Chapter 2

Basic Equations, Boundary Conditions and non-dimensional Parameters

In this chapter, the required basic equations (i.e., continuity equation, momentum equation, energy equation, magnetic induction equation and equation of state), the relevant boundary conditions (i.e., velocity, interface, thermal and magnetic boundary conditions), the various non-dimensional parameters (such as Reynolds number, Hartmann number, Grashof number, Prandtl number, Nusselt number, magnetic Reynolds number, porous parameter, couple stress parameter and ratio of viscosities) that occur in the problems investigated in this thesis are discussed in detail. Also governing equations for the motion of viscous, incompressible, electrically conducting fluids through porous media in the presence of transverse magnetic field experiencing Lorentz force are discussed and the methods considered in this thesis are discuss in detail.

2.1 Basic Equations

In any theoretical investigation in fluid dynamics we consider some basic axioms, called conservation laws are supposed to hold universally. These conservation laws are

(i) Conservation of Mass,
(ii) Conservation of momentum,
(iii) Conservation of energy,
(iv) Maxwell’s Equations.

Besides these four basic laws there is another basic equation from thermodynamics of fluid, namely the equation of state
\[ f(p, \rho, T) = 0 \]  
\[ p \rho^{-\gamma} = \text{constant} \]

Where \( \gamma = \frac{C_p}{C_v} \), \( C_v \) is the specific heat at constant volume and \( C_p \) is the specific heat at constant pressure.

The basic equations of fluid dynamics consist of the so-called field equations and the constitutive equations of fluids. The field equations are the equations of continuum mechanics derived from the fundamental laws of conservation of mass, conservation of momentum, conservation of energy and conservation of magnetic field.

To obtain basic equations suitable to problems investigated in this thesis, the following approximations have been made:

1. The Boussinesq approximation is assumed to be valid. That is, density is constant everywhere in the momentum equation except in the buoyancy force \( \rho \vec{g} \) in which variations in density are brought about by temperature. This is valid only when the speed of the fluid is much less than that of the sound and all accelerations are slow compared with those associated with sound waves. The basic idea of this approximation is to filter out high frequency phenomenon such as sound waves since these are thought to be unimportant in the transport processes.

2. The fluid properties namely the kinematic viscosity and thermal diffusivity are assumed to be constants.

3. The viscous, thermal and Darcy dissipations, radioactive effects and the work done pressure changes are neglected.

4. The porous medium is fluid saturated, homogeneous and isotropic so that the permeability and porosity are constants.

5. Low magnetic Reynolds number approximation. Under this approximation the induced magnetic field is neglected with respect to the externally imposed uniform magnetic field and therefore the induction equation is eliminated.

6. The gravity acts vertically downwards.
2.1.1 Continuity equation (Conservation of mass)

The conservation of mass states that the amount of fluid flowing in to a volume must be equal to the amount of fluid flowing out of that volume which is expressed mathematically by the continuity equation.

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \bar{q}) = 0. \tag{2.1.3}
\]

Using the Dupuit-Forchheimer relation \( \bar{q} = \phi \bar{q}_f \), (where, \( \bar{q} \) and \( \bar{q}_f \) be the averages of the fluid velocity over a volume element \( V_m \), of the medium and volume element \( V_f \) of the fluid ) the continuity equation for a porous medium case becomes

\[
\phi \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \bar{q}) = 0. \tag{2.1.4}
\]

For an incompressible fluid Eq. (2.1.4) reduces to

\[
\nabla \cdot \bar{q} = 0. \tag{2.1.5}
\]

2.1.2 Momentum equation (Conservation of momentum)

Many physical problems of interest in the field of engineering are described by the Navier-Stokes equations. Among other applications, they are used in the design of more efficient vehicles and aircrafts, in oceanographic and meteorological studies, in the prediction of pollutant diffusion, the motion of blood in the body, in optimization process in the power generation, etc.

The equation of motion is derived on the basis of conservation of momentum, given by the Newton’s second law, for Newtonian fluids. The momentum equation for the fluid is the general Navier-Stokes equation. As there are various forms of momentum equations depending on the nature of the fluid like Newtonian and non-Newtonian are discussed below, which are analogous to the Navier-Stokes equation. As we know Newtonian fluid is a fluid in which there is a linear relation between stress and strain. Generally most common fluids, both liquid and gases are Newtonian. Examples: water, air and oil. The fluid is said to be non-Newtonian fluid in which the relation between
stress and strain rate is not linear. In such type of fluids there are many constant value of 
viscosity. Examples: Polymer solutions, blood, paint etc.

Mathematically speaking, the compressible Navier-Stokes equations are a set of 
second order partial differential equations that can be described as a combination of 
hyperbolic and parabolic equations in time. This system has five independent variables, 
three spatial co-ordinates \( x, y \) and \( z \) and the time \( t \) and five dependent variables. Usually 
these later are the three components of velocity vector \( u, v \) and \( w \) and two of the 
following thermodynamic variables; density \( \rho \), pressure \( p \) and temperature \( T \). the 
relations between these variables is given by the aforementioned state equation.

A porous medium is an assemblage of solid particles or grains which encloses a 
system of inter-connected voids through which the fluids may flow under the influence of 
external forces (either surface or body forces or both). Such a medium is therefore 
particulate in nature and any attempt to describe their internal geometry in detail is 
fraught with virtually insurmountable complexity. It is impossible to know the nature of 
fluid flow through each pore because of the microscopic nature and many degrees of 
freedom. Also solutions to problems in porous media, as in any other dynamical systems, 
require a statement of boundary geometry. Due to many intermediate boundary 
conditions of the flow geometry in porous media, a complete mathematical solution of 
the microscopic flow through the pores is highly complex.

In dealing with problems involving the transport of momentum, a remarkable 
degree of success has been achieved during the past century or so by neglecting the 
complexity of the internal geometry and adopting the concept of ‘equivalent continuum’. 
Usually a representative sample of this continuum is considered and average values of the 
required physical properties are obtained (see Cheng, 1978 and Neild and Bejan, 2006). 
Following these work, in this thesis, a well established macroscopic approach in studying 
the different transport phenomena in a porous medium is adopted.
When a fluid permeates through a porous material, the actual path of an individual fluid particle cannot be followed analytically. The Gross effect, as the fluid slowly percolates through the pores of a porous medium must be represented by a macroscopic law which is applicable to masses of fluid large compared to the dimensions of the porous structure of a medium.

The momentum equation for the clear fluid is the usual Navier-stokes equation, while the momentum equation for the fluid saturated porous media will depend on the nature of packing of the porous media. In what follows, some of the equations which have been used to describe the flow in a porous medium are listed.

