blacksquare\]

**Illustration 4.3.2.** Cordial labeling of the graph \( K_{1,7} \ast K_{1,7} \) is shown in **Figure 4.2**.

**Theorem 4.3.3.** \( D_2(B_{n,n}) \) is a cordial graph.

**Proof.** Consider two copies of \( B_{n,n} \). Let \( \{u, v, u_i, v_i : 1 \leq i \leq n\} \) and \( \{u', v', u'_i, v'_i : 1 \leq i \leq n\} \) be the corresponding vertex sets of each copy of \( B_{n,n} \). Let \( G \) be the graph \( D_2(B_{n,n}) \) then \( |V(G)| = 4(n + 1) \) and \( |E(G)| = 4(2n + 1) \). We define vertex labeling \( f : V(G) \rightarrow \{0, 1\} \) as follows:
In view of the above defined labeling pattern we have,

\[ v_f(0) = 2(n + 1) = v_f(1) \text{ and } e_f(0) = 4n + 2 = e_f(1). \]

Thus we proved that \(|v_f(0) - v_f(1)| \leq 1\) and \(|e_f(0) - e_f(1)| \leq 1\).

Hence, \(D_2(B_{n,n})\) is a cordial graph.

**Illustration 4.3.4.** Cordial labeling of the graph \(D_2(B_{5,5})\) is shown in Figure 4.3.

**Theorem 4.3.5.** \(S'(B_{n,n})\) is a cordial graph.

**Proof.** Consider \(B_{n,n}\) with vertex set \(\{u, v, u_i, v_i : 1 \leq i \leq n\}\) where \(u_i, v_i\) are pendant vertices. In order to obtain \(S'(B_{n,n})\), add \(u', v', u'_i, v'_i\) vertices corresponding to \(u, v, u_i, v_i\)
where, \(1 \leq i \leq n\). If \(G = S'(B_{n,n})\) then \(|V(G)| = 4(n+1)\) and \(|E(G)| = 6n + 3\). We define vertex labeling \(f : V(G) \to \{0, 1\}\) as follows:

\[
\begin{align*}
    f(u) &= 0, \\
    f(u') &= 1, \\
    f(v) &= 0, \\
    f(v') &= 1, \\
    f(u_i) &= 0; \quad 1 \leq i \leq n \\
    f(u'_i) &= 0; \quad 1 \leq i \leq n \\
    f(v_i) &= 1; \quad 1 \leq i \leq n \\
    f(v'_i) &= 1; \quad 1 \leq i \leq n
\end{align*}
\]

In view of the above labeling pattern we have, \(v_f(0) = 2n + 2 = v_f(1)\) and \(e_f(0) = 3n + 1, e_f(1) = 3n + 2\).
Thus we proved that \(|v_f(0) - v_f(1)| \leq 1\) and \(|e_f(0) - e_f(1)| \leq 1\).

Hence, \(S'(B_{n,n})\) is a cordial graph.

\textbf{Illustration 4.3.6.} Cordial labeling of the graph \(S'(B_{6,6})\) is shown in Figure 4.4.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{S66.png}
\caption{\(S'(B_{6,6})\) and its cordial labeling}
\end{figure}
Theorem 4.3.7. $DS(B_{n,n})$ is a cordial graph.

Proof. Consider $B_{n,n}$ with $V(B_{n,n}) = \{u, v, u_i, v_i : 1 \leq i \leq n\}$, where $u_i, v_i$ are pendant vertices. Here $V(B_{n,n}) = V_1 \cup V_2$, where $V_1 = \{u_i, v_i : 1 \leq i \leq n\}$ and $V_2 = \{u, v\}$. Now in order to obtain $DS(B_{n,n})$ from $G$, we add $w_1, w_2$ corresponding to $V_1, V_2$. Then $|V(DS(B_{n,n}))| = 2n + 4$ and $E(DS(B_{n,n})) = \{uv, uw_2, vw_2\} \cup \{uu_i, vv_i, w_1u_i, w_1v_i : 1 \leq i \leq n\}$ so $|E(DS(B_{n,n}))| = 4n + 3$. We define vertex labeling $f : V(DS(B_{n,n})) \rightarrow \{0, 1\}$ as follows:

$$
\begin{align*}
    f(u) &= 0, \\
    f(v) &= 0, \\
    f(w_1) &= 1, \\
    f(w_2) &= 1, \\
    f(u_i) &= 0; \quad 1 \leq i \leq n \\
    f(v_i) &= 1; \quad 1 \leq i \leq n
\end{align*}
$$

In view of the above defined labeling pattern we have,

$v_f(0) = n + 2 = v_f(1)$ and $e_f(0) = 2n + 1, e_f(1) = 2n + 2$.

Thus we proved that $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$.

Hence, $DS(B_{n,n})$ is a cordial graph. 

Illustration 4.3.8. Cordial labeling of the graph $DS(B_{5,5})$ is shown in Figure 4.5.

![Figure 4.5: DS(B_{5,5}) and its cordial labeling](image-url)
4.4 Cordial labeling of snake related graphs

Definition 4.4.1. An alternate triangular snake $A(T_n)$ is obtained from a path $u_1, u_2, \ldots, u_n$ by joining $u_i$ and $u_{i+1}$ (alternately) to a new vertex $v_i$. That is every alternate edge of path is replaced by $C_3$.

Definition 4.4.2. An alternate quadrilateral snake $A(QS_n)$ is obtained from a path $u_1, u_2, \ldots, u_n$ by joining $u_i, u_{i+1}$ to new vertices $v_i, w_i$ respectively and then joining $v_i$ and $w_i$. That is every alternate edge of path is replaced by $C_4$.

Definition 4.4.3. A double alternate triangular snake $DA(T_n)$ consists of two alternate triangular snakes that have a common path. That is, double alternate triangular snake is obtained from a path $u_1, u_2, \ldots, u_n$ by joining $u_i$ and $u_{i+1}$ (alternately) to new vertices $v_i$ and $w_i$.

Definition 4.4.4. A double alternate quadrilateral snake $DA(QS_n)$ consists of two alternate quadrilateral snakes that have a common path. That is, it is obtained from a path $u_1, u_2, \ldots, u_n$ by joining $u_i$ and $u_{i+1}$ (alternately) to new vertices $v_i, v'_i$ and $w_i, w'_i$ respectively and adding the edges $v_iw_i$ and $v'_iw'_i$.

This section is aimed to report some results on cordial labeling of snake related graphs which are investigated by us.

Theorem 4.4.5. $A(T_n)$ admits cordial labeling.

Proof. Let $A(T_n)$ be alternate triangular snake obtained from a path $u_1, u_2, \ldots, u_n$ by joining $u_i$ and $u_{i+1}$ (alternately) to new vertex $v_i$ where $1 \leq i \leq n-1$ for even $n$ and $1 \leq i \leq n-2$ for odd $n$. Therefore $V(A(T_n)) = \{u_i, v_j/1 \leq i \leq n, 1 \leq j \leq [\frac{n}{2}]\}$. We note that

$$|V(A(T_n))| = \begin{cases} \frac{3n}{2}, & n \equiv 0 \pmod{2} \\ \frac{3n-1}{2}, & n \equiv 1 \pmod{2} \end{cases} \text{ and } |E(A(T_n))| = \begin{cases} 2n-1, & n \equiv 0 \pmod{2} \\ 2n-2, & n \equiv 1 \pmod{2} \end{cases}.$$ 

To define vertex labeling $f : V(A(T_n)) \to \{0, 1\}$ we consider following five cases.
Case 1: \( n = 2, 3 \).

For \( n = 2 \), \( A(T_2) = C_3 \), which is a cordial graph as proved by Ho et al. [45].

For \( n = 3 \), \( f(u_1) = 0, f(u_2) = 1, f(u_3) = 1 \) and \( f(v_1) = 0 \). Then \( v_f(0) = 2, v_f(1) = 2 \) and \( e_f(0) = 2 = e_f(1) \). Hence, \( A(T_3) \) admits cordial labeling.

Case 2: \( n \equiv 0 (\text{mod} \ 4) \).

Let \( n = 4k \),

\[
\begin{align*}
  f(u_{1+4i}) &= 0; \quad 0 \leq i \leq k - 1 \\
  f(u_{2+4i}) &= 1; \quad 0 \leq i \leq k - 1 \\
  f(u_{3+4i}) &= 1; \quad 0 \leq i \leq k - 1 \\
  f(u_{4+4i}) &= 0; \quad 0 \leq i \leq k - 1 \\
  f(v_{1+2i}) &= 1; \quad 0 \leq i \leq \frac{n}{4} - 1 \\
  f(v_{2+2i}) &= 0; \quad 0 \leq i \leq \frac{n}{4} - 1
\end{align*}
\]

In view of the above defined labeling pattern, \( v_f(0) = \frac{3n}{4} = v_f(1) \) and \( e_f(0) = n - 1 \), \( e_f(1) = n \).

Case 3: \( n \equiv 1 (\text{mod} \ 4) \).

Let \( n = 4k + 1 \),

\[
\begin{align*}
  f(u_{1+4i}) &= 0; \quad 0 \leq i \leq k - 1 \\
  f(u_{2+4i}) &= 1; \quad 0 \leq i \leq k - 1 \\
  f(u_{3+4i}) &= 1; \quad 0 \leq i \leq k - 1 \\
  f(u_{4+4i}) &= 0; \quad 0 \leq i \leq k - 1 \\
  f(u_n) &= 0; \\
  f(v_{1+2i}) &= 1; \quad 0 \leq i \leq \frac{n-1}{4} - 1 \\
  f(v_{2+2i}) &= 0; \quad 0 \leq i \leq \frac{n-1}{4} - 1
\end{align*}
\]

In view of the above defined labeling pattern, \( v_f(0) = \left\lceil \frac{3n-1}{4} \right\rceil \), \( v_f(1) = \left\lfloor \frac{3n-1}{4} \right\rfloor \) and \( e_f(0) = n - 1 = e_f(1) \).
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**Case 4:** \( n \equiv 2 \pmod{4} \).

