Chapter 4

Peristaltic transport of a Bingham fluid through a porous medium in a channel under the effect of a magnetic field

4.1 Introduction

Physiological fluids in animal and human bodies are in general, pumped by the continuous contraction and expansion of the ducts. These contractions and expansion are expected to be caused by peristaltic waves that propagate along the walls of the ducts. In general during peristaltic the fluid is pumped from lower pressure to higher pressure. Peristaltic transport occurs widely in the stomach, ureter, bile duct, small vessels etc. The principle of peristalsis is used by roller pumps for pumping fluids without being contaminated due to the contact with the pumping machinery. The initial work on peristalsis is done by using lab frame analysis. The important characteristics of peristaltic pumping namely trapping and reflux are studied in detail by Shapiro et al. [70] for the peristaltic flow of a viscous fluid through a tube and a channel.

Peristaltic flow of non-Newtonian fluids in a tube was first studied by Raju and Devanathan [64], considering the blood as a power-law fluid and they obtained the solution for the stream function as a power series in terms of the amplitude of deformation and numerically evaluated the stream function and velocity components. Subsequently Raju and Devanathan [65] have investigated the peristaltic transport of a viscoelastic fluid in a tube considering the constitute equation of simple fluid with fading memory. Peristaltic flow of a power-law fluid in a channel under long wavelength approximation was studied by Radhakrishnamacharya [63]. Srivastava and Srivastava [81] was investigated the peristaltic flow of blood in small blood
vessels using the Casson fluid model. Peristaltic transport of chyme modeled by a non-Newtonian fluid or power-law fluid in small intestine and oesophagus has been studied by Lew et al. [49], Srivastava and Srivastava [82] and Misra and Pandey [59]. The peristaltic transport of a power-law fluid in the male reproductive tract was discussed by Srivastava and Srivastava [83]. Peristaltic pumping of two layered power-law fluids in cylindrical tube was investigated by Usha and Ramachandra Rao [90]. Kapur [46] have made theoretical investigations regarding blood as a Casson and Herschel–Bulkley fluids. Lew et al. [49] reported that chyme is a non-Newtonian material having plastic like properties. Recently, Sreenadh et al. [76] have studied the effect of yield stress on peristaltic pumping of non-Newtonian fluids in a channel. The non-Newtonian fluids are Bingham and Herschel-Bulkley fluids. Vajravelu et al[91],[92] made a detailed study on the effect of yield stress on peristaltic pumping of a Herschel–Bulkley fluid in an inclined tube and a channel. All these investigations are confined to hydrodynamic study of a physiological fluid obeying some yield stress model.

So far, no investigation is made to the peristaltic flow of MHD Non-Newtonian fluid filling the porous space in a channel. Since, the MHD effect on peristaltic flow is important in technology (MHD pumps) and biology (blood flow). Such analysis is of great value in medical research. The influence of moving magnetic field on blood flow was studied by Stud et al. [84] and they have observed that the effect of suitable moving magnetic field accelerates the speed of blood. Srivastava and Agrawal [80] and Prasad and Ramacharyulu [62] have observed that by considering the blood as an electrically conducting fluid constitutes a suspension of red cell in plasma. Also, Agrawal and Anwaruddin [3] studied the effect of magnetic field on the peristaltic flow of blood using long wavelength approximation method.
and observed for the flow of blood in arteries with arterial stenosis or arteriosclerosis, that the influence of magnetic field may be utilized as blood pump in carrying out cardiac operations. Li et al. [51] have studied an impulsive magnetic filed in the combined therapy of patients with stone fragments in the upper urinary tract. It was found that the Impulsive Magnetic Field activates the impulsive activity of the ureteral smooth muscles in 100% of cases. The peristaltic transport of blood under effect of a magnetic field in non uniform channels was studied by Mekheimer [56]. Elshahed and Haroun [11] discussed peristaltic flow of a Johnson Segalman fluid under the effect of a magnetic field. Hayat and Ali [37] studied peristaltic flow of Jeffrey fluid under the effect of a magnetic field in tube. Further more, flow through a porous medium has practical applications especially in geophysical fluid dynamics. Examples of natural porous medium are beach sand, sandstone, limestone, rye bread, wood, the human lung, bile duct, gall bladder with stones and in small blood vessels. In view of this, El Shehawey eta l. [12] investigated the peristaltic flow of a Newtonian fluid through a porous medium. Mekheimer and Al-Arabi [55] have discussed the peristaltic flow of a Newtonian fluid through a porous medium in a channel under the effect of magnetic field. The peristaltic flow of electrically conducting fluid through a porous medium in a planar channel was investigated by Hayat et al. [35]. Sudhakar Reddy et al. [87] have studied peristaltic motion of a Carreau fluid through a porous medium in a channel under the effect of a magnetic field.