**Darcy equation**

Henry Darcy formulated a law based on the experimental results on the flow of water through beds of sand and on steady state three-dimensional flow in a uniform medium revealed proportionality between flow rate, the applied pressure difference and gravitational force. Darcy’s law is a constitutive equation which describes the flow of a fluid through a porous medium. For densely packed porous media with low Reynolds number the Darcy equation is

\[
\frac{\partial \bar{q}}{\partial t} = -\nabla p + \rho \bar{g} - \frac{\mu}{K} \bar{q},
\]

(2.1.6)

Where \( \bar{q} \) the flux, \( p \) is is the pressure gradient in the flow direction, \( \mu \) is the viscosity of the fluid, \( \bar{g} \) is the gravitational acceleration and \( K \) is the permeability of the porous medium which is independent of the nature of the fluid, but it depends on the geometry of the medium and has a dimension of length squared. The negative sign indicates the flow of fluid from high pressure to low pressure. This may be considered as the dynamics basis for the study of motion of Newtonian fluids. This law has been verified by various experimental results and also theoretically in various ways with the aid of either deterministic or statistical models.
Darcy-Lapwood(DL) equation

For densely packed porous media with moderate Reynolds number, the Darcy-Lapwood (DL) equation given below is valid.

\[ \rho_0 \left[ \frac{1}{\phi} \frac{\partial \tilde{q}}{\partial t} + \frac{1}{\phi^2} (\tilde{q} \cdot \nabla) \tilde{q} \right] = -\nabla p + \rho \tilde{g} - \frac{\mu}{K} \tilde{q} \]  
\[ (2.1.7) \]

Darcy-Forchheimer equation

The Darcy-Forchheimer equation is given by

\[ \rho_0 \left[ \frac{1}{\phi} \frac{\partial \tilde{q}}{\partial t} + \frac{\rho_0 \nu_k}{\sqrt{k}} |\tilde{q}| \tilde{q} \right] = -\nabla p + \rho \tilde{g} - \frac{\mu}{K} \tilde{q}. \]  
\[ (2.1.8) \]

Brinkman equation

The Darcy, Darcy-Lapwood, Darcy-Forchheimer equations are independent of boundary effects. Brinkmann modified Darcy- equation in order to fulfill this gap about the boundaries. The Darcy-Brinkmann equation is a governing equation for the flow through a porous medium with an extra Laplacian (viscous) added to classical Darcy equation. To incorporate boundary effects the following Brinkman extended Darcy equation is used

\[ \frac{\rho_0}{\phi} \frac{\partial \tilde{q}}{\partial t} = -\nabla p + \rho \tilde{g} - \frac{\mu}{K} \tilde{q} + \mu_e \nabla^2 \tilde{q}. \]  
\[ (2.1.9) \]

Darcy-Lapwood-Brinkman equation

\[ \rho_0 \left[ \frac{1}{\phi} \frac{\partial \tilde{q}}{\partial t} + \frac{1}{\phi^2} (\tilde{q} \cdot \nabla) \tilde{q} \right] = -\nabla p + \rho \tilde{g} - \frac{\mu}{K} \tilde{q} + \mu_e \nabla^2 \tilde{q}. \]  
\[ (2.1.10) \]

This is known as Darcy-Lapwood-Brinkman equation and is valid for \( \phi \) as small as 0.8 (see Nield and Bejan 1991). Equation 2.1.9 has a parameter \( k \), the permeability such that the equation reduces to a form of Navier-Stoke’s equation as \( k \rightarrow \infty \) and to the Darcy equation \( k \rightarrow 0 \). It is important to note that, most of the works based on \( \mu = \mu_e \). However, the effective viscosity \( \mu_e \) may have a different value than fluid viscosity \( \mu \). A detailed discussion on the values of \( \mu_e \) has been presented by Lundgren (1973).
range of values for the ratio of viscosities $\Lambda = \frac{\mu_r}{\mu}$ is given by Lauriat and Prasad (1987) as $0.5 \leq \Lambda \leq 2.5$ and later Givler and Altobelli (1994) showed that this range can be extended to $0.5 \leq \Lambda \leq 10.9$.

**Effect of rotation**

The equation of motion is valid for an inertial coordinate system. In meteorology, the rotating earth is taken as reference body and thus we have take in to account the rotation and spherical shape of the earth. The vertical structure that commonly occurs in the atmosphere permits making changes in the equations of motions when they are used to study large scale force.

The effect of using a rotating frame of reference is well known from the mechanics of solid system, there are accelerations associated with the use of a non-inertial frame that can be taken in to account by introducing centrifugal and coriolis forces. This statement can be expressed as

$$
\left( \frac{D\ddot{q}}{Dt} \right)_I = \left( \frac{D\ddot{q}}{Dt} \right)_R + \ddot{\Omega} \times (\ddot{\Omega} \times \dot{r}) + 2\ddot{\Omega} \times \ddot{r},
$$

(2.1.11)

The subscript $I$ and $R$ refer to inertial and rotating frames of reference. $\left( \frac{D\ddot{q}}{Dt} \right)_I$ is the actual acceleration that a fluid particle is experiencing and so $\rho \left( \frac{D\ddot{q}}{Dt} \right)_I$ is the quantity to be equated with the sum of the various forces acting on the fluid particles. $\left( \frac{D\ddot{q}}{Dt} \right)_R$ is the acceleration in rotating frame and can thus be expanded in the usual way

$$
\left( \frac{D\ddot{q}}{Dt} \right)_R = \left( \frac{\partial\ddot{q}}{\partial t} + (\ddot{q} \cdot \nabla) \ddot{q} \right)_R,
$$

(2.1.12)

Dropping the subscript $R$, the equation of motion is
\[ \rho \left( \frac{\partial \vec{q}}{\partial t} + (\vec{q} \cdot \nabla) \vec{q} \right) = -\nabla p - \vec{\Omega} \times (\vec{\Omega} \times \vec{r}) - 2\rho \vec{\Omega} \times \vec{q} + \nabla^2 \vec{q}. \] (2.1.13)