Let \( n = 4k + 2 \),
\[
\begin{align*}
    f(u_{1+4i}) & = 0; \quad 0 \leq i \leq k-1 \\
    f(u_{2+4i}) & = 1; \quad 0 \leq i \leq k-1 \\
    f(u_{3+4i}) & = 1; \quad 0 \leq i \leq k-1 \\
    f(u_{4+4i}) & = 0; \quad 0 \leq i \leq k-1 \\
    f(u_{n-1}) & = 0; \\
    f(u_{n}) & = 1; \\
    f(v_{1+2i}) & = 1; \quad 0 \leq i \leq \frac{n-2}{4} - 1 \\
    f(v_{2+2i}) & = 0; \quad 0 \leq i \leq \frac{n-2}{4} - 1 \\
    f(v_{\frac{n}{2}}) & = 1;
\end{align*}
\]
In view of the above defined labeling pattern, \( v_f(0) = \left\lfloor \frac{3n}{4} \right\rfloor \), \( v_f(1) = \left\lceil \frac{3n}{4} \right\rceil \) and \( e_f(0) = n-1, e_f(1) = n \).

**Case 5:** \( n \equiv 3 \pmod{4} \).

Let \( n = 4k + 3 \),
\[
\begin{align*}
    f(u_{1+4i}) & = 0; \quad 0 \leq i \leq k-1 \\
    f(u_{2+4i}) & = 1; \quad 0 \leq i \leq k-1 \\
    f(u_{3+4i}) & = 1; \quad 0 \leq i \leq k-1 \\
    f(u_{4+4i}) & = 0; \quad 0 \leq i \leq k-1 \\
    f(u_{n-2}) & = 0; \\
    f(u_{n-1}) & = 1; \\
    f(u_{n}) & = 1; \\
    f(v_{1+2i}) & = 1; \quad 0 \leq i \leq \frac{n-3}{4} - 1 \\
    f(v_{2+2i}) & = 0; \quad 0 \leq i \leq \frac{n-3}{4} - 1 \\
    f(v_{\frac{n-1}{2}}) & = 0;
\end{align*}
\]
In view of the above defined labeling pattern, \( v_f(0) = \frac{3n-1}{4} = v_f(1) \) and \( e_f(0) = n-1 = e_f(1) \).
Thus, in each case we have \( |v_f(0) - v_f(1)| \leq 1 \) and \( |e_f(0) - e_f(1)| \leq 1 \).

Hence, \( A(T_n) \) admits cordial labeling.

**Illustration 4.4.6.** A cordial labeling of \( A(T_{11}) \) is shown in **Figure 4.6.**

![Figure 4.6: A(T_{11}) and its cordial labeling](image)

**Theorem 4.4.7.** \( A(QS_n) \) admits cordial labeling.

**Proof.** Let \( A(QS_n) \) be an alternate quadrilateral snake obtained from a path \( u_1, u_2, \ldots, u_n \) by joining \( u_i, u_{i+1} \) (alternately) to new vertices \( v_i, w_i \) respectively and then joining \( v_i \) and \( w_i \) where \( 1 \leq i \leq n-1 \) for even \( n \) and \( 1 \leq i \leq n-2 \) for odd \( n \). Therefore \( V(A(T_n)) = \{u_i, v_j, w_j/1 \leq i \leq n, 1 \leq j \leq \lfloor \frac{n}{2} \rfloor \} \). We note that

\[
|V(A(QS_n))| = \begin{cases} 
2n & , n \equiv 0 \pmod{2}; \\
2n - 1 & , n \equiv 1 \pmod{2}.
\end{cases}
\]

\[
|E(A(QS_n))| = \begin{cases} 
\frac{5n-2}{2} & , n \equiv 0 \pmod{2}; \\
\frac{5n-5}{2} & , n \equiv 1 \pmod{2}.
\end{cases}
\]

To define vertex labeling \( f : V(A(QS_n)) \to \{0, 1\} \) we consider following five cases.

**Case 1:** \( n = 2,3 \).

For \( n = 2, A(QS_2) = C_4 \), which is a cordial graph as proved by Ho et al. [45].

For \( n = 3, f(u_1) = 1, f(u_2) = 0, f(u_3) = 1 \) and \( f(v_1) = 1, f(w_1) = 0 \). Then \( v_f(0) = 2, v_f(1) = 3 \) and \( e_f(0) = 2, e_f(1) = 3 \).

Hence, \( A(QS_3) \) admits cordial labeling.
Case 2: \( n \equiv 0 (\text{mod } 4) \).

Let \( n = 4k \),

\[
\begin{align*}
    f(u_{1+4i}) &= 0; \quad 0 \leq i \leq k - 1 \\
    f(u_{2+4i}) &= 0; \quad 0 \leq i \leq k - 1 \\
    f(u_{3+4i}) &= 1; \quad 0 \leq i \leq k - 1 \\
    f(u_{4+4i}) &= 0; \quad 0 \leq i \leq k - 1 \\
    f(v_i) &= 1; \quad 1 \leq i \leq \frac{n}{2} \\
    f(w_{1+2i}) &= 1; \quad 0 \leq i \leq \frac{n}{4} - 1 \\
    f(w_{2+2i}) &= 0; \quad 0 \leq i \leq \frac{n}{4} - 1
\end{align*}
\]

In view of the above defined labeling pattern, \( v_f(0) = n = v_f(1) \) and \( e_f(0) = n + \frac{n}{4} - 1 \) and \( e_f(1) = n + \frac{n}{4} \).

Case 3: \( n \equiv 1 (\text{mod } 4) \).

Let \( n = 4k + 1 \),

\[
\begin{align*}
    f(u_{1+4i}) &= 0; \quad 0 \leq i \leq k - 1 \\
    f(u_{2+4i}) &= 0; \quad 0 \leq i \leq k - 1 \\
    f(u_{3+4i}) &= 1; \quad 0 \leq i \leq k - 1 \\
    f(u_{4+4i}) &= 0; \quad 0 \leq i \leq k - 1 \\
    f(u_n) &= 0; \\
    f(v_i) &= 1; \quad 1 \leq i \leq \frac{n-1}{2} \\
    f(w_{1+2i}) &= 1; \quad 0 \leq i \leq \frac{n-1}{4} - 1 \\
    f(w_{2+2i}) &= 0; \quad 0 \leq i \leq \frac{n-1}{4} - 1
\end{align*}
\]

In view of the above defined labeling pattern, \( v_f(0) = n \), \( v_f(1) = n - 1 \) and \( e_f(0) = n - 1 + \frac{n-1}{4} = \frac{5n-5}{4} = e_f(1) \).
**Case 4:** \( n \equiv 2 \pmod{4} \).

Let \( n = 4k + 2 \),

\[
\begin{align*}
    f(u_{1+4i}) &= 0; \quad 0 \leq i \leq k \\
    f(u_{2+4i}) &= 0; \quad 0 \leq i \leq k \\
    f(u_{3+4i}) &= 1; \quad 0 \leq i \leq k - 1 \\
    f(u_{4+4i}) &= 0; \quad 0 \leq i \leq k - 1 \\
    f(v_i) &= 1; \quad 1 \leq i \leq \frac{n}{2} \\
    f(w_{1+2i}) &= 1; \quad 0 \leq i \leq \frac{n-2}{4} \\
    f(w_{2+2i}) &= 0; \quad 0 \leq i \leq \frac{n-2}{4} - 1
\end{align*}
\]

In view of the above defined labeling pattern, \( v_f(0) = n = v_f(1) \) and \( e_f(0) = \frac{5n - 2}{4} = e_f(1) \).

**Case 5:** \( n \equiv 3 \pmod{4} \).

Let \( n = 4k + 3 \),

\[
\begin{align*}
    f(u_{1+4i}) &= 0; \quad 0 \leq i \leq k \\
    f(u_{2+4i}) &= 0; \quad 0 \leq i \leq k \\
    f(u_{3+4i}) &= 1; \quad 0 \leq i \leq k - 1 \\
    f(u_{4+4i}) &= 0; \quad 0 \leq i \leq k - 1 \\
    f(u_n) &= 1; \\
    f(v_i) &= 1; \quad 1 \leq i \leq \frac{n-1}{2} \\
    f(w_{1+2i}) &= 1; \quad 0 \leq i \leq \frac{n-3}{4} \\
    f(w_{2+2i}) &= 0; \quad 0 \leq i \leq \frac{n-3}{4} - 1
\end{align*}
\]

In view of the above defined labeling pattern, \( v_f(0) = n - 1, v_f(1) = n \) and \( e_f(0) = \frac{5n - 7}{4}, e_f(1) = \frac{5n - 3}{4} \).

Thus in each case we have \( |v_f(0) - v_f(1)| \leq 1 \) and \( |e_f(0) - e_f(1)| \leq 1 \).

Hence, \( A(QS_n) \) admits cordial labeling.
Illustration 4.4.8. A cordial labeling of $A(QS_8)$ is shown in Figure 4.7.

\[ \begin{align*} &\quad v_1 \quad w_1 \quad v_2 \quad w_2 \quad v_3 \quad w_3 \quad v_4 \quad w_4 \\ & u_1 \quad 0 \quad u_2 \quad 1 \quad u_3 \quad 0 \quad u_4 \quad 1 \quad u_5 \quad 0 \quad u_6 \quad 1 \quad u_7 \quad 0 \quad u_8 \quad 1 \quad \end{align*} \]

\[ \text{Figure 4.7: } A(QS_8) \text{ and its cordial labeling} \]

Theorem 4.4.9. $DA(T_n)$ admits cordial labeling.