Motivated by these, the peristaltic transport of a conducting Bingham fluid through a porous medium in a channel is studied under long wavelength and low Reynolds number assumptions. The expressions for the velocity field in the plug flow and non- plug flow regions, the pressure rise in the channel and the volume flow rate
are obtained analytically. The effects of the magnetic field, Darcy number, yield stress and amplitude ratio on the axial pressure gradient, pumping characteristics and frictional force are discussed through graphs in detail.

4.2 Mathematical formulation

We consider the peristaltic pumping of a conducting Bingham fluid flow through a porous medium in a channel of half-width $a$. A longitudinal train of progressive sinusoidal waves takes place on the upper and lower walls of the channel. For simplicity, we restrict our discussion to the half-width of the channel as shown in the figure. The region between $y = 0$ and $y = y_0$ is called plug flow region. In the plug flow region, $|\tau_{yx}| \leq \tau_y$. In the region between $y = y_0$ and $y = H$, $|\tau_{yx}| > \tau_y$. Fig. 4.1 shows the physical model of the problem. The wall deformation is given by

$$H(X, t) = a + b \cos \left[ \frac{2\pi}{\lambda} (X - ct) \right]$$

where $b$ is the amplitude, $\lambda$ the wavelength and $c$ is the wave speed.

Under the assumptions that the channel length is an integral multiple of the wavelength $\lambda$ and the pressure difference across the ends of the channel is a constant, the flow becomes steady in the wave frame $(x, y)$ moving with velocity $c$ away from the fixed (laboratory) frame $(X, Y)$. The transformation between these two frames is given by

$$x = X - ct, y = Y, u(x, y) = U - c, \text{ and } v(x, y) = V$$

where $U$ and $V$ are velocity components in the laboratory frame and $u$ and $v$ are velocity components in the wave frame.
Fig: 4.1 The Physical model

The equations governing the flow in wave frame are given by

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

(4.2.3)

$$\rho \left( u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} - \sigma B^2_0(u + c) - \frac{\mu}{k}(u + c)$$

(4.2.4)

$$\rho \left( u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} \right) = -\frac{\partial p}{\partial y} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} - \frac{\mu}{k} v$$

(4.2.5)

where $\tau_{ij}$ denote the stresses. For the Bingham Plastic these are related to the strains through the constitutive model,

$$\tau_{ij} = \left( \mu + \frac{\tau_y}{\dot{\gamma}} \right) \dot{\gamma}_{ij} \quad \text{for} \quad \tau \geq \tau_y$$

(4.2.6)

and

$$\tau_{ij} = \dot{\gamma}_{ij} = 0 \quad \text{for} \quad \tau < \tau_y$$

(4.2.7)
where $\dot{y}_{ij}$ is the rate of strain tensor,

$$
\dot{y}_{ij} = \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i},
$$

(4.2.8)

$$
\tau = \sqrt{\frac{1}{2} \tau_{ij} \tau_{ij}} \quad \text{and} \quad \dot{\gamma} = \sqrt{\frac{1}{2} \sum \sum \dot{y}_{ij} \dot{y}_{ij}} = \sqrt{2 \left[ \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial v}{\partial y} \right)^2 \right] + \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)^2}
$$