The term \( 2\vec{\Omega} \times \vec{q} \) is referred to as the coriolis force, \( \vec{\Omega} \times (\vec{\Omega} \times \vec{r}) \) as the centrifugal force and it can be expressed as the gradient of a scalar quantity that is

\[ \vec{\Omega} \times (\vec{\Omega} \times \vec{r}) = -\nabla \left( \frac{1}{2} \Omega^2 r^2 \right), \] (2.1.14)

where \( r' \) is the distance from the axis of rotation. Hence replacing the pressure by \( \left( p - \frac{1}{2} \Omega^2 r^2 \right) \) reduces the problem to one that is identical except that the centrifugal force is absent. By right hand rule \( -\vec{\Omega} \times (\vec{\Omega} \times \vec{r}) \) is always perpendicular to the earth’s axis and directed outward. This force is apparently the cause of the equatorial bulge of the earth and directed outward. An observer on the earth cannot distinguish between the gravitational attraction of the earth and the centrifugal force. Thus one can experience only the resultant force \( -\vec{g} \). Thus equation of motion when earth’s rotation effect is considered as

\[ \rho \left( \frac{\partial \vec{q}}{\partial t} + (\vec{q} \cdot \nabla) \vec{q} \right) = -\nabla p - \rho \vec{g} - 2\rho \vec{\Omega} \times \vec{q} + \nabla^2 \vec{q}. \] (2.1.15)

### 2.1.3 Heat Transport Equation

Since the transport of heat through a porous medium involves two substances, fluid and the porous matrix, the process will be characterized by specific parameters of these substances. Generally, the properties of these two substances are different and we have to consider two phases for heat transport in a porous medium; one for the fluid and the other for the porous matrix. This model is known as two-phase model for heat transport process in a porous medium. However, a well established model for heat transport in a porous medium is one phase model.

The conservation of energy for the mixture of solid-fluid phase of the medium is
\[
\left( \rho \mathcal{C}\right)_{m} \frac{\partial T}{\partial t} + \left( \rho \mathcal{C}_{p}\right)_{f} (\mathbf{q} \cdot \nabla) T = \nabla \cdot (k \nabla T). \tag{2.1.16}
\]

Equation (2.1.11) with the assumption that \( k \) is a constant, can be written as

\[
A_{h} \frac{\partial T}{\partial t} + (\mathbf{q} \cdot \nabla) T = \kappa \nabla^{2} T. \tag{2.1.17}
\]

The equation that expresses the first law of thermodynamics in a porous medium is focused here. In most of the cases studied in the literature, it has been assumed that the porous matrix and the fluid flowing through it are in local thermal equilibrium (LTE) so that \( T_{f} = T_{s} = T \), where \( T_{f} \) and \( T_{s} \) are the temperatures of the fluid and solid phases, respectively. The LTE case assumes that the difference between the volume-averaged fluid and solid temperatures is negligible. At the microscopic level, the temperature and the rate of heat flux at the interface between solid and fluid phases must be identical, but the average value over a representative elementary volume may not yield locally equal temperatures for the two phases. In this case the two phases are in local thermal non-equilibrium (LTNE) state. Studies by Vafai and Sozen (1990), Amiri and Vafai (1994, 1998), and Lee and Vafai (1999) have shown that the assumption of LTE fails in a substantial number of applications. Minkowycz et al. (1999) have established one such area of failure corresponding to the presence of a rapidly changing surface heat flux. In their review, Rees and Pop (2005) have summarized most of the recent developments, including various models used for LTNE and their application to free, mixed and forced convective flows and to stability analyses. Under the circumstances, it is warranted to investigate convective instability problems involving ferro fluids using a LTNE model and cases where LTE holds, may always be obtained as special cases.

Following Nield and Bejan (2006), the simplest way to model the LTNE is to use two thermal balance equations, one for the fluid phase and the other for the solid phase. Taking averages over an elementary volume of the medium by neglecting the viscous and dissipation terms and assuming that there is no local heat generation, the energy equation for the fluid phase is
\[ \varepsilon \rho_0 c_p \frac{\partial T_f}{\partial t} + \rho_0 c_p \frac{\partial \bar{q} \cdot \nabla T_f}{\partial t} = \varepsilon k_f \nabla^2 T_f + h \left( T_s - T_f \right) \]  
(2.1.18)

and for the solid phase is

\[ (1 - \varepsilon) \rho_0 c_s \frac{\partial T_s}{\partial t} = (1 - \varepsilon) k_s \nabla^2 T_s + h \left( T_f - T_s \right). \]  
(2.1.19)

### 2.1.4 Maxwell’s Equations

Magnetohydrodynamics (MHD) is the physical mathematical frame work that concerns the dynamics of magnetic field in electrically conducting fluids. The presence of magnetic field leads to the forces that in turn act on the fluid, thereby potentially altering the geometry (Topology). For an electrically conducting fluids, when an external constraints like magnetic field is applied then we need to use the constitutive equations which are derived from the Faraday’s principle of dynamics and also using Maxwell’s equations. These Maxwell’s equations are a set of partial differential equations that, together with a Lorentz force law, form the foundation of classical electrodynamics. Both together describe how electric and magnetic fields are generated and altered by each other and by charges and currents. They are named after the Scottish physicist and Mathematician James Clarke Maxwell. The set of equations which describe MHD are a combination of the Navier-Stokes equations of fluid dynamics and Maxwell’s equations of electromagnetism.

By neglecting displacement current and free charges, the basic equations of MHD are given by

\[ \nabla \times \vec{E} = 0, \]  
(2.1.20)

\[ \nabla \cdot \vec{B} = 0, \]  
(2.1.21)

Faraday’s law:
\[ \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad (2.1.22) \]

Ampere’s law:
\[ \nabla \times \vec{B} = \vec{J} \quad (2.1.23) \]

Where \( \vec{J} \) is the current density, \( \vec{B} \) is the magnetic field, \( \vec{E} \) is the electric field. Ohm’s law is given by
\[ \vec{J} = \sigma \left( \vec{E} + \left( \hat{q} \times \vec{B} \right) \right), \quad (2.1.24) \]

When the conduction current is negligible compared to \( \sigma \left( \hat{q} \times \vec{B} \right) \), Ohm’s law becomes
\[ \vec{J} = \sigma \left( \hat{q} \times \vec{B} \right). \quad (2.1.25) \]

### 2.1.5 Equation of State

These equations are supplemented by the equation of state and the same for thermal system under Boussinesq approximation is
\[ \rho = \rho_0 \left[ 1 - \alpha (T - T_0) \right]. \quad (2.1.26) \]

### 2.1.6 Momentum Equation for Couple- stress fluid

The popular micropolar and couple-stress fluid theories were proposed by Eringen (1964) and Stokes (1966), respectively. Five decades have passed ever since the theory of couple-stresses was first considered in isolation by assuming that the fluid has no microstructure at the kinematic level, so that the kinematics as motion is fully determined by the velocity field. In the theory of couple-stress fluids the intrinsic angular momentum and the kinetic energy of spin density are not taken into account, whereas couple-stresses will be considered and assumed that the fluid will not have any microstructure. This is the simplest theory that shows all the important features, effects
of couple-stresses and results in equations that are similar to the Navier-Stokes equations, thereby facilitating a comparison with the results for the classical non-polar case. The main effect of couple-stresses is to introduce a size dependent effect based on the material constant and dynamic viscosity, which is not present in the classical viscous fluid theories. The fluids consisting of rigid, randomly oriented particles suspended in a viscous medium, such as blood, lubricants containing small amount of polymer additive, electro-rheological fluids and synthetic fluids are some of the examples for these fluids. Also, some more examples are the extrusion of polymer fluids, solidification of liquid crystals, cooling of metallic plate in a bath and colloidal solutions, etc.