Proof. Let $G$ be a double alternate triangular snake $DA(T_n)$ then $V(G) = \{u_i,v_j,w_j/1 \leq i \leq n, 1 \leq j \leq \lfloor \frac{n}{2} \rfloor \}$. We note that

\[ |V(G)| = \begin{cases} 2n, & n \equiv 0 \pmod{2} \\ 2n - 1, & n \equiv 1 \pmod{2} \end{cases} \quad \text{and} \quad |E(G)| = \begin{cases} 3n - 1, & n \equiv 0 \pmod{2} \\ 3n - 3, & n \equiv 1 \pmod{2} \end{cases} \]

To define vertex labeling $f : V(DA(T_n)) \to \{0,1\}$ we consider following five cases.

Case 1: $n = 2,3$.

For $n = 2$, $f(u_1) = 0, f(u_2) = 1$ and $f(v_1) = 0, f(w_1) = 1$. Then $v_f(0) = 2 = v_f(1)$ and $e_f(0) = 2, e_f(1) = 3$. Hence, $DA(T_2)$ admits cordial labeling.

For $n = 3$, $f(u_1) = 0, f(u_2) = 1, f(u_3) = 1$ and $f(v_1) = 0, f(w_1) = 1$. Then $v_f(0) = 2, v_f(1) = 3$ and $e_f(0) = 3 = e_f(1)$. Hence, $DA(T_3)$ admits cordial labeling.

Case 2: $n \equiv 0 \pmod{4}$.

Let $n = 4k$,

\[ f(u_{i+4i}) = 0; \quad 0 \leq i \leq k - 1 \]
\[ f(u_{2+4i}) = 1; \quad 0 \leq i \leq k - 1 \]
\[ f(u_{3+4i}) = 1; \quad 0 \leq i \leq k - 1 \]
\[ f(u_{4+4i}) = 0; \quad 0 \leq i \leq k - 1 \]
\[ f(v_i) = 0; \quad 1 \leq i \leq \frac{n}{2} \]
\[ f(w_i) = 1; \quad 1 \leq i \leq \frac{n}{2} \]
In view of the above defined labeling pattern, \( v_f(0) = n = v_f(1) \) and \( e_f(0) = \frac{3n - 2}{2} \) and \( e_f(1) = \frac{3n}{2} \).

**Case 3:** \( n \equiv 1(\text{mod } 4) \).

Let \( n = 4k + 1 \),

\[
\begin{align*}
  f(u_{1+4i}) &= 0; \quad 0 \leq i \leq k - 1 \\
  f(u_{2+4i}) &= 1; \quad 0 \leq i \leq k - 1 \\
  f(u_{3+4i}) &= 1; \quad 0 \leq i \leq k - 1 \\
  f(u_{4+4i}) &= 0; \quad 0 \leq i \leq k - 1 \\
  f(u_n) &= 0; \\
  f(v_i) &= 0; \quad 1 \leq i \leq \frac{n-1}{2} \\
  f(w_i) &= 1; \quad 1 \leq i \leq \frac{n-1}{2}
\end{align*}
\]

In view of the above defined labeling pattern, \( v_f(0) = n, v_f(1) = n - 1 \) and \( e_f(0) = \frac{3n - 3}{2}, e_f(1) = \frac{3n - 3}{2} \).

**Case 4:** \( n \equiv 2(\text{mod } 4) \).

Let \( n = 4k + 2 \),

\[
\begin{align*}
  f(u_{1+4i}) &= 0; \quad 0 \leq i \leq k \\
  f(u_{2+4i}) &= 1; \quad 0 \leq i \leq k \\
  f(u_{3+4i}) &= 1; \quad 0 \leq i \leq k - 1 \\
  f(u_{4+4i}) &= 0; \quad 0 \leq i \leq k - 1 \\
  f(v_i) &= 0; \quad 1 \leq i \leq \frac{n}{2} \\
  f(w_i) &= 1; \quad 1 \leq i \leq \frac{n}{2}
\end{align*}
\]

In view of the above defined labeling pattern, \( v_f(0) = n = v_f(1) \) and \( e_f(0) = \frac{3n}{2} - 1, e_f(1) = \frac{3n}{2} \).
**Case 5**: $n \equiv 3 \,(\text{mod} \, 4)$.

Let $n = 4k + 3$,

\[
\begin{align*}
    f(u_{1+4i}) &= 0; \quad 0 \leq i \leq k \\
    f(u_{2+4i}) &= 1; \quad 0 \leq i \leq k \\
    f(u_{3+4i}) &= 1; \quad 0 \leq i \leq k - 1 \\
    f(u_{4+4i}) &= 0; \quad 0 \leq i \leq k - 1 \\
    f(u_n) &= 1; \\
    f(v_j) &= 0; \quad 1 \leq i \leq \frac{n-1}{2} \\
    f(w_j) &= 1; \quad 1 \leq i \leq \frac{n-1}{2}
\end{align*}
\]

In view of the above defined labeling pattern, $v_f(0) = n - 1, v_f(1) = n$ and $e_f(0) = \frac{3n - 3}{2} = e_f(1)$.

Thus, in each case we have $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$.

Hence, $DA(T_n)$ admits cordial labeling.

**Illustration 4.4.10.** A cordial labeling of $DA(T_{10})$ is shown in Figure 4.8.

\[\begin{array}{cccccc}
  v_1 & 0 & 1 & 0 & 0 & 1 \\
  u_1 & \downarrow & u_2 & \downarrow & u_3 & \downarrow \\
  w_1 & 1 & 0 & 1 & 0 & 1 \\
  v_2 & 0 & 1 & 0 & 0 & 1 \\
  u_4 & \downarrow & u_5 & \downarrow & u_6 & \downarrow \\
  w_2 & 1 & 0 & 1 & 0 & 1 \\
  v_3 & 0 & 1 & 0 & 0 & 1 \\
  u_8 & \downarrow & u_9 & \downarrow & u_{10} & \downarrow \\
  w_3 & 1 & 0 & 1 & 0 & 1 \\
  v_4 & 0 & 1 & 0 & 0 & 1 \\
  u_10 & \downarrow & u_11 & \downarrow & u_{12} & \downarrow \\
  w_4 & 1 & 0 & 1 & 0 & 1 \\
  v_5 & 0 & 1 & 0 & 0 & 1 \\
  u_{12} & \downarrow & u_{13} & \downarrow & u_{14} & \downarrow \\
  w_5 & 1 & 0 & 1 & 0 & 1 \\
  v_6 & 0 & 1 & 0 & 0 & 1 \\
  u_{14} & \downarrow & u_{15} & \downarrow & u_{16} & \downarrow \\
  w_6 & 1 & 0 & 1 & 0 & 1 \\
  v_7 & 0 & 1 & 0 & 0 & 1 \\
  u_{16} & \downarrow & u_{17} & \downarrow & u_{18} & \downarrow \\
  w_7 & 1 & 0 & 1 & 0 & 1 \\
  v_8 & 0 & 1 & 0 & 0 & 1 \\
  u_{18} & \downarrow & u_{19} & \downarrow & u_{20} & \downarrow \\
  w_8 & 1 & 0 & 1 & 0 & 1 \\
  v_9 & 0 & 1 & 0 & 0 & 1 \\
  u_{20} & \downarrow & u_{21} & \downarrow & u_{22} & \downarrow \\
  w_9 & 1 & 0 & 1 & 0 & 1 \\
  v_{10} & 0 & 1 & 0 & 0 & 1 \\
  u_{22} & \downarrow & u_{23} & \downarrow & u_{24} & \downarrow \\
  w_{10} & 1 & 0 & 1 & 0 & 1 \\
\end{array}\]

**Theorem 4.4.11.** $DA(QS_n)$ admits cordial labeling.

**Proof.** Let $G$ be a double alternate quadrilateral snake $DA(QS_n)$ then $V(G) = \{u_i, v_j, w_j, v'_j, w'_j/1 \leq i \leq n, 1 \leq j \leq \lfloor \frac{n}{2} \rfloor \}$. We note that

\[
|V(G)| = \begin{cases} 
3n, & n \equiv 0 \,(\text{mod} \, 2) \\
3n - 2, & n \equiv 1 \,(\text{mod} \, 2)
\end{cases} \quad \text{and} \quad |E(G)| = \begin{cases} 
4n - 1, & n \equiv 0 \,(\text{mod} \, 2) \\
4n - 4, & n \equiv 1 \,(\text{mod} \, 2)
\end{cases}
\]
Chapter 4. Cordial and 3-equitable labelings

To define vertex labeling $f : V(G) \rightarrow \{0, 1\}$ we consider following two cases.

**Case 1:** $n \equiv 0 \pmod{2}$.

Let $n = 2k$,

- $f(u_{1+2i}) = 0; \quad 0 \leq i \leq \frac{n}{2} - 1$
- $f(u_{2i}) = 1; \quad 1 \leq i \leq \frac{n}{2}$
- $f(v_i) = 1; \quad 1 \leq i \leq \frac{n}{2}$
- $f(w_i) = 1; \quad 1 \leq i \leq \frac{n}{2}$
- $f(v'_i) = 0; \quad 1 \leq i \leq \frac{n}{2}$
- $f(w'_i) = 0; \quad 1 \leq i \leq \frac{n}{2}$

In view of the above defined labeling pattern, $v_f(0) = \frac{3n}{2} = v_f(1)$ and $e_f(0) = 2n, e_f(1) = 2n - 1$.

