Chapter- 6

The effect of Transverse Magnetic Field on the Stability of Natural Convection in a Vertical Porous Layer of Electrically Conducting Fluid

6.1 Introduction

The stability of flow driven by the combined shear and buoyancy forces is one of the classical problems with significance for fundamental fluid mechanics as well as for geophysical and engineering applications. Among engineering applications there are materials processing, crystal growth, and cooling systems for nuclear reactors, solar energy collectors, manufacturing and welding. The stability of a natural convection is relevant to many industrial and manufacturing processes such as reactor core, extrusion, drawing etc. many geophysical phenomenons like motion in between earths tectonic plates are maintained by buoyancy forces, strongly modified by the coexisting shear. The recognition of high free convection heat transfer rates in atomic reactors, electrical transformers and other engineering applications prompted many to understand and study the natural convection.

The stability of natural convection in a vertical slot, the most common type of body force which acts on a fluid, is due to gravity, so that the body force can be defined as in magnitude and direction by the acceleration due to gravity. The stability of buoyancy driven flow between two vertical plates maintained constant but different temperatures has been studied analytically and numerically by many researchers (see, for example Vest and Arpacı (1969), Korpela et.al. (1973) and Gershuni and Zhukhovitskii (1976). The most interesting thing here is finding the threshold
value of the Prandtl number at which instability switches from stationary to travelling wave modes. Later Takashima and Hamabata (1984) showed that the threshold value of Pr is always lies between 12.4 and 12.5. The characteristics and the stability of two-dimensional buoyant-thermo capillary-driven flows in finite shallow cavities have been investigated for low-Prandtl-number fluids by Ben Hadid & Roux (1992). This study has shown the existence of complex flow structures in the form of multicellular steady states and the transition to oscillatory convection has been observed.

The interaction between magnetic field and fluid motion is observed in many fields of science and engineering. Depending on the nature of fluids, the effect of magnetic field becomes important. Takashima (1994) studied the effect of transverse magnetic field on the stability of natural convection of an electrically conducting fluid is confined between two parallel vertical plates. He concluded that the magnetic field stabilizes the convective flow due to the value of Prandtl number Pr at the point of transition from stationary to travelling wave mode was found to decrease with increasing Hartmann number M. Hence the magnetic field effects are dominant if the fluid is highly electrically conducting. A recent numerical study (Gelfgat & Bar-Yoseph 2001) is also devoted to the effect of a magnetic field on the onset of oscillatory instability, but concerns the convective flows in a two-dimensional rectangular cavity.

Coupled heat and mass transfer in fluid saturated porous media finds applications in a variety of engineering processes such as in heat exchanger devices, insulation systems, petroleum reservoirs, magnetohydrodynamic accelerators and generators, filtration, chemical catalytic reactors and processes, nuclear waste repositories and problems of soil contamination by crude oil. Besides, the study is of importance in many engineering and technological areas. This includes high performance insulation for building and cold storage, cooling of nuclear fuel in shipping flasks and water filled storage bays. The instability of buoyancy opposed mixed convection in a vertical channel filled with a fluid - saturated porous medium is studied by Bera and Khalili (2006). Recently Shankar et.al. (2015) have studied the effect of horizontal alternating current field on the stability of natural convection in a dielectric fluid saturated vertical porous layer.

Porous materials used in many technological applications such as heat exchangers, chemical reactors and fluid filters possess high permeability and porosity. For example, for
permeabilities of compressed foams as high as $8 \times 10^{-6}$ m$^2$ and for a 1 mm thick foam layer, the equivalent Darcy number is equal to 8. In such situations, the Darcy model does not give satisfactory results and there is a need to consider non-Darcian effects to study the problem. Then the consideration of inertial effects is inevitable and they exhibit profound influence on the stability of convective flows. In addition, for a high porosity porous medium, Givler and Altobelli (1994) determined experimentally that $5.1 \leq \mu_c / \mu \leq 10.9$, where $\mu_c$ is the effective viscosity or the Brinkman viscosity and $\mu$ is the fluid viscosity. Therefore, it is instructive to consider the ratio of these two viscosities is different from unity. Hill and Straughan (2010) clearly mention that stability of flow in a porous channel cannot be discussed on the basis of an analogy with a magneto-hydrodynamic problem. This is because, in the resulting stability equation the terms containing porous parameter and Hartmann number are not same unlike in thermal convection problems.