Recently, the study of couple-stress fluid flows has been the subject of great interest due to its widespread industrial and scientific applications as in the case of micropolar fluids. Important fields where the couple-stress fluids have applications include squeezing and lubrication, bio-fluid mechanics, magnetohydrodynamic flows and synthesis of the plasticity of chemical compounds, polymer solutions and melts, protein solutions and so on.

The equation of motion for couple-stress fluids are based on the constitutive equations which are given by Stokes (1966). The stress tensor \( \tau_{ij} \) consists of symmetric and anti-symmetric parts and the couple-stress tensor \( M_{ij} \) has the linear constitutive relation. They are respectively given by

\[
\tau_{ij} = \left[ (-p + \lambda D_{kk})\delta_{ij} + 2\eta D_{ij} \right] - \left[ 2\eta c W_{ij,kl} + \frac{1}{2} \varepsilon_{ikl} \left( \rho G_k + m_x \right) \right]
\]  

(2.1.27)

and

\[
M_{ij} = \frac{1}{3} m \delta_{ij} + 4\eta c \omega_{ij} + 4\eta' \omega_{ij}.
\]

(2.1.28)

In the above equations, \( p \) is the pressure, \( D_{ij} = (q_{i,j} + q_{j,i}) / 2 \) is the rate of deformation tensor, \( W_{ij} = -(q_{i,j} - q_{j,i}) / 2 \) is the vorticity tensor, \( \rho G_k \) is the body couple vector, \( q_i \) is the velocity vector, \( m \) is the trace of the couple-stress tensor, \( \lambda \) and \( \eta \) are the material constants having dimensions of viscosity, while \( \eta_c \) and \( \eta' \) are material constants having
dimensions of momentum and \( \omega = \varepsilon_{ik} q_{j,k} / 2 \) is the spin vector. In the vector notation, the equation of motion of an incompressible couple-stress fluid in the presence of body force and in the absence of body couples is then given by

\[
\rho_0 \left[ \frac{\partial \tilde{q}}{\partial t} + (\tilde{q} \cdot \nabla) \tilde{q} \right] = -\nabla p + \rho \tilde{g} + \left( \eta - \eta_c \nabla^2 \right) \nabla^2 \tilde{q}.
\] (2.1.29)

where, \( \tilde{g} \) is the gravitational acceleration and \( \eta_c \) is the couple stress viscosity.

In the absence of couple stresses, (2.1.29) reduces to usual Navier-Stokes equation.

If the couple-stress fluid layer is rotating with a constant angular velocity \( \tilde{\Omega} \) then (2.1.29) takes the following form

\[
\rho_0 \left[ \frac{\partial \tilde{q}}{\partial t} + (\tilde{q} \cdot \nabla) \tilde{q} \right] = -\nabla p + \rho \tilde{g} + \left( \eta - \eta_c \nabla^2 \right) \nabla^2 \tilde{q} + 2 \rho_0 (\tilde{q} \times \tilde{\Omega}).
\] (2.1.30)

### 2.2 Boundary Conditions

The basic equations in section 2.1 have been to be solved subject to various boundary conditions on velocity, temperature and magnetic field. These are discussed in this section.

#### 2.2.1 Velocity Boundary Conditions

The boundary conditions on velocity are obtained from mass balance, the no-slip condition and the stress principle of Cauchy. This will depend on the nature of bounding surfaces of the fluid. The bounding surfaces may be either rigid or free with or without deformation.

(i) **Rigid Boundary**

If the fluid layer is bounded above and below by horizontal rigid boundaries, the no-slip condition is valid at the boundaries and we have

\[
u = v = w = 0
\] (2.2.1)

if the boundaries are at rest. If the layer is of an infinite extent in the horizontal, \( x \) and \( y \) directions, then
\[ u = v = 0 \]  \hspace{1cm} (2.2.2)

for all \( x \) and \( y \). Hence, from the equation of continuity (2.1.4) it follows that

\[ \frac{\partial w}{\partial z} = 0 \]  \hspace{1cm} (2.2.3)

at the boundaries. Thus in the case of rigid boundaries, the boundary conditions on velocity are:

\[ w = \frac{\partial w}{\partial z} = 0 . \]  \hspace{1cm} (2.2.4)

(ii) **Free boundary**

If the fluid layer is bounded below and above by free boundaries, then the boundary conditions on velocity depend on whether we consider surface tension effect or not. If the free surface does not deform in the direction normal to itself, then it is required that

\[ w = 0 . \]  \hspace{1cm} (2.2.5)

In the absence of surface tension, the free surface is free from the shear stresses so that

\[ \frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = 0 . \]  \hspace{1cm} (2.2.6)

From the equation of continuity (2.1.4), it follows that

\[ \frac{\partial^2 w}{\partial z^2} = 0 . \]  \hspace{1cm} (2.2.7)

Thus in the absence of surface tension, the conditions on velocity to be imposed at a horizontal free surface (i.e., the stress free boundaries) are:

\[ w = \frac{\partial^2 w}{\partial z^2} = 0 . \]  \hspace{1cm} (2.2.8)

### 2.2.2 Thermal Boundary Conditions

In the study of convection flow problems, the conditions applied at the upper and lower boundaries of the porous layer are based on the nature of boundaries. The following three kinds of thermal boundary conditions are used:

(i) **Isothermal conditions**, where the temperatures are specified on the boundaries called Dirichlet problem.
(ii) Adiabatic or insulating conditions, where the heat flux is specified on the boundaries called Neumann problem.

(iii) Radiation conditions, where the combination of temperature gradient and temperature is specified on the boundaries, sometimes called Churchill problem.

In this thesis, the first type of temperature boundary condition is imposed. That is,

\[ T = 0. \] (2.2.9)

### 2.2.3 Boundary Conditions on Magnetic field

The boundary conditions on magnetic field will depend on whether the boundaries are perfectly conducting or not. The following conditions on perturbed magnetic field \( \vec{B} \), are used in the study:

\[ B = 0 \] (2.2.10)

on the rigid boundary.