**Case 2:** $n \equiv 1 \pmod{2}$.

Let $n = 2k + 1$,

- $f(u_{1+2i}) = 0; \quad 0 \leq i \leq \frac{n-1}{2}$
- $f(u_{2i}) = 1; \quad 1 \leq i \leq \frac{n-1}{2}$
- $f(v_i) = 1; \quad 1 \leq i \leq \frac{n-1}{2}$
- $f(w_i) = 1; \quad 1 \leq i \leq \frac{n-1}{2}$
- $f(v'_i) = 0; \quad 1 \leq i \leq \frac{n-1}{2}$
- $f(w'_i) = 0; \quad 1 \leq i \leq \frac{n-1}{2}$

In view of the above defined labeling pattern, $v_f(0) = \frac{3n - 1}{2}, v_f(1) = \frac{3n - 3}{2}$ and $e_f(0) = 2n - 2 = e_f(1)$.

Thus, in both the cases we have $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$.

Hence, $DA(QS_n)$ admits cordial labeling. ■
Illustration 4.4.12. A cordial labeling of $DA(QS_9)$ is shown in Figure 4.9.

![Figure 4.9: DA(QS_9) and its cordial labeling](image)

4.5 Cordial labeling of degree splitting graphs

In this section we report some results on cordial labeling of degree splitting graphs of some graphs which are investigated by us.

Theorem 4.5.1. $DS(P_n)$ is a cordial graph.

**Proof.** Consider $P_n$ with $V(P_n) = \{v_i : 1 \leq i \leq n\}$. Here $V(P_n) = S_1 \cup S_2$, where $S_1 = \{v_i : 2 \leq i \leq n - 1\}$ and $S_2 = \{v_1, v_n\}$. Now in order to obtain $DS(P_n)$ from $P_n$, we add $w_1, w_2$ corresponding to $S_1, S_2$. Then $|V(DS(P_n))| = n + 2$ and $E(DS(P_n)) = E(P_n) \cup \{v_1w_2, v_2w_2\} \cup \{w_1v_i : 2 \leq i \leq n - 1\}$ so $|E(DS(P_n))| = 2n - 1$. To define vertex labeling $f : V(DS(P_n)) \rightarrow \{0, 1\}$ we consider following five cases.

Case 1: $n = 4, 5$.

The graphs $DS(P_4)$ and $DS(P_5)$ are to be dealt separately and their cordial labeling is shown in Figure 4.10.
Case 2: $n \equiv 0 \pmod{4}$ and $n \geq 6$.

Let $n = 4k$,

\[
\begin{align*}
  f(w_1) &= 0, \\
  f(w_2) &= 1, \\
  f(v_1) &= 0, \\
  f(v_{2+4i}) &= 0; \quad 0 \leq i < k-1 \\
  f(v_{3+4i}) &= 0; \quad 0 \leq i < k-1 \\
  f(v_{4+4i}) &= 1; \quad 0 \leq i < k-1 \\
  f(v_{5+4i}) &= 1; \quad 0 \leq i < k-1 \\
  f(v_{n-2}) &= 0, \\
  f(v_{n-1}) &= 1, \\
  f(v_n) &= 1,
\end{align*}
\]

In view of the above defined labeling pattern for this case,

we have $v_f(0) = \frac{n+2}{2} = v_f(1)$ and $e_f(0) = \left\lceil \frac{2n-1}{2} \right\rceil$ and $e_f(1) = \left\lfloor \frac{2n-1}{2} \right\rfloor$.

Case 3: $n \equiv 1 \pmod{4}$ and $n \geq 6$.

Let $n = 4k + 1$, 

\[
\begin{align*}
  f(w_1) &= 0, \\
  f(w_2) &= 1, \\
  f(v_1) &= 0, \\
  f(v_{2+4i}) &= 0; \quad 0 \leq i < k-1 \\
  f(v_{3+4i}) &= 0; \quad 0 \leq i < k-1 \\
  f(v_{4+4i}) &= 1; \quad 0 \leq i < k-1 \\
  f(v_{5+4i}) &= 1; \quad 0 \leq i < k-1 \\
  f(v_{n-2}) &= 0, \\
  f(v_{n-1}) &= 1, \\
  f(v_n) &= 1,
\end{align*}
\]

In view of the above defined labeling pattern for this case,

we have $v_f(0) = \frac{n+3}{2} = v_f(1)$ and $e_f(0) = \left\lceil \frac{2n-1}{2} \right\rceil$ and $e_f(1) = \left\lfloor \frac{2n-1}{2} \right\rfloor$. 

\[DF(P_4)\] and \[DF(P_5)\] and their cordial labeling
In view of the above defined labeling pattern for this case,

we have \( v_f(0) = \left\lfloor \frac{n + 2}{2} \right\rfloor, v_f(1) = \left\lceil \frac{n + 2}{2} \right\rceil \) and \( e_f(0) = n \) and \( e_f(1) = n - 1 \).

**Case 4: \( n \equiv 2(\text{mod } 4) \) and \( n \geq 6 \).**

Let \( n = 4k + 2 \),

\[
\begin{align*}
f(w_1) &= 0, \\
f(w_2) &= 1, \\
f(v_1) &= 1, \\
f(v_{2+4i}) &= 0; \ 0 \leq i < k - 1 \\
f(v_{3+4i}) &= 0; \ 0 \leq i < k - 1 \\
f(v_{4+4i}) &= 1; \ 0 \leq i < k - 1 \\
f(v_{5+4i}) &= 1; \ 0 \leq i < k - 1 \\
f(v_{n-3}) &= 1, \\
f(v_{n-2}) &= 1, \\
f(v_{n-1}) &= 0, \\
f(v_n) &= 0, \\
f(v_{n-3}) &= 1, \\
f(v_{n-2}) &= 1, \\
f(v_{n-1}) &= 0, \\
f(v_n) &= 0,
\end{align*}
\]

In view of the above defined labeling pattern for this case,

we have \( v_f(0) = \frac{n + 2}{2} = v_f(1) \) and \( e_f(0) = n - 1 \) and \( e_f(1) = n \).
Case 5: $n \equiv 3(\text{mod } 4)$ and $n \geq 6$.

Let $n = 4k + 3$,

\[
\begin{align*}
  f(w_1) &= 0, \\
  f(w_2) &= 0, \\
  f(v_1) &= 1, \\
  f(v_{2+4i}) &= 0; \ 0 \leq i < k - 1 \\
  f(v_{3+4i}) &= 0; \ 0 \leq i < k - 1 \\
  f(v_{4+4i}) &= 1; \ 0 \leq i < k - 1 \\
  f(v_{5+4i}) &= 1; \ 0 \leq i < k - 1 \\
  f(v_{n-1}) &= 1, \\
  f(v_n) &= 1,
\end{align*}
\]

In view of the above defined labeling pattern for this case,

we have $v_f(0) + 1 = \left\lceil \frac{n+2}{2} \right\rceil = v_f(1)$ and $e_f(0) = n - 1$ and $e_f(1) = n$.

Thus, in each case we have $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$.

Hence, $DS(P_n)$ is a cordial graph. \hfill \blacksquare

Illustration 4.5.2. A Cordial cordial labeling of the graph $DS(P_{15})$ is shown in Figure 4.11.

Theorem 4.5.3. $DS(S_n)$ is a cordial graph.

Proof. Consider $S_n$ with $V(S_n) = \{v_i : 1 \leq i \leq n\}$ where $v_1$ is the apex vertex and $E(S_n) = \{v_1v_n, v_iv_{i+1} : 1 \leq i \leq n - 1\} \cup \{v_1v_j : 3 \leq j \leq n - 1\}$. There are three types of vertices in $S_n$,

(i) vertices of degree 2 namely $v_2, v_n$,

(ii) vertices of degree 3 namely $v_3, v_4, \ldots, v_{n-1}$,

(iii) a vertex of degree $n - 1$ namely $v_1$.
Figure 4.11: $DS(P_{15})$ and its cordial labeling

Thus $V(S_n) = S_1 \cup S_2 \cup T$, where $S_1 = \{v_2, v_n\}$, $S_2 = \{v_3, v_4, \ldots, v_{n-1}\}$ and $T = \{v_1\}$.

Now in order to obtain $DS(S_n)$ from $S_n$, we add $w_1, w_2$ corresponding to $S_1, S_2$. Then $V(DS(S_n)) = V(S_n) \cup \{w_1, w_2\}$ so $|V(DS(S_n))| = n + 2$ and $E(DS(S_n)) = E(S_n) \cup \{v_2w_1, v_nw_1\} \cup \{v_iw_2 : 3 \leq i \leq n - 1\}$ so $|E(DS(S_n))| = 9$ and $|E(DS(S_n))| = 3n - 4, n \geq 5$.

To define vertex labeling $f : V(DS(S_n)) \to \{0, 1\}$ we consider following five cases.

Case 1: $n = 4$.

The graphs $DS(S_4)$ is to be dealt separately and its cordial labeling is shown in Figure 4.12.
Case 2: $n \equiv 0 \pmod{4}$ and $n \geq 5$.

Let $n = 4k$,

\[
\begin{align*}
f(v_{1+4i}) &= 0; \quad 0 \leq i < k \\
f(v_{2+4i}) &= 0; \quad 0 \leq i < k \\
f(v_{3+4i}) &= 1; \quad 0 \leq i < k \\
f(v_{4+4i}) &= 1; \quad 0 \leq i < k \\
f(w_1) &= 0, \\
f(w_2) &= 0,
\end{align*}
\]

In view of the above defined labeling pattern for this case,

we have $v_f(0) = \frac{n + 2}{2} = v_f(1)$ and $e_f(0) = \frac{3n - 4}{2} = e_f(1)$.