(4.2.9)

Introducing the non-dimensional quantities

$$
\bar{X} = \frac{x}{\lambda}, \quad \bar{Y} = \frac{y}{a}, \quad \bar{U} = \frac{u}{c}, \quad \bar{V} = \frac{v}{c}, \quad p = \frac{\rho a^2}{\mu \lambda}, \quad \bar{t} = \frac{c t}{\lambda}, \quad h = \frac{H}{a}, \phi = \frac{b}{a},
$$

$$
\bar{q} = \frac{q}{ac}
$$

Into Eqs. (4.2.3)-(4.2.5), we get (dropping the bars)

$$
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0
$$

(4.2.10)

$$
Re \delta \left( u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = - \frac{\partial p}{\partial x} + \delta^2 \frac{\partial^2 u}{\partial x^2} + \frac{\partial \tau_{xx}}{\partial y} - \left[ M^2 + \frac{1}{Da} \right] (u + 1)
$$

(4.2.11)

$$
Re \delta^2 \left( u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} \right) = - \frac{\partial p}{\partial y} + \delta^2 \frac{\partial^2 v}{\partial x^2} + \frac{\partial \tau_{xy}}{\partial y} - \frac{\delta^2}{Da} v
$$

(4.2.12)

where

$$
\tau_{ij} = \left( \mu + \frac{\tau_0}{y} \right) \dot{y}_{ij} \quad \text{for} \quad \tau \geq \tau_y
$$

(4.2.13)

$$
\tau_{ij} = \dot{y}_{ij} = 0 \quad \text{for} \quad \tau < \tau_y
$$

(4.2.14)

$$
\dot{\gamma}_{xy} = \dot{\gamma}_{yx} = \frac{\partial u}{\partial y} + \delta^2 \frac{\partial v}{\partial x}
$$

(4.2.15)

$$
\dot{\gamma}_{xx} = 2 \delta \frac{\partial u}{\partial x}, \quad \dot{\gamma}_{yy} = 2 \delta \frac{\partial v}{\partial y}
$$

(4.2.16)

$$
\dot{\gamma} = \sqrt{2 \delta^2 \left[ \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial v}{\partial y} \right)^2 \right] + \left( \frac{\partial u}{\partial y} + \delta^2 \frac{\partial v}{\partial x} \right)^2}
$$

(4.2.17)

$$
\tau = \sqrt{\tau_{xy}^2 + \delta^2 \tau_{xx}^2}
$$

(4.2.18)

$$
Re = \frac{\rho ac}{\mu} \quad \text{is the Reynolds number,} \quad M = \sqrt{\frac{c}{\mu} B_0 a} \quad \text{is the Hartman number and} \quad Da = \frac{k}{a^2} \quad \text{is the Darcy number}.
$$
Under the assumptions of long wavelength and low Reynolds number, the Eqs. (4.2.11) and (4.2.12) reduce to

\[
0 = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} - N^2(u + 1) \tag{4.2.19}
\]

\[
0 = -\frac{\partial p}{\partial y} \tag{4.2.20}
\]

where \( N^2 = M^2 + \frac{1}{\partial a} \)

\[
\tau_{xy} = \frac{\partial u}{\partial y} - \tau_y \quad \text{for} \quad \tau \geq \tau_y \tag{4.2.21}
\]

\[
\tau_{xy} = 0 \quad \text{for} \quad \tau < \tau_y \tag{4.2.22}
\]

Here \( \tau_y \) is the yield stress.

Here Eq. (4.2.20) indicates that \( p \) is independent of \( y \) and depends only upon \( x \).