### 2.3 Non-Dimensional Parameters

The non-dimensional parameters which appear in the resulting non-dimensional equations are the parameters of the solutions and are the key factors in determining the qualitative and quantitative nature of the flow phenomena. These parameters are used to indicate dominant physical factors for mathematical simplifications and data correlations and to design proper theoretical and experimental models. They are also useful in understanding the physics of the complex problem. The Dimensionless parameters can be found by reducing the governing differential equations to non-dimensional form. Physical process may be described by an equation among dimensional physical quantities or variables. Through dimensional analysis these quantities are arranged in dimensionless groups. Dimensionless analysis is a procedure that allows us to formulate a functional relationship between a set of non-dimensionless parameters numbering less than variables and arranges them into dimensionless groups. In applying it is necessary that all the
dimensional variables affecting the process be known. The non-dimensional parameters so obtained may be viewed as a ration of pair of fluid forces where relative magnitude indicates the relative importance of one of the forces, with respect to the other. Consequently one should always introduce dimensionless form brings out the behavior of the system. The following are the important dimensionless parameters that govern the system investigated in this thesis.

2.3.1 Reynolds number

It is defined as the ratio of inertia force to viscous force. Mathematically

\[ Re = \frac{\rho V h}{\mu} = \frac{V h}{\nu} \]

where \( V \) is the velocity of the flow, \( h \) is the characteristic length and \( \rho \), \( \mu \) and \( \nu \) are the density, dynamic viscosity and kinematic viscosity of the fluid respectively. If \( Re \) is very small, there is an indication that the viscous forces are dominant compared to inertia forces. Such types of flow are commonly referred to as creeping or viscous flows. Conversely, for large \( Re \), viscous forces are small compared to inertial effects and such flows are characterized as inviscid analysis. This number is very important to study the transition between the laminar and turbulent flow regimes. When viscous forces are dominant it is a laminar flow and when inertial forces are dominant it is a turbulent flow.

2.3.2 Prandtl number

The measure of momentum diffusivity (kinematic viscosity) to thermal diffusivity is given by the Prandtl number \( Pr \) and is defined as

\[ Pr = \frac{\nu}{\kappa \theta^2}. \]

This is a physical parameter, which depends on the intrinsic property of the fluid and not of the flow. Physically, it represents the ratio of viscous force to thermal force. At Prandtl numbers of order of unity and lower, the convective terms cause a transition from steady convective rolls to time-dependent oscillatory convection.
Depending on the magnitude of the Prandtl number, liquids are divided into three categories; liquids with $Pr \ll 1$ (liquids metals), heat-carrying liquids with $Pr = 1$ (non-metallic at high temperature and gases), liquids with $Pr > 1$ (non-metallic liquids which are poor conductors of electrically). $Pr > 1$ implies $\nu > \kappa$, i.e., momentum disturbance propagates further into the free stream. For the non porous case, $\phi = 1$.

### 2.3.3 Porous Parameter

The porous parameter is inversely proportional to the Darcy number, which is denoted by $\sigma_p$ and is given by

$$\sigma_p = \frac{d}{\sqrt{k}}.$$

Physically, this represents the scale factor, which describes the extent of the division of porous structure (permeability) as compared to the vertical extent of the porous layer. When the permeability is very high the resistance to the flow becomes effectively controlled by ordinary viscous resistance. In that case, the convection phenomenon is similar to that in an ordinary layer. But in most of the problems, either the viscous force is negligible or is of comparable order to the Darcy resistance. This is an appropriate assumption as long as the size of the pore is relatively small compared to the dimension of the layer. But when the pore size is not small when compared with the dimension of the layer, one has to consider the viscous resistance to the flow.

### 2.3.4 Grashof number

It is defined as the ratio of buoyancy force to viscous force in natural convection flows. Mathematically

$$G = \frac{\alpha \Delta T gh^3}{\nu^2}$$

where, $\alpha$ is the coefficient of thermal expansion, $\nu$ is the kinematic viscosity, $h$ is the characteristic length, $g$ is the gravitational force. In natural convection the Grashof number plays the same role is played by the Reynolds number in forced convection. When the buoyancy forces overcome the viscous forces the flow starts a transition to turbulent.
2.3.5 Viscosity ratio

The viscosity ratio is denoted by $\Lambda$ and is given by

$$\Lambda = \frac{\mu_e}{\mu_f}.$$

Where $\mu_e$ is the effective viscosity and $\mu_f$ is the fluid viscosity. For a high permeability porous media takes the value up to 11 (Gilvler and Allobelli, 1994).

2.3.6 Magnetic Reynolds number

The magnetic Reynolds number is a non dimensional number that occurs in Magneto hydrodynamics. It gives estimates of the effect of magnetic advection to magnetic diffusion and it is defined as

$$Rm = \frac{U_0 h}{\eta}$$

where $\eta$ is the magnetic diffusivity, $U_0$ is the characteristic velocity scale of the flow and $h$ is the characteristic length scale of the flow. In astrophysics, $Rm$ is usually extremely large, which implies that the magnetic field is almost frozen within the flow (this behavior is analogous to the freezing of vorticity in an inviscid fluid). In earth’s core, $Rm$ is approximately $10^3$ to $10^4$, which is enough to allow for the dynamo effect (see, for instance, Roberts 1967), which self-sustains earth’s magnetic field. At the laboratory scale, $Rm$ is typically of the order of $10^{-4}$ or $10^{-2}$ for electrically conducting liquids such as molten metals. In that case, the fluctuating magnetic field $\overline{B}$ induced by the fluctuating current is rapidly smoothed out by the strong magnetic diffusivity. These fluctuations, as well as those of the electric field, however, can be measured and related to the fluctuating velocity field $\overline{u}$. with $\overline{B}$ as the applied magnetic field, the fluctuations $\overline{B}$ are of order $Rm \overline{B}$, which means they are negligible in comparison with the applied magnetic field $\overline{B}$. 
2.3.7 Hartmann number

The non-dimensional constant $M$ is called the Hartmann Number, which indicates the ratio between the magnetic viscosity and the ordinary viscosity in the flow.

$$M = B_0 h \sqrt{\frac{\sigma}{\mu}}$$

Where, $B_0$ is the magnetic field, $h$ is the characteristic length, $\sigma$ is the electrical conductivity and $\mu$ is the dynamic viscosity. In the equation of motion, the magnetic force and the viscous force have the same dimensions. The ratio of these is just $M^2$. The case $M = 0$ corresponds to no magnetic field and normal viscous flow. In this case we find the parabolic velocity profile is expected. As the Hartmann number increases, the velocity profile changes from a parabola to a constant value, showing the effect of dragging the magnetic field.