Case 3: $n \equiv 1 \pmod{4}$ and $n \geq 5$.

Let $n = 4k + 1$,

\[
\begin{align*}
f(v_{1+4i}) &= 0; \quad 0 \leq i < k \\
f(v_{2+4i}) &= 0; \quad 0 \leq i < k \\
f(v_{3+4i}) &= 1; \quad 0 \leq i < k \\
f(v_{4+4i}) &= 1; \quad 0 \leq i < k \\
f(v_n) &= 1, \\
f(w_1) &= 0, \\
f(w_2) &= 1,
\end{align*}
\]

In view of the above defined labeling pattern for this case,

we have $v_f(0) = \left\lfloor \frac{n + 2}{2} \right\rfloor$, $v_f(1) = \left\lceil \frac{n + 2}{2} \right\rceil$ and $e_f(0) = \left\lceil \frac{3n - 4}{2} \right\rceil$ and $e_f(1) = \left\lfloor \frac{3n - 4}{2} \right\rfloor$.

Case 4: $n \equiv 2 \pmod{4}$ and $n \geq 5$.

Let $n = 4k + 2$,
In view of the above defined labeling pattern for this case, we have \( v_f(0) = \frac{n+2}{2} = v_f(1) \) and \( e_f(0) = \frac{3n-4}{2} = e_f(1) \).

**Case 5:** \( n \equiv 3 (\text{mod} \ 4) \) and \( n \geq 5 \).

Let \( n = 4k + 3 \),

\[
\begin{align*}
f(v_{1+4i}) &= 0; \quad 0 \leq i < k \\
f(v_{2+4i}) &= 0; \quad 0 \leq i < k \\
f(v_{3+4i}) &= 1; \quad 0 \leq i < k \\
f(v_{4+4i}) &= 1; \quad 0 \leq i < k \\
f(v_{n-2}) &= 0, \\
f(v_{n-1}) &= 0, \\
f(v_n) &= 1, \\
f(w_1) &= 0, \\
f(w_2) &= 1,
\end{align*}
\]

In view of the above defined labeling pattern for this case, we have \( v_f(0) = \left\lfloor \frac{n+2}{2} \right\rfloor, v_f(1) = \left\lceil \frac{n+2}{2} \right\rceil \) and \( e_f(0) = \left\lfloor \frac{3n-4}{2} \right\rfloor \) and \( e_f(1) = \left\lceil \frac{3n-4}{2} \right\rceil \).

Thus, in each case we have \( |v_f(0) - v_f(1)| \leq 1 \) and \( |e_f(0) - e_f(1)| \leq 1 \).

Hence, \( DS(S_n) \) is a cordial graph. \( \blacksquare \)
Illustration 4.5.4. A cordial labeling of the graph $DS(S_9)$ is shown in Figure 4.13.

Theorem 4.5.5. $DS(H_n)$ is a cordial graph.

Proof. Consider $H_n$ with $V(S_n) = \{v, v_i, u_i : 1 \leq i \leq n\}$ where $v$ is the apex vertex, $v_i$'s are rim vertices and $u_i$'s are pendant vertices. There are three types of vertices in $H_n$,

(i) vertices of degree 4 namely $v_1, v_2, \ldots, v_n$,

(ii) vertices of degree 1 namely $u_1, u_2, \ldots, u_n$,

(iii) a vertex of degree $n$ namely $v$.

Thus $V(S_n) = S_1 \cup S_2 \cup T$, where $S_1 = \{v_1, v_2, \ldots, v_n\}, S_2 = \{u_1, u_2, \ldots, u_n\}$ and $T = \{v\}$.

Now in order to obtain $DS(H_n)$ from $H_n$, we add $w_1, w_2$ corresponding to $S_1, S_2$. Then $V(DS(H_n)) = V(H_n) \cup \{w_1, w_2\}$ so $|V(DS(H_n))| = 2n + 3$ and $E(DS(H_n)) = E(H_n) \cup \{v_iw_1 : 1 \leq i \leq n\} \cup \{u_iw_2 : 1 \leq i \leq n\}$ so $|E(DS(H_n))| = 5n$. To define vertex labeling $f : V(DS(H_n)) \to \{0, 1\}$ we consider following five cases.

Case 1: $n = 3$.

The graphs $DS(H_3)$ is to be dealt separately and its cordial labeling is shown in Figure 4.14.
Figure 4.14: DS($H_3$) and its cordial labeling

**Case 2:** $n \equiv 0 \pmod{4}$.

Let $n = 4k$,

\[
\begin{align*}
  f(v_{1+4i}) &= 0; \quad 0 \leq i < k \\
  f(v_{2+4i}) &= 0; \quad 0 \leq i < k \\
  f(v_{3+4i}) &= 1; \quad 0 \leq i < k \\
  f(v_{4+4i}) &= 1; \quad 0 \leq i < k \\
  f(v) &= 0, \\
  f(u_{1+2i}) &= 0; \quad 0 \leq i < \frac{n}{2} \\
  f(u_{2+2i}) &= 1; \quad 0 \leq i < \frac{n}{2} \\
  f(w_1) &= 1, \\
  f(w_2) &= 0,
\end{align*}
\]

In view of the above defined labeling pattern for this case,

we have $v_f(0) = \left\lceil \frac{2n + 3}{2} \right\rceil$, $v_f(1) = \left\lfloor \frac{2n + 3}{2} \right\rfloor$ and $e_f(0) = \frac{5n}{2} = e_f(1)$. 
Case 3: $n \equiv 1 (\text{mod} \ 4)$.

Let $n = 4k + 1$,

\[
\begin{align*}
f(v_{1+4i}) &= 0; \quad 0 \leq i < k \\
f(v_{2+4i}) &= 0; \quad 0 \leq i < k \\
f(v_{3+4i}) &= 1; \quad 0 \leq i < k \\
f(v_{4+4i}) &= 1; \quad 0 \leq i < k \\
f(v_n) &= 1, \\
f(u_{1+2i}) &= 0; \quad 0 \leq i < \frac{n-1}{2} \\
f(u_{2+2i}) &= 1; \quad 0 \leq i < \frac{n-1}{2} \\
f(u_n) &= 0, \\
f(v) &= 0, \\
f(w_1) &= 1, \\
f(w_2) &= 0,
\end{align*}
\]

In view of the above defined labeling pattern for this case,

we have $v_f(0) - 1 = \left\lfloor \frac{2n + 3}{2} \right\rfloor = v_f(1)$ and $e_f(0) = \left\lceil \frac{5n}{2} \right\rceil, e_f(1) = \left\lceil \frac{5n}{2} \right\rceil$.

Case 4: $n \equiv 2 (\text{mod} \ 4)$.

Let $n = 4k + 2$,

\[
\begin{align*}
f(v_{1+4i}) &= 0; \quad 0 \leq i < k \\
f(v_{2+4i}) &= 0; \quad 0 \leq i < k \\
f(v_{3+4i}) &= 1; \quad 0 \leq i < k \\
f(v_{4+4i}) &= 1; \quad 0 \leq i < k \\
f(v_{n-1}) &= 1, \\
f(v_n) &= 0, \\
f(u_{1+2i}) &= 0; \quad 0 \leq i < \frac{n-2}{2}
\end{align*}
\]
f(\(u_{2+2i}\)) = 1; \quad 0 \leq i < \frac{n-2}{2}
f(\(u_{n-1}\)) = 0,
f(\(u_n\)) = 1,
f(\(v\)) = 0,
f(\(w_1\)) = 1,
f(\(w_2\)) = 0,

we have \(v_f(0) - 1 = \left\lceil \frac{2n+3}{2} \right\rceil = v_f(1)\) and \(e_f(0) = \frac{5n}{2} = e_f(1)\).

Case 5: \(n \equiv 3 \pmod{4}\).

Let \(n = 4k + 3\),

\[
\begin{align*}
f(v_{1+4i}) & = 0; \quad 0 \leq i < k \\
f(v_{2+4i}) & = 0; \quad 0 \leq i < k \\
f(v_{3+4i}) & = 1; \quad 0 \leq i < k \\
f(v_{4+4i}) & = 1; \quad 0 \leq i < k \\
f(v_{n-2}) & = 0, \\
f(v_{n-1}) & = 0, \\
f(v_n) & = 1, \\
f(u_{1+2i}) & = 0; \quad 0 \leq i < \frac{n-1}{2} \\
f(u_{2+2i}) & = 1; \quad 0 \leq i < \frac{n-1}{2} \\
f(u_n) & = 1, \\
f(v) & = 0, \\
f(w_1) & = 1, \\
f(w_2) & = 0,
\end{align*}
\]

we have \(v_f(0) = \left\lfloor \frac{2n+3}{2} \right\rfloor, v_f(1) = \left\lfloor \frac{2n+3}{2} \right\rfloor\) and \(e_f(0) = \left\lceil \frac{5n}{2} \right\rceil, e_f(1) = \left\lceil \frac{5n}{2} \right\rceil\).

Thus, in each case we have \(|v_f(0) - v_f(1)| \leq 1\) and \(|e_f(0) - e_f(1)| \leq 1\).

Hence, \(DS(H_n)\) is a cordial graph.
Illustration 4.5.6. A cordial cordial labeling of the graph $DS(H_{13})$ is shown in Figure 4.15.