Therefore, Eq. (4.2.19), can be rewritten as

\[
\frac{d^2u}{dy^2} - N^2 u = \frac{\partial p}{\partial x} + N^2 \tag{4.2.23}
\]

The non-dimensional boundary conditions are

\[
\frac{\partial u}{\partial y} = \tau_y \quad \text{at} \quad y = 0 \tag{4.2.24}
\]

\[
u = -1 \quad \text{at} \quad y = h = 1 + \phi \cos 2\pi x \tag{4.2.25}
\]

The volume flux \( q \) through each cross section in the wave frame is given by

\[
q = \int_{y_0}^{y_0 + \phi \cos 2\pi x} u_pdy + \int_{y_0}^{h} udy \tag{4.2.26}
\]

The instantaneous volume flow rate \( Q(X, t) \) in the laboratory frame between the centre line and the wall is

\[
Q(X, t) = \int_{0}^{\phi \cos 2\pi x} U dY = \int_{0}^{h} (u + 1)dy = q + h \tag{4.2.27}
\]

The average volume flow rate \( \bar{Q} \) over one wave period \( (T = \frac{\lambda}{c}) \) of the peristaltic wave is defined as

\[
\bar{Q} = \frac{1}{t} \int_{0}^{T} Q(X, t) dt = q + 1 \tag{4.2.28}
\]
4.3 Solution

Solving equation (4.2.23) using the boundary conditions (4.2.24) and (4.2.25) we obtain the velocity as

\[ u = \frac{1}{N^2} \frac{dp}{dx} \left[ \cosh Ny - 1 \right] - \tau_y \tanh Nh \cosh Ny + \frac{\tau_x}{N} \sinh Ny - 1 \]  \hspace{1cm} (4.3.1)

We find the upper limit of plug flow region using the boundary condition

\[ \frac{\partial u}{\partial y} = 0 \text{ at } y = y_0. \] So, we have

\[ \tau_y = - \frac{1}{N} \frac{dp}{dx} \frac{\sinh Ny_0}{\cosh N(h-y_0)} \]  \hspace{1cm} (4.3.2)

Taking \( y = y_0 \) in equation (4.3.1), we get the velocity in plug flow region as

\[ u_p = \frac{1}{N^2} \frac{dp}{dx} \left[ 1 - \cosh N(h-y_0) \right] - 1 \]  \hspace{1cm} (4.3.3)

The volume flux \( q \) through each cross section in the wave frame is given by

\[ q = \int_{y_0}^{y_0} u_p \, dy + \int_{y_0}^{h} u \, dy = \frac{1}{N^2} \frac{dp}{dx} \left[ A_1 A_2 + A_3 - h + y_0 \right] - h \]  \hspace{1cm} (4.3.4)

where \( A_1 = \frac{1 - \cosh N(h-y_0)}{\cosh N(h-y_0)} \), \( A_2 = y_0 - \frac{\sinh Ny_0}{N \cosh Nh} \) and \( A_3 = \frac{\sinh Nh - \sinh Ny_0}{N \cosh Nh} \)

From equation (3.4), we have

\[ \frac{dp}{dx} = \frac{N^2(q+h)}{A_1 A_2 + A_3 - h + y_0} \]  \hspace{1cm} (4.3.5)

The pressure rise and frictional force over one wavelength of the peristaltic are given by

\[ \Delta p = \int_{y_0}^{1} \frac{dp}{dx} \, dx \]  \hspace{1cm} (4.3.6)

\[ F = \int_{y_0}^{1} h \left( - \frac{dp}{dx} \right) \, dx \]  \hspace{1cm} (4.3.7)

The above integrals numerically evaluated using the MATLAB software.
4.4 Discussion of the results

In order to see the effects of $Da, M, y_0$ and $\phi$ on the axial pressure gradient, the pumping characteristics and the frictional force, we have plotted Figs. 4.2-4.13.

Fig. 4.2 shows the variation of axial pressure gradient $\frac{dp}{dx}$ with Darcy number $Da$ for $\phi = 0.6$, $\overline{Q} = -1, M = 1$ and $y_0 = 0.2$. It is found that increasing the Darcy number $Da$ decreases the axial pressure gradient.