2.3.8 Couple stress Parameter

The couple stress parameter is a non dimensional quantity and it defined as

$$\Lambda = h \sqrt{\frac{\mu_d}{\lambda}}$$

Where, $\mu_d$ is the dynamic viscosity, $\lambda$ couple stress viscosity and $h$ is the characteristic length. The couple stress parameter represents the extent of variation of the velocity due to the presence of suspended particles.
2.4 Methods to study the stability of shear flows.

In this thesis we used the following methods to study the linear stability analysis of some shear flows

1. Normal mode analysis
2. Energy method
3. Galerkin method with Legendre based polynomials
4. Chebyshev collocation method

Normal mode analysis:

The classical and standard method to study the stability of shear flows is the normal mode analysis. It is the generalization of the Liapunov’s first method (i.e., the method of small oscillations) in the theory of ordinary differential equations. In this method, it is assumed that the disturbance of each quantity is resolved into independent components or modes with time varying like \( e^{ct} \), for some \( c \) which is in general complex. Thus, for the example of a parallel flow, two-dimensional disturbances are assumed to be of the form (function of \( y \)) \( e^{ikx+ct} \), where \( k \) is the real wave number. The linear stability problems, thereby, reduces to an eigenvalue problem for \( c \). If there are eigenvalues \( c \) with \( \text{Re}(c) > 0 \), then the flow is unstable. Otherwise, it is stable. This classical method has been extensively used by many authors to analyze the linear stability of many fluid flows.

One of the classical applications of solving the Orr-Sommerfeld equation is for the case of plane Poiseuille flow, which we can determine if a given base flow is stable to sinusoidal perturbations. In the case of temporal stability, the Reynolds number \( Re \) and the wave number \( \alpha \) are given as parameters, while the eigenvalue, \( c \) needs to be determined. In this case wave number \( \alpha \) is to be real while \( c \) can be complex. In the case of spatial stability, instead of \( \alpha \) being a parameter, the frequency \( \omega \) is the parameter along with the Reynolds number. The wave number \( \alpha \) is then allowed to be complex. The mathematical problem is to find this sequence, which has a different functional form for spatial versus temporal growth of disturbances.
Temporal growth: \( f(Re, \alpha, c_r, c_i) = 0 \)

Spatial growth: \( g(Re, \alpha, c_r, c_i) = 0 \)

Of particular interest is the case of neutral stability: \( c_i = 0 \) for temporal case and \( \alpha = 0 \) for spatial neutral growth.

**Energy method:**

Energy method is the analytical method for obtaining sufficient conditions for stability of shear flows has been given by Synge (1935) and later his results have been extended and improved by Joseph (1969). In this method we first multiply the Orr-Sommerfeld equation by \( \phi^* \) (complex conjugate of \( \phi \)) and integrating over the interval (-1, 1) to obtain

\[
\left( I_z^2 + 2\alpha^2 I_z^2 + \alpha^4 I_z^2 \right) = -i\alpha Re Q + ic\alpha Re \left( I_z^2 + \alpha^2 I_z^2 \right) 
\]

(2.4.1)

Where we take \( z_1 = -1, z_2 = 1, \)

\[
I_n^2 = \int_{-1}^{1} |D^n \phi|^2 \, dz \quad (n = 0 \text{ to } 2)
\]

and \( Q = \int_{-1}^{1} \left[ u_b |D\phi|^2 + \left( D^2 u_b + \alpha^2 u_b \right) |\phi|^2 \right] \, dz + \int_{-1}^{1} \phi (Du_b) (D\phi) \, dz. \)

From this equation we immediately have

\[
c_r = \frac{Q_r}{\left( I_z^2 + \alpha^2 I_z^2 \right)}, \quad (2.4.2)
\]

\[
c_i = \frac{1}{\left( I_z^2 + \alpha^2 I_z^2 \right)} \left[ Q_r - \frac{I_z^2 + 2\alpha^2 I_z^2 + \alpha^4 I_z^2}{\alpha Re} \right], \quad (2.4.3)
\]

where

\[
Q_r = \text{Re}(Q) = \int_{-1}^{1} \left[ u_b |D\phi|^2 + \left( \frac{1}{2} D^2 u_b + \alpha^2 u_b \right) |\phi|^2 \right] \, dz
\]

and \( Q_i = \text{Im}(Q) = \frac{i}{2} \int_{-1}^{1} (\phi D\phi^* - \phi^* D\phi) Du_b \, dz. \)

Equation (2.4.3) is simply the energy equation for two-dimensional disturbances propagating in the direction of the basic flow \( u_b(z) \). Observe now that
\[ |\text{Im}(Q)| \leq \int_1^1 |\phi| |D\phi| |Du_b| \, dz , \]

and hence, by Schwarz’s inequality,

\[ |\text{Im}(Q)| \leq I_0 I_1 q , \text{ where } q = \max_{-1 \leq z \leq 1} |Du_b| , \]

This gives the upper bound for \( c_i \):

\[ c_i \leq \frac{1}{I_1^2 + \alpha^2 I_0^2} \left[ qI_0 I_1 - \frac{I_1^2 + 2\alpha^2 I_1^2 + \alpha^4 I_0^2}{\alpha \text{Re}} \right] , \tag{2.4.4} \]

from which it follows that a sufficient condition for stability is

\[ \alpha \text{Re} < \frac{I_1^2 + 2\alpha^2 I_1^2 + \alpha^4 I_0^2}{(qI_0 I_1)} . \tag{2.4.5} \]

**Galerkin Method:**

The Galerkin method for solving differential equations is a very general one and it may be shown to be equivalent to the Ritz method. When the considered equations are derivable from an extremum principle. Apart of this possible identification, it belongs with others (tau, pseudo spectral, and collocation) to the set of the so called spectral methods, which have been systematically investigated recently. All of them have the common feature of searching the desired solution by expanding it into a complete set of functions. They differentiate themselves by how the boundary conditions of the problem and the algebraic equations for the expansion coefficients are treated. Among them, the method of Galerkin is certainly the simplest of all.

A Legendre polynomial based Galerkin method is presented here for solving the following eigenvalue problem.

\[ u'' + \lambda u = 0 , \tag{2.4.6} \]

where \( u \in H_0^1(\Omega) \) an Hilbert space and \( u = 0 \) at \( z = \pm 1 \).

Equation (2.4.6) can be solved numerically by replacing the infinite dimensional space \( H_0^1(\Omega) \) by a finite dimensional space \( S_N \subset H_0^1(\Omega) \) of dimension \( N \subset \mathbb{N} \). Assuming
that a basis $\phi_1, \phi_2, \ldots, \phi_N$ of $S_N$ can be constructed, the solution of $u$ in equation (2.4.6) may be approximated by $u = \sum_{k=1}^{N} u_k \phi_k$ and then equation (2.4.6) is replaced by

$$\sum_{k=1}^{N} u_k \phi_k'' + \lambda \sum_{k=1}^{N} u_k \phi_k = 0,$$  \hspace{1cm} (2.4.7)

where $u_k$ are the coefficients.