![Figure 4.15: $DS(H_{13})$ and its cordial labeling](image)

**Theorem 4.5.7.** $DS(G_n)$ is a cordial graph.

**Proof.** Let $W_n$ be the wheel with apex vertex $v$ and rim vertices $v_1, v_2, \ldots, v_n$. To obtain the gear graph $G_n$ subdivide each rim edge of wheel by the vertices $u_1, u_2, \ldots, u_n$. Where each $u_i$ is added between $v_i$ and $v_{i+1}$ for $i = 1, 2, \ldots, n-1$ and $u_n$ is added between $v_1$ and $v_n$. There are three types of vertices in $G_n$,

(i) vertices of degree 3 namely $v_1, v_2, \ldots, v_n$,

(ii) vertices of degree 2 namely $u_1, u_2, \ldots, u_n$,

(iii) a vertex of degree $n$ namely $v$.

Thus $V(S_n) = S_1 \cup S_2 \cup T$, where $S_1 = \{v_1, v_2, \ldots, v_n\}$, $S_2 = \{u_1, u_2, \ldots, u_n\}$ and $T = \{v\}$. Now in order to obtain $DS(G_n)$ from $G_n$, we add $w_1, w_2$ corresponding to $S_1, S_2$. Then $V(DS(G_n)) = V(G_n) \cup \{w_1, w_2\}$ so $|V(DS(G_n))| = 2n + 3$ and $E(DS(G_n)) = E(G_n) \cup$
\{v_iw_1 : 1 \leq i \leq n\} \cup \{u_iw_2 : 1 \leq i \leq n\} \text{ so } |E(DS(G_n))| = 5n. To define vertex labeling 
f : V(DS(G_n)) \rightarrow \{0, 1\} we consider following three cases.

**Case 1:** \(n \equiv 0 (mod\ 4)\) or \(n \equiv 2 (mod\ 4)\).

In this case \(n\) is even, so \(2n \equiv 0 (mod\ 4)\).

Therefore \(2n = 4k,\) for some \(k \in \mathbb{N}\).

\[
\begin{align*}
f(v) &= 0, \\
f(v_{1+2i}) &= 0; \quad 0 \leq i < k \\
f(v_{2+2i}) &= 1; \quad 0 \leq i < k \\
f(u_{1+2i}) &= 0; \quad 0 \leq i < k \\
f(u_{2+2i}) &= 1; \quad 0 \leq i < k \\
f(w_1) &= 1, \\
f(w_2) &= 0,
\end{align*}
\]

In view of the above defined labeling pattern for this case,

we have \(v_f(0) = n + 2, v_f(1) = n + 1\) and \(e_f(0) = \frac{5n}{2} = e_f(1)\).

**Case 2:** \(n \equiv 1 (mod\ 4)\).

\(n \equiv 1 (mod\ 4)\) so \(2n \equiv 2 (mod\ 4)\),

Therefore \(2n = 4k + 2,\) for some \(k \in \mathbb{N}\).

\[
\begin{align*}
f(v) &= 0, \\
f(v_{1+2i}) &= 0; \quad 0 \leq i < k \\
f(v_{2+2i}) &= 1; \quad 0 \leq i < k \\
f(v_n) &= 0,
\end{align*}
\]
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\[ f(u_{1+2i}) = 0; \quad 0 \leq i < k \]
\[ f(u_{2+2i}) = 1; \quad 0 \leq i < k \]
\[ f(u_n) = 1, \]
\[ f(w_1) = 1, \]
\[ f(w_2) = 1, \]

In view of the above defined labeling pattern for this case,

we have \( v_f(0) = n + 1, v_f(1) = n + 2 \) and \( e_f(0) = \left\lfloor \frac{5n}{2} \right\rfloor, e_f(1) = \left\lceil \frac{5n}{2} \right\rceil \).

**Case 3**: \( n \equiv 3(\text{mod } 4) \).

\( n \equiv 3(\text{mod } 4) \) so \( 2n \equiv 2(\text{mod } 4) \),

Therefore \( 2n = 4k + 2 \), for some \( k \in \mathbb{N} \).

\[ f(v) = 0, \]
\[ f(v_{1+2i}) = 1; \quad 0 \leq i < k \]
\[ f(v_{2+2i}) = 0; \quad 0 \leq i < k \]
\[ f(v_n) = 1, \]
\[ f(u_{1+2i}) = 1; \quad 0 \leq i < k \]
\[ f(u_{2+2i}) = 0; \quad 0 \leq i < k \]
\[ f(u_n) = 0, \]
\[ f(w_1) = 1, \]
\[ f(w_2) = 0, \]

In view of the above defined labeling pattern for this case,

we have \( v_f(0) = n + 1, v_f(1) = n + 2 \) and \( e_f(0) = \left\lfloor \frac{5n}{2} \right\rfloor, e_f(1) = \left\lceil \frac{5n}{2} \right\rceil \).

Thus, in each case we have \( |v_f(0) - v_f(1)| \leq 1 \) and \( |e_f(0) - e_f(1)| \leq 1 \).

Hence, \( DS(G_n) \) is a cordial graph. \( \blacksquare \)

**Illustration 4.5.8.** A cordial cordial labeling of the graph \( DS(G_{10}) \) is shown in Figure 4.16.
4.6 3-Equitable labeling of graphs

In 1990 Cahit[26] proposed the idea of distributing the vertex and the edge labels among \{0, 1, 2, \ldots, k−1\} as evenly as possible to obtain a generalization of graceful labeling and named it as \(k\)-equitable labeling which is defined as follows.

**4.6.1 \(k\)-equitable labeling**

A vertex labeling of a graph \(G = (V(G), E(G))\) is a function \(f : V(G) \rightarrow \{0, 1, 2, \ldots, k−1\}\) and the value \(f(u)\) is called label of vertex \(u\). For the vertex labeling function \(f : V(G) \rightarrow \{0, 1, \ldots, k−1\}\), the induced function \(f^* : E(G) \rightarrow \{0, 1, \ldots, k−1\}\) defined as
\[ f^*(e = uv) = |f(u) - f(v)| \]

which satisfies the conditions

\[
\begin{align*}
|v_f(i) - v_f(j)| &\leq 1 \\
|e_f(i) - e_f(j)| &\leq 1
\end{align*}
\]

where \( 0 \leq i, j \leq k - 1 \)

where \( v_f(i) \) and \( e_f(i) \) denotes the number of vertices and the number of edges having label \( i \) under \( f \) and \( f^* \) respectively. Such labeling \( f \) is called \( k \)-equitable labeling for the graph \( G \). A graph which admits \( k \)-equitable labeling is called \( k \)-equitable graph.

Note that \( G(V; E) \) is graceful if and only if it is \(|E| + 1\)-equitable,

Obviously 2-equitable labeling is cordial labeling which is already discussed in section 4.2.

When \( k = 3 \) the labeling is called 3-equitable labeling. The remaining part of this chapter is devoted to the discussion of 3-equitable labeling of graphs.

**Illustration 4.6.1.** A 3-Equitable labeling of \( W_4 \) is shown in [Figure 4.17].

![Figure 4.17: W_4 and its 3-equitable labeling](image)

### 4.6.2 Some known results on 3-Equitable labeling

- Cahit[25],[26] proved that
  - \( C_n \) is 3-equitable if and only if \( n \not\equiv 3(mod\ 6) \).
  - An Eulerian graph with \( q \equiv 3(mod\ 6) \) is not 3-equitable where \( q \) is the number of edges.
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- All caterpillars are 3-equitable.
- A triangular cactus with \( n \) blocks is 3-equitable if and only if \( n \) is even. (Conjecture)
- Every tree with fewer than five end vertices has a 3-equitable labeling.

- Seoud and Abdel Maqsoud[66] proved that
  - A graph with \( p \) vertices and \( q \) edges in which every vertex has odd degree is not 3-equitable if \( p \equiv 0(\text{mod}3) \) and \( q \equiv 3(\text{mod}6) \).
  - All fans except \( P_2 + K_1 \) are 3-equitable.
  - \( P_n^2 \) is 3-equitable for all \( n \) except 3.
  - \( K_{m,n} \) (where \( 3 \leq m \leq n \)) is 3-equitable if and only if \( (m,n) = (4,4) \).

  - Helms \( H_n \) (where \( n \geq 4 \)) are 3-equitable graphs.
  - Flower graph admit 3-equitable labeling
  - The one-point union of any number of helms is 3-equitable graph.
  - The one-point union of any number of copies of \( K_4 \) is a 3-equitable graph.

- Youssef[100] proved that \( W_n = C_n + K_1 \) is 3-equitable for all \( n \geq 4 \).

- Vaidya et al.[82, 83] have discussed wheel related 3-equitable graphs.

- Vaidya et al.[80] have discussed some shell related 3-equitable graphs.

- Vaidya et al.[79, 81] have discussed some star related 3-equitable graphs.

- Vaidya et al.[94] proved following graphs are 3-equitable.
  - The shadow graph of \( C_n \) is 3-equitable graph except for \( n = 3 \) and 5.
  - The shadow graph of \( P_n \) is 3-equitable graph except for \( n = 3 \).
  - The middle graph of \( P_n \) is 3-equitable graph.
  - The middle graph of \( C_n \) is 3-equitable graph for \( n \) even and not 3-equitable for \( n \) odd.
4.7 3-Equitable labeling of some star and bistar related graphs

This section is aimed to report some results on 3-equitable labeling of star and bistar related graphs which are investigated by us.

**Theorem 4.7.1.** $S'(K_{1,n})$ is 3-equitable graph.

**Proof.** Let $v_1, v_2, v_3, \ldots, v_n$ be the pendant vertices and $v$ be the apex vertex of $K_{1,n}$ and $u, u_1, u_2, u_3, \ldots, u_n$ are added vertices corresponding to $v, v_1, v_2, v_3, \ldots, v_n$ to obtain $S'(K_{1,n})$. Let $G$ be the graph $S'(K_{1,n})$ then $|V(G)| = 2n + 2$ and $|E(G)| = 3n$. To define $f: V(G) \to \{0, 1, 2\}$ we consider following three cases.