The variation of axial pressure gradient $\frac{dp}{dx}$ with Hartmann number $M$ for $\phi = 0.6$, $\overline{Q} = -1, Da = 0.1$ and $y_0 = 0.2$ is depicted in Fig. 4.3. It is observed that the axial pressure gradient increases with increasing Hartmann number $M$.

Fig. 4.4 illustrates the variation of axial pressure gradient $\frac{dp}{dx}$ with half width of the plug flow region $y_0$ for $\phi = 0.6, \overline{Q} = -1, M = 1$ and $Da = 0.1$. It is found that the axial pressure gradient increases with an increase in $y_0$.

The variation of axial pressure gradient $\frac{dp}{dx}$ with amplitude ratio $\phi$ for $Da = 0.1, \overline{Q} = -1, M = 1$ and $y_0 = 0.2$ is shown in Fig. 4.5. It is observed that the axial pressure gradient increases with increasing amplitude ratio $\phi$.

Fig. 4.6 depicts the variation of pressure rise $\Delta p$ with $\overline{Q}$ for different values of Darcy number $Da$ with $\phi = 0.6$, $M = 1$, and $y_0 = 0.2$. It is found that in the pumping region ($\Delta p > 0$) the time averaged flux $\overline{Q}$ decreases with increasing Darcy number $Da$, while it increases with increasing $Da$ in both the free pumping ($\Delta p = 0$) and co-pumping ($\Delta p < 0$) regions.

The variation of pressure rise $\Delta p$ with $\overline{Q}$ for different values of Hartmann number $M$ with $\phi = 0.6$, $Da = 0.1$, and $y_0 = 0.2$ is presented in Fig. 4.7. It is noted that in the pumping region the time averaged flux $\overline{Q}$ increases with an increasing Hartmann number $M$, while it decreases with increasing $M$. 
Fig. 4.8 shows the variation of pressure rise $\Delta p$ with $\bar{Q}$ for different values of half width of the plug flow region $y_0$ with $\phi = 0.6$, $Da = 0.1$, and $M = 1$. It is found that the time averaged flux increases with increasing $y_0$ in both the pumping and free pumping regions, while in the co-pumping region it decreases with increasing $y_0$ for appropriately chosen $\Delta p (\approx -6.157)$. The variation of pressure rise $\Delta p$ with $\bar{Q}$ for different values of amplitude ratio $\phi$ with $M = 1$, $Da = 0.1$, and $y_0 = 0.2$ is depicted in Fig. 4.9. It is observed that the time averaged flux $\bar{Q}$ increases with increasing $\phi$ in both pumping and free pumping regions, while it decreases with increasing $\phi$ in the co-pumping region for appropriately chosen $\Delta p (\approx -11.121)$. Fig. 4.10 illustrates the variation of frictional force on inner tube $F$ with $\bar{Q}$ for different values of Darcy number $Da$ with $\phi = 0.6$, $M = 1$, and $y_0 = 0.2$. It is found that the frictional force on the channel wall $F$ decreases with increasing Darcy number $Da$. The variation of frictional force on inner tube $F$ with $\bar{Q}$ for different values of Hartmann number $M$ with $\phi = 0.6$, $Da = 0.1$, and $y_0 = 0.2$ is shown in Fig. 4.11. It is observed that the frictional force on the channel wall $F$ initially decreases and then increases with an increase in Hartmann number $M$. Fig. 4.12 depicts the variation of frictional force on inner tube $F$ with $\bar{Q}$ for different values of half width of the plug flow region $y_0$ with $\phi = 0.6$, $M = 1$, and $Da = 0.1$. It is found that the frictional force on the channel wall $F$ first decreases and then increases with increasing $y_0$. The variation of frictional force on inner tube $F$ with $\bar{Q}$ for different values of amplitude ratio $\phi$ with $M = 1$, $Da = 0.1$, and $y_0 = 0.2$ is presented in Fig. 4.13. It is
noted that the frictional force on the channel wall $F$ first decreases and then increases with increasing amplitude ratio $\phi$.