Let $L_i$ be the $i^{th}$ Legendre polynomial on $(-1,1)$ with $S_N = P^{N+1}(\Omega) \cap H^1_0(\Omega)$, where $P^p(\Omega)$ denotes the polynomial in $p$ on $\Omega$. Using the identity

$$(2i + 1)L_i(z) = L'_{i+1}(z) - L'_{i-1}(z),$$  \hspace{1cm} (2.4.8)

We define a basis function for $p \geq 2$

$$\phi_i(z) = \int_{-1}^{z} L_i(s) ds = \frac{L_{i+1} - L_{i-1}}{2i + 1}, \quad i = 1, \ldots, p - 1,$$  \hspace{1cm} (2.4.9)

By the definition of Legendre polynomials the basis functions $\phi_i$ are linearly independent, such that

$S_N = \text{Span}(\phi_i), i = 1, \ldots, N$, with $N = \dim(S_N)$. A crucial aspect of these basis functions is their inclusion in the space $H^1_0(\Omega)$ or, more specifically in this context, that $\phi_i(1) = \phi_i(-1) = 0$. This follows when utilizing the relation $L_i(\pm 1) = (\pm 1)^i$. This inherent structure clearly avoids the need for the superimposition of the standard homogeneous boundary conditions in the resulting a matrix.

To solve equation (2.4.7), we multiplied by $\phi_i$ and integrate over $\Omega$ to find

$$\left\langle \sum_{k=1}^{N} u_k \phi_k'', \phi_i \right\rangle + \lambda \left\langle \sum_{k=1}^{N} u_k \phi_k, \phi_i \right\rangle = 0, \quad i = 1, \ldots, N$$  \hspace{1cm} (2.4.10)

By making use of the divergence theorem and utilizing Eq. (2.4.9), we can observe that

$$\left\langle \sum_{k=1}^{N} u_k \phi_k'', \phi_i \right\rangle = -\sum_{k=1}^{N} u_k \left\langle \phi_k'', \phi_i' \right\rangle = -\left\langle \sum_{k=1}^{N} u_k L_k, L_i \right\rangle.$$  \hspace{1cm} (2.4.11)
Eq. (2.4.10), with the rearrangement of Eq. (2.4.11), may be solved by utilizing the inherent orthogonality of Legendre polynomials within the specified inner product, where

$$\left( L_i, L_j \right) = \int_\Omega L_i(z) L_j(z) \, dz = \begin{cases} \frac{2}{2i+1}, & i = j \\ 0, & i \neq j \end{cases}.$$ (2.4.12)

This procedure leads to a generalized eigenvalue problem of the form

$$A u = \lambda B u$$ (2.4.13)

where \( u = (u_1, u_2, \ldots, u_N)^T \). By using the orthogonal behavior shown in equation (2.4.13) the matrix \( A \) can be derived from equation (2.4.12) yielding diagonal elements \( A_{i,i} = \frac{2}{2i+1} \), where \( i = 1, \ldots, N \). This desirable feature that the matrix \( A \) is diagonal is due to the fact that \( \phi_i \) is selected so that \( \phi_i' = L_i \).

Similarly

$$\left\langle \sum_{k=1}^{N} u_k \phi_k, \phi_i \right\rangle = \left\langle \sum_{k=1}^{N} u_k \left( \frac{L_{k+1} - L_{k-1}}{2k+1}, \frac{L_{i+1} - L_{i-1}}{2i+1} \right), \phi_i \right\rangle,$$ (2.4.14)

Which, utilizing equation (2.4.14), yields symmetric banded matrix \( B \) with elements

$$B_{i,j} = \begin{cases} 4 & j = i, \quad i = 1, \ldots, N \\ \frac{4}{(2j-1)(2j+1)(2j+3)} & j = i+2, i = 1, \ldots, N-2 \\ \frac{4}{(2j-3)(2j-1)(2j+1)} & \end{cases}$$

which is of bandwidth 4. The equivalent procedure with the Chebyshev tau approach yields matrices \( A \) and \( B \) which are not a banded structure as they are here. System (2.4.13) is sparse eigenvalue problem making it ideal for specific sparse iterative solvers such as the implicitly restarted Arnoldi method as presented in the ARPACK package. This reduces computational and storage requirements needed by the QZ algorithm (see e.g. Golub and Vanloan, 1996), which is necessary for a technique like the Chebyshev tau method.
Chebyshev collocation method:

Collocation methods are a general class of numerical methods that can be applied to many different kinds of ordinary differential equations. The general idea of collocation methods is to find a solution to an ODE by approximating the solution as a linear combination of some assumed set of orthogonal functions. This linear combination of orthogonal functions are required to satisfy the ODE at a set of collocation points. Mathematically this looks the following

\[ \sum_{j=1}^{n} c_j (L v_j)(t_i) = w(t_i), \quad \text{where } i = 1, \ldots, n. \]

In the above equation, \( L \) is a general linear operator, \( v_j \) is a set of functions, and \( t_i \) are the collocation points. There exist a large set of possible choices of \( v_j \), and some common functions that can be picked are the set of polynomials \( 1, \; t, \; t^2, \ldots \) or B-splines. Specifically with the Orr-Sommerfeld equation, Chebyshev polynomials were found by Orzag (1971) to obtain highly accurate results of the eigenvalues for plane Poiseuille flow using many less discretization points than required for finite difference methods. As such, Chebyshev collocation methods have become one of the standard methods for solving the eigenvalues of the Orr-Sommerfeld equation.