**Case 1:** $n \equiv 0 (\text{mod } 3)$

\[
\begin{align*}
  f(v) &= 2, \\
  f(u) &= 0, \\
  f(v_i) &= 0; \quad 1 \leq i \leq \frac{n}{3} + 1 \\
  f\left(\frac{v_n}{3} + i\right) &= 1; \quad 1 \leq i \leq \frac{n}{3} - 1 \\
  f\left(\frac{2v_n}{3} + i\right) &= 2; \quad 1 \leq i \leq \frac{n}{3} \\
  f(u_i) &= 0; \quad 1 \leq i \leq \frac{n}{3} - 1 \\
  f\left(\frac{u_n}{3} + i\right) &= 1; \quad 1 \leq i \leq \frac{n}{3} + 2 \\
  f\left(\frac{2u_n}{3} + i\right) &= 2; \quad 1 \leq i \leq \frac{n}{3} - 1
\end{align*}
\]

In view of the above labeling pattern we have

\[
\begin{align*}
  v_f(0) &= v_f(1) = \frac{2n}{3} + 1 = v_f(2) + 1 \\
  e_f(0) &= e_f(1) = e_f(2) = n
\end{align*}
\]
Case 2: 

\[ n \equiv 1 \pmod{3} \]

Since \( n \equiv 1 \pmod{3} \), \( n = 3k + 1 \) some \( k \in \mathbb{N} \).

\[
\begin{align*}
    f(v) &= 2, \\
    f(u) &= 0, \\
    f(v_i) &= 0; \quad 1 \leq i \leq k + 1 \\
    f(v_{k+1+i}) &= 1; \quad 1 \leq i \leq k - 1 \\
    f(v_{2k+i}) &= 2; \quad 1 \leq i \leq k + 1 \\
    f(u_i) &= 0; \quad 1 \leq i \leq k - 1 \\
    f(u_{k-1+i}) &= 1; \quad 1 \leq i \leq k + 3 \\
    f(u_{2k+2+i}) &= 2; \quad 1 \leq i \leq k - 1
\end{align*}
\]

In view of the above labeling pattern we have

\[
\begin{align*}
    v_f(0) &= v_f(2) = \frac{2n + 1}{3} = v_f(1) - 1 \\
    e_f(0) &= e_f(1) = e_f(2) = n
\end{align*}
\]

Case 3: 

\[ n \equiv 2 \pmod{3} \]

Since \( n \equiv 2 \pmod{3} \), \( n = 3k + 2 \) some \( k \in \mathbb{N} \).

\[
\begin{align*}
    f(v) &= 2, \\
    f(u) &= 0, \\
    f(v_i) &= 0; \quad 1 \leq i \leq k + 1 \\
    f(v_{k+1+i}) &= 1; \quad 1 \leq i \leq k \\
    f(v_{2k+1+i}) &= 2; \quad 1 \leq i \leq k + 1 \\
    f(u_i) &= 0; \quad 1 \leq i \leq k \\
    f(u_{k+i}) &= 1; \quad 1 \leq i \leq k + 2 \\
    f(u_{2k+2+i}) &= 2; \quad 1 \leq i \leq k
\end{align*}
\]

In view of the above labeling pattern we have
Thus in each case we have \(|v_f(i) - v_f(j)| \leq 1\) and \(|e_f(i) - e_f(j)| \leq 1\), for all \(0 \leq i, j \leq 2\).

Hence, \(S'(K_{1,n})\) is a 3-equitable graph.

**Illustration 4.7.2.** 3-equitable labeling of the graph \(S'(K_{1,7})\) is shown in Figure 4.18.

**Theorem 4.7.3.** \(S'(B_{n,n})\) is 3-equitable graph.

**Proof.** Consider \(B_{n,n}\) with vertex set \(\{u, v, u_i, v_i, 1 \leq i \leq n\}\) where \(u_i, v_i\) are pendant vertices. In order to obtain \(S'(B_{n,n})\), add \(u', v', u'_i, v'_i\) vertices corresponding to \(u, v, u_i, v_i\) where, \(1 \leq i \leq n\). If \(G = S'(B_{n,n})\) then \(|V(G)| = 4(n+1)\) and \(|E(G)| = 6n+3\). To define \(f : V(G) \to \{0, 1, 2\}\) we consider following four cases.

**Case 1: \(n = 2, 5\)**

The graphs \(S'(B_{2,2})\) and \(S'(B_{5,5})\) are to be dealt separately and their 3-equitable labeling is shown in Figure 4.19 and Figure 4.20.
Figure 4.19: $S'(B_{2,2})$ and its 3-equitable labeling

Figure 4.20: $S'(B_{5,5})$ and its 3-equitable labeling
Case 2: \( n \equiv 0 \pmod{3} \)

Since \( n \equiv 0 \pmod{3} \), \( n = 3k \) some \( k \in \mathbb{N} \).

\[
\begin{align*}
    f(u) &= 0, \\
    f(u') &= 0, \\
    f(v) &= 1, \\
    f(v') &= 1, \\
    f(u_i) &= 2; \quad 1 \leq i \leq n \\
    f(u'_1) &= 2; \\
    f(u'_1 + i) &= 1; \quad 1 \leq i \leq k \\
    f(u'_{k+1+i}) &= 0; \quad 1 \leq i \leq n-k-1 \\
    f(v_i) &= 0; \quad 1 \leq i \leq 2(k-1) \\
    f(v_{2k-2+i}) &= 1; \quad 1 \leq i \leq n-2k+2 \\
    f(v'_1) &= 2; \quad 1 \leq i \leq k \\
    f(v'_{k+i}) &= 1; \quad 1 \leq i \leq 2k-2 \\
    f(v'_{2k}) &= 0; \quad i = n, n-1
\end{align*}
\]

In view of the above labeling pattern we have

\[
\begin{align*}
    v_f(0) &= v_f(2) = 4k + 1 = v_f(1) - 1 \\
    e_f(0) &= e_f(1) = e_f(2) = 2n + 1
\end{align*}
\]

Case 3: \( n \equiv 1 \pmod{3} \)

Since \( n \equiv 1 \pmod{3} \), \( n = 3k + 1 \) some \( k \in \mathbb{N} \).

\[
\begin{align*}
    f(u) &= 0, \\
    f(u') &= 0, \\
    f(v) &= 1, \\
    f(v') &= 1,
\end{align*}
\]
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\[ f(u_i) = 2; \quad 1 \leq i \leq n \]
\[ f(u'_1) = 2; \]
\[ f(u'_{1+i}) = 1; \quad 1 \leq i \leq k \]
\[ f(u'_{k+1+i}) = 0; \quad 1 \leq i \leq 2k \]
\[ f(v_i) = 1; \quad 1 \leq i \leq k+2 \]
\[ f(v_{k+i}) = 0; \quad 1 \leq i \leq 2k-1 \]
\[ f(v'_i) = 2; \quad 1 \leq i \leq k \]
\[ f(v'_{k+i}) = 1; \quad 1 \leq i \leq 2k-1 \]
\[ f(v'_i) = 0; \quad i = n, n-1 \]

In view of the above labeling pattern we have

\[ v_f(0) = v_f(1) = 4k + 3 = v_f(2) + 1 \]
\[ e_f(0) = e_f(1) = e_f(2) = 2n + 1 \]

**Case 4: \( n \equiv 2 \ (\text{mod } 3), n \neq 5. \)**

Since \( n \equiv 2 \ (\text{mod } 3), n = 3k + 2 \) some \( k \in \mathbb{N} - \{1\}. \)

\[ f(u) = 0, \]
\[ f(u') = 0, \]
\[ f(v) = 1, \]
\[ f(v') = 1, \]
\[ f(u_i) = 2; \quad 1 \leq i \leq n \]
\[ f(u'_1) = 2; \]
\[ f(u'_{1+i}) = 1; \quad 1 \leq i \leq k+1 \]
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\[ f(u'_{k+2+i}) = 0; \quad 1 \leq i \leq 2k \]
\[ f(v_i) = 1; \quad 1 \leq i \leq k + 4 \]
\[ f(v_{k+4+i}) = 0; \quad 1 \leq i \leq 2k - 2 \]
\[ f(v'_i) = 2; \quad 1 \leq i \leq k + 1 \]
\[ f(v'_{k+1+i}) = 1; \quad 1 \leq i \leq 2k - 3 \]
\[ f(v'_i) = 0; \quad n - 3 \leq i \leq n \]

In view of the above labeling pattern we have

\[ v_f(0) = v_f(1) = 4(k + 1) = v_f(2) \]
\[ e_f(0) = e_f(1) = e_f(2) = 2n + 1 \]

Thus in each case we have \(|v_f(i) - v_f(j)| \leq 1\) and \(|e_f(i) - e_f(j)| \leq 1\), for all \(0 \leq i, j \leq 2\).

Hence, \(S'(B_{n,n})\) is a 3-equitable graph. ■

**Illustration 4.7.4.** 3-equitable labeling of the graph \(S'(B_{6,6})\) is shown in Figure 4.21.

---

**Figure 4.21:** \(S'(B_{6,6})\) and its 3-equitable labeling
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**Theorem 4.7.5.** $D_2(B_{n,n})$ is a 3-equitable graph.

**Proof.** Consider two copies of $B_{n,n}$. Let $\{u, v, u_i, v_i, 1 \leq i \leq n\}$ and $\{u', v', u'_i, v'_i, 1 \leq i \leq n\}$ be the corresponding vertex sets of each copy of $B_{n,n}$. Let $G$ be the graph $D_2(B_{n,n})$ then $|V(G)| = 4(n + 1)$ and $|E(G)| = 4(2n + 1)$.