4.5 Conclusions

We studied the MHD peristaltic flow of a Bingham fluid through a porous medium in a channel under assumptions of low Reynolds number and long wavelength. The expressions for the velocity and pressure gradient are obtained analytically. It is found that the pressure gradient and the time averaged flux increases with increasing Hartmann number $M$, half width of the plug flow region $y_0$ and amplitude ratio $\phi$ while they decreases Darcy number $Da$. The frictional force initially decreases with increasing Hartmann number $M$, half width of the plug flow region $y_0$ and amplitude ratio $\phi$, while it decreases with increasing Darcy number $Da$. 
Fig: 4.2 The variation of axial pressure gradient $\frac{dp}{dx}$ with $Da$ for $\phi = 0.6$, $\bar{Q} = -1$, $M = 1$ and $y_0 = 0.2$. 

$Da = 0.01, 0.1, 10, 100$
Fig: 4.3 The variation of axial pressure gradient $\frac{dp}{dx}$ with $M$ for $\phi = 0.6$, $\bar{Q} = -1$, $Da = 0.1$ and $\gamma_0 = 0.2$. 


dp

dx

$M = 4.2, 0$

$\bar{Q} = -1$, $Da = 0.1$ and $\gamma_0 = 0.2$. 

Fig: 4.4 The variation of axial pressure gradient $\frac{dp}{dx}$ with $y_0$ for $\phi = 0.6$, $\bar{Q} = -1$, $M = 1$ and $Da = 0.1$. 

The figure shows the variation of the axial pressure gradient $\frac{dp}{dx}$ with the parameter $y_0$ for different values of $y_0$. The curves demonstrate how the pressure gradient changes along the axis ($x$) for the given parameters.
Fig: 4.5 The variation of axial pressure gradient $\frac{dp}{dx}$ with $\phi$ for $Da = 0.1$, $\tilde{Q} = -1$, $M = 1$ and $y_0 = 0.2$. 
Fig: 4.6 The variation of pressure rise $\Delta p$ with $\bar{Q}$ for different values of Darcy number $Da$ with $\phi = 0.6$, $M = 1$, and $y_0 = 0.2$. 
Fig: 4.7 The variation of pressure rise $\Delta p$ with $\bar{Q}$ for different values of Hartmann number $M$ with $\phi = 0.6$, $Da = 0.1$, and $y_0 = 0.2$. 
**Fig: 4.8** The variation of pressure rise $\Delta p$ with $\bar{Q}$ for different values of $y_0$ with $\phi = 0.6$, $Da = 0.1$, and $M = 1$. 
Fig. 4.9 The variation of pressure rise $\Delta p$ with $\bar{Q}$ for different values of amplitude ratio $\phi$ with $M = 1$, $D\alpha = 0.1$, and $y_0 = 0.2$. 
Fig: 4.10 The variation of frictional force on inner tube $F$ with $\bar{Q}$ for different Values of Darcy number $Da$ with $\phi = 0.6$, $M = 1$, and $\gamma_0 = 0.2$. 
Fig 4.11 The variation of frictional force on inner tube $F$ with $\bar{Q}$ for different values of Hartmann number $M$ with $\phi = 0.6$, $Da = 0.1$, and $y_0 = 0.2$. 
Fig: 4.12 The variation of frictional force on inner tube $F$ with $\bar{Q}$ for different values of $y_0$ with $\phi = 0.6$, $M = 1$, and $Da = 0.1$. 

$y_0 = 0.2, 0.1, 0$
Fig: 4.13 The variation of frictional force on inner tube $F$ with $\bar{Q}$ for different values of amplitude ratio $\phi$ with $M = 1$, $Da = 0.1$, and $y_0 = 0.2$. 