To apply Chebyshev collocation method to solve Orr-Sommerfeld equation, we follow Takashima (1994) and first expand the perturbations \( \phi \) in terms of Chebyshev polynomials as follows,

\[ \phi = \sum_{k=0}^{N} a_k T_k(y) \]  \hspace{1cm} (2.4.15)

The classical Orr-Sommerfeld equation for the hydrodynamic stability problem is

\[ (u_b - c)(\phi' - \alpha^2 \phi) - u''_b \phi - \frac{1}{i \alpha \text{Re}} (\phi'' - 2 \alpha^2 \phi' + \alpha^4 \phi) \]  \hspace{1cm} (2.4.16)

If the boundaries are rigid then the appropriate boundary conditions are:

\[ \phi = D \phi = 0 \quad \text{at} \; y = \pm 1. \]  \hspace{1cm} (2.4.17)

After incorporating equation (2.4.15) in equations (2.4.16) and (2.4.17), we arrive at the following Chebyshev discretization of the Orr-Sommerfeld equation,
\[
\left( U(y)\alpha^2 - U'(y) - \frac{\alpha^3}{i\Re} \right) \sum_{k=0}^{N} a_k T_k(y) + \left( U(y)\alpha^2 + \frac{2\alpha}{i\Re} \right) \sum_{k=0}^{N} a_k T_k'(y)
\]
\[
- \frac{1}{i\alpha \Re} \sum_{k=0}^{N} a_k T_k^{''}(y) = c \left( \sum_{k=0}^{N} a_k T_k(y) - \alpha^2 \sum_{k=0}^{N} a_k T_k'(y) \right)
\]

(2.4.18)

Along with boundary conditions given as
\[
\sum_{k=0}^{N} a_k T_k(\pm) = 0, \quad \sum_{k=0}^{N} a_k T_k'(\pm) = 0
\]

(2.4.19)

To numerically solve equation (2.4.18) along with the boundary conditions (2.4.19), we can rewrite the equations as a matrix equation of the form
\[
A \hat{\phi} = c B \hat{\phi}
\]

(2.4.20)

where \( A \) and \( B \) are the complex matrices, \( c \) is the eigenvalue and \( \hat{\phi} \) is the eigenvector. Here \( A \) has the following form
\[
A = \begin{pmatrix}
T_0(1) & T_1(1) & \ldots & \ldots \\
T_0'(1) & T_1'(1) & \ldots & \ldots \\
A_2 T_0(y2) + B_2 T_0'(y2) - C_2 T_0^{''}(y2) & A_2 T_1(y2) + B_2 T_1'(y2) - C_2 T_1^{''}(y2) & \ldots & \ldots \\
\vdots & \vdots & \ddots & \ldots \\
\end{pmatrix}
\]

Where \( A_2, B_2 \) and \( C_2 \) are as follows,
\[
A_2 = U(y)\alpha^2 - U'(y) - \frac{\alpha^3}{i\Re}
\]
\[
B_2 = U(y)\alpha^2 + \frac{2\alpha}{i\Re}
\]
\[
C_2 = \frac{1}{i\alpha \Re}
\]

(2.4.21)

\( B \) has the following form
\[ B = \begin{pmatrix}
T_0(1) & T_1(1) & \cdots \\
T_0(1) & T_1(1) & \cdots \\
T_0(y_2) - \alpha^2 T_0(y_2) & T_1(y_2) - \alpha^2 T_1(y_2) & \cdots \\
\vdots & \vdots & \ddots & \cdots \\
T_0(y_{n-2}) - \alpha^2 T_0(y_{n-2}) & T_1(y_{n-2}) - \alpha^2 T_1(y_{n-2}) & \cdots \\
\vdots & \vdots & \ddots & \ddots 
\end{pmatrix} \]

An easier method of writing both \( A \) and \( B \) as follows,

\[ A = A_0 D^0 + B_0 D^2 - C_0 D^4 \]
\[ B = D^0 + \alpha^2 D^2 \quad (2.4.22) \]

Here, \( D^K \) is known as a differentiation matrix and is simply a full matrix that has the derivatives of the Chebyshev polynomials evaluated at \( N \) collocation points \( y_1, y_2, y_3, \ldots, y_N \), where \( y_m = \cos \left( \frac{m-1}{2N-1} \pi \right), \quad m = 1, 2, 3, \ldots, N \). One of the primary advantages of writing the matrix problem using differentiation matrices is that a recursion algorithm can be used to obtain the \( n \)th derivative matrix efficiently. To solve the generalized eigenvalue problem (2.4.20), we use the following procedure.

For fixed values of \( Re \) and \( \alpha \), the values of \( c \) which ensure a non-trivial solution of Eq. (2.4.20) are obtained as the eigenvalues of the matrix \( B^{-1} A \). From \( N \) eigenvalues \( k(1), k(2), k(3), \ldots, k(N) \), the one having largest imaginary part (\( k(p) \), say) is selected. In order to obtain the neutral stability curve, the value of \( Re \) for which the imaginary part of \( k(p) \) vanishes must be sought. Let this value of \( Re \) be denoted by \( Re^n \). The lowest point of \( Re^n \) as a function of \( \alpha \) gives the critical Reynolds number \( Re_c \) and critical wave number \( \alpha_c \). The real part of \( k(p) \) corresponding to \( Re_c \) and \( \alpha_c \) gives the critical wave speed \( c_c \).
2.5 Nomenclature

\( \vec{q} = (u, v, w) \) velocity vector

\( \vec{B} = (B_x, B_y, B_z) \) magnetic field vector

\( \vec{j} \) current density

\( (x, y, z) \) cartesian co-ordinates

\( c \) wave speed

\( c_r \) phase velocity

\( c_i \) growth rate

\( \ddot{g} \) acceleration due to gravity

\( G \) Grashof number

\( h \) thickness of the fluid layer

\( \hat{i} \) unit vector in \( x \)-direction

\( k \) permeability

\( \hat{k} \) unit vector in \( z \)-direction

\( M \) Hartmann number

\( P \) total pressure

\( Pr \) Prandtl number

\( Pr_m \) magnetic Prandtl number

\( Re \) Reynolds number

\( t \) time

\( T \) temperature

\( T_1 \) temperature of the left boundary

\( T_2 \) temperature of the right boundary

\( w_b \) basic velocity

\( p \) pressure

\( Al \) Alfven number

\( N \) magnetic interaction parameter
Greek symbols

$\alpha$  
stream wise wave number

$\alpha_T$  
volumetric thermal expansion coefficient

$\beta$  
temperature gradient

$\chi$  
 ratio of heat capacities

$\varepsilon$  
porosity of the porous medium

$\kappa$  
thermometric conductivity

$\mu$  
magnetic permeability

$\mu_f$  
fluid viscosity

$\mu_e$  
effective viscosity

$\nu$  
kinematic viscosity

$\sigma$  
electrical conductivity

$\sigma_p$  
porous parameter

$\Lambda$  
 ratio of effective viscosity to the fluid viscosity

$\theta$  
amplitude of perturbed temperature

$\rho$  
fluid density

$\rho_0$  
reference density at $T_0$

$\psi$  
amplitude of perturbed magnetic field

$\phi$  
amplitude of perturbed velocity

$\hat{\phi}(x,z,t)$  
stream function

$\mu_d$  
dynamic viscosity

$\lambda$  
couple stress viscosity

$\Lambda_c$  
couple stress parameter

$\lambda_r$  
 ratio of effective couple stress to the couple stress viscosity