To define $f : V(G) \to \{0, 1, 2\}$ we consider following three cases.

**Case 1: $n \equiv 0 (mod \ 3)$**

Since $n \equiv 0 (mod \ 3)$, $n = 3k$ some $k \in \mathbb{N}$.

$$f(u) = 0,$$
$$f(u') = 2,$$
$$f(v) = 0,$$
$$f(v') = 2,$$
$$f(u_i) = 0; \quad 1 \leq i \leq 2k + 1$$
$$f(u_{2k+1+i}) = 1; \quad 1 \leq i \leq k - 1$$
$$f(u'_i) = 1; \quad 1 \leq i \leq 2(k - 1)$$
$$f(u'_{2(k-1)+i}) = 2; \quad 1 \leq i \leq k + 2$$

$$f(v_i) = 0; \quad 1 \leq i \leq 2k - 1$$
$$f(v_{2k-1+i}) = 1; \quad 1 \leq i \leq k + 1$$
$$f(v'_i) = 1; \quad 1 \leq i \leq 3$$
$$f(v'_{3+i}) = 2; \quad 1 \leq i \leq n - 3$$

$$v_f(1) = v_f(2) = \frac{4(n + 1) - 1}{3} = v_f(0) - 1$$
$$e_f(0) = e_f(2) = \frac{8n + 3}{3} = e_f(1) - 1$$

**Case 2: $n \equiv 1 (mod \ 3)$**

Since $n \equiv 1 (mod \ 3)$, $n = 3k + 1$ some $k \in \mathbb{N}$.
\begin{align*}
&f(u) = 0, \\
&f(u') = 2, \\
&f(v) = 0, \\
&f(v') = 2, \\
&f(u_i) = 0; \quad 1 \leq i \leq n \\
&f(u'_i) = 2; \quad 1 \leq i \leq n \\
&f(v_i) = 0; \quad 1 \leq i \leq k \\
&f(v_{k+i}) = 1; \quad 1 \leq i \leq n-k \\
&f(v'_j) = 1; \quad 1 \leq i \leq 2k+1 \\
&f(v'_{2k+1+i}) = 2; \quad 1 \leq i \leq k \\
&\end{align*}

\begin{align*}
v_f(0) &= v_f(2) = \frac{4(n+1) + 1}{3} = v_f(1) + 1 \\
e_f(0) &= e_f(2) = e_f(1) = \frac{8n+4}{3} \\
\end{align*}

**Case 3: \( n \equiv 2 (mod \ 3) \)**

Since \( n \equiv 2 (mod \ 3) \), \( n = 3k + 2 \) some \( k \in \mathbb{N} \cup \{0\} \).

\begin{align*}
&f(u) = 0, \\
&f(u') = 0, \\
&f(v) = 0, \\
&f(v') = 1, \\
&f(u_i) = 0; \quad 1 \leq i \leq k+1 \\
&f(u_{k+1+i}) = 1; \quad 1 \leq i \leq k \\
&f(u_{2k+1+i}) = 2; \quad 1 \leq i \leq k+1 \\
&f(u'_i) = 2; \quad 1 \leq i \leq n \\
&f(v_i) = 0; \quad 1 \leq i \leq n-2 \\
&f(v_{n-1}) = 1; \\
&f(v_n) = 2; \\
&f(v'_i) = 1; \quad 1 \leq i \leq n \end{align*}
\[ v_f(0) = v_f(1) = v_f(2) = \frac{4(n+1)}{3} \]
\[ e_f(0) = e_f(2) = \frac{8n+4+1}{3} = e_f(1) + 1 \]

Thus in each case we have \(|v_f(i) - v_f(j)| \leq 1\) and \(|e_f(i) - e_f(j)| \leq 1\), for all \(0 \leq i, j \leq 2\).

Hence, \(D_2(B_{n,n})\) is a 3-equitable graph.

\[ \blacksquare \]

**Illustration 4.7.6.** 3-equitable labeling of the graph \(D_2(B_{5,5})\) is shown in FIGURE 4.22.

\[ \text{FIGURE 4.22: } D_2(B_{5,5}) \text{ and its 3-equitable labeling} \]

**Theorem 4.7.7.** \(K_{1,n} \ast K_{1,n}\) is 3-equitable graph for \(n \equiv 0 (mod\ 3)\) and \(n \equiv 1 (mod\ 3)\).

**Proof.** Consider two copies of \(K_{1,n}\) with vertex sets \(\{u, u_i : 1 \leq i \leq n\}\) and \(\{v, v_i : 1 \leq i \leq n\}\) respectively where \(u_i, v_i, 1 \leq i \leq n\) are the pendant vertices. Let \(G\) be the graph \(K_{1,n} \ast K_{1,n}\) with \(V(G) = \{u, v, u_i, v_i : 1 \leq i \leq n\}\) and \(E(G) = \{uv, uu_i, uv_i, vv_i, vu_i : 1 \leq i \leq n\}\) then \(|V(G)| = 2n + 2\) and \(|E(G)| = 4n + 1\). To define \(f : V(G) \rightarrow \{0, 1, 2\}\), we consider following two cases.
Case 1: $n \equiv 0 \pmod{3}$

Since $n \equiv 0 \pmod{3}$, $n = 3k$ some $k \in \mathbb{N}$.

$$f(u) = 2,$$
$$f(v) = 0,$$
$$f(u_i) = 1; \quad 1 \leq i \leq k$$
$$f(u_{k+i}) = 2; \quad 1 \leq i \leq 2k$$
$$f(v_i) = 0; \quad 1 \leq i \leq 2k$$
$$f(v_{2k+i}) = 1; \quad 1 \leq i \leq k$$

$$v_f(0) = v_f(2) = \frac{2n + 3}{3} = v_f(1) + 1$$
$$e_f(0) = e_f(1) = \frac{4n}{3} = e_f(2) - 1$$

Case 2: $n \equiv 1 \pmod{3}$

Since $n \equiv 1 \pmod{3}$, $n = 3k + 1$ some $k \in \mathbb{N}$.

$$f(u) = 2,$$
$$f(v) = 0,$$
$$f(u_i) = 1; \quad 1 \leq i \leq k$$
$$f(u_{k+i}) = 2; \quad 1 \leq i \leq 2k + 1$$
$$f(v_i) = 0; \quad 1 \leq i \leq 2k$$
$$f(v_{2k+i}) = 1; \quad 1 \leq i \leq k + 1$$

$$v_f(0) = v_f(1) = \frac{2n + 1}{3} = v_f(2) - 1$$
$$e_f(1) = e_f(2) = \frac{4n + 2}{3} = e_f(0) + 1$$

Thus in both the cases we have $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$, for all $0 \leq i, j \leq 2$.

Hence, $K_{1,n} \ast K_{1,n}$ is a 3-equitable graph for $n \equiv 0 \pmod{3}$ and $n \equiv 1 \pmod{3}$. ⊡
Illustration 4.7.8. 3-equitable labeling of the graph $K_{1,7} \ast K_{1,7}$ is shown in Figure 4.23.

Theorem 4.7.9. $K_{1,n} \ast K_{1,n}$ is not a 3-equitable graph for $n \equiv 2 \pmod{3}$.

Proof. Let $G$ be the graph $K_{1,n} \ast K_{1,n}$ then $|V(G)| = 2n + 2$ and $|E(G)| = 4n + 1$. Here $n \equiv 2 \pmod{3}$ therefore $n = 3k_1 + 2$ for some $k_1 \in \mathbb{N}$. Hence $|V(G)| = 3k$ and $|E(G)| = 6k - 3$ where $k = 2k_1 + 2$. So if $K_{1,n} \ast K_{1,n}$ is 3-equitable then we must have $v_f(0) = v_f(1) = v_f(2) = k$ and $e_f(0) = e_f(1) = e_f(2) = 2k - 1$.

In $K_{1,n} \ast K_{1,n}$, note that each $u_i$ and $v_i$ $(1 \leq i \leq n)$ are adjacent to $u$ and $v$ both moreover $u$ and $v$ are adjacent vertices. It is obvious that any edge will have label 1 if it is incident to one vertex with label 1. Following Table 4.1 shows all possible assignments of vertex label. From the Table 4.1 (column 6) we can observe that the edge condition violates in all the possible assignments.

Hence, $K_{1,n} \ast K_{1,n}$ is not a 3-equitable graph for $n \equiv 2 \pmod{3}$.

\[ \Box \]
### Table 4.1

<table>
<thead>
<tr>
<th>$u$</th>
<th>$v$</th>
<th>$u_i$’s and $v_i$’s</th>
<th>$e_f(1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>vertex label</td>
<td>vertex label</td>
<td>$v'_f(0)$</td>
<td>$v'_f(1)$</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>$k - 2$</td>
<td>$k$</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>$k - 1$</td>
<td>$k - 1$</td>
</tr>
<tr>
<td>0</td>
<td>2</td>
<td>$k - 1$</td>
<td>$k$</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>$k$</td>
<td>$k - 2$</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>$k$</td>
<td>$k - 1$</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>$k$</td>
<td>$k$</td>
</tr>
</tbody>
</table>

In above table $v'_f(j)$ = number of vertices having label $j$ for $u_i$ and $v_i$ where $1 \leq i \leq n$ and $0 \leq j \leq 2$.

### 4.8 Conclusion and Scope of Further Research

This chapter was focused to investigate equitable labeling for larger graphs obtained from the standard graph by means of various graph operations.

To investigate necessary and sufficient conditions for a particular graph to admit cordial, 3-equitable (in general $k$-equitable) labeling is an open area of research.

The next chapter is targeted to discuss some variants of cordial labeling.