The aim of this chapter is to investigate the effect of a transverse magnetic on the stability of natural convection in a vertical channel filled with a saturated porous media of electrically conducting fluid with fluid viscosity different from effective viscosity. The study is conducted through a numerical linear stability analysis that gives the variation of the critical characteristics of the different types of instabilities when the intensity of the magnetic field is increased. The plan of the paper is as follows. Section 6.2 presents the mathematical formulation of the problem and basic state solution is obtained in section 6.3. Section 6.4 contains details of the linear stability analysis. Section 6.5 discusses the numerical solution for the eigenvalue problem similar to Orr-Sommerfeld equation for the present study. Results and discussion are presented in section 6.6.
6.2 Mathematical Formulation

We consider an infinite sparsely packed porous layer of an incompressible electrically conducting fluid confined between two rigid parallel vertical plates at $x = -h$ and $x = h$, which we shall assume to be electrically non-conducting. The plates are maintained at constant and different temperatures $T_i$ and $T_o$, respectively, where the $x$ - axis of the rectangular coordinates is taken perpendicularly to the plates and $z$ - axis vertically upward. The fluid layer is permeated by a uniform externally applied magnetic field $B_o = (B_o, 0, 0)$ perpendicular to the channel axis. Figure 6.1 illustrate the geometric arrangement of the problem schematically.

The governing equations for the stated problem are:

**Flow continuity:**

$$\nabla \cdot \vec{q} = 0 ,$$  \hspace{1cm} (6.2.1)

**Momentum equation:**

$$\rho_o \left[ \frac{1}{\varepsilon} \frac{\partial \vec{q}}{\partial t} + \frac{1}{\varepsilon} (\vec{q} \cdot \nabla) \vec{q} \right] = \rho \vec{g} - \nabla P + \mu_v \nabla^2 \vec{q} + \frac{1}{\mu} (\vec{B} \cdot \nabla) \vec{B} - \frac{\mu_f}{k} \vec{q} .$$  \hspace{1cm} (6.2.2)

**Magnetic continuity:**

$$\nabla \cdot \vec{B} = 0 .$$  \hspace{1cm} (6.2.3)

**Magnetic field equation:**

$$\frac{\partial \vec{B}}{\partial t} + \frac{1}{\varepsilon} (\vec{q} \cdot \nabla) \vec{B} = \frac{1}{\varepsilon} (\vec{B} \cdot \nabla) \vec{q} + \frac{1}{\sigma \mu} \nabla^2 \vec{B} .$$  \hspace{1cm} (6.2.4)

**Energy equation:**

$$\frac{\partial T}{\partial t} + \frac{1}{\varepsilon} (\vec{q} \cdot \nabla) T = \kappa \nabla^2 T .$$  \hspace{1cm} (6.2.5)

**Equation of state:**

$$\rho = \rho_o \left[ 1 - \alpha_f (T - T_o) \right] .$$  \hspace{1cm} (6.2.6)

where $\vec{q} = (u, v, w)$ the flow velocity, $\vec{B} = (B_x, B_y, B_z)$ the magnetic field vector, $\vec{g} = (0, 0, -g)$ the gravitational acceleration, $\rho$ the fluid density, $P = P + B^2 / 2\mu$ the total pressure, $\mu$ the magnetic permeability, $\mu_f$ the fluid viscosity, $\mu_e$ the effective viscosity, $k$ the permeability, $\varepsilon$
the porosity of the porous medium, $\sigma$, the electrical conductivity, $\rho_0$, the reference density at $T_o$, $T$ the temperature, $T_e = (T_1 + T_2)/2$ the temperature at $x=0$, $\alpha_r$ the volumetric thermal expansion coefficient, $\kappa$ the thermometric conductivity and $t$ the time.

### 6.3 Base Flow

For channels with constant cross section, as the one depicted in Fig. 6.1, a fully developed equilibrium flow is established. In this case, the flow velocity $\vec{q} = w_b(x) \hat{k}$ has only on component depending on the coordinate $x$. The magnetic field is decomposed into two contributions, one due to the external imposed magnetic field and the other cause by the magnetic field induced by the flow $\vec{B} = B_{ji} \hat{j} + B_b \hat{k}$. With these assumptions, Eqs. (6.2.2), (6.2.4), (6.2.5) and (6.2.6) reduces to

\[
0 = -\rho_b \frac{dP_b}{dz} + \rho \frac{d^2w_b}{dx^2} + \frac{\mu}{\mu} \frac{dB_b}{dx} \frac{\mu}{k} \frac{w_b}{k}, \tag{6.3.1}
\]

\[
0 = \frac{B_b}{\varepsilon} \frac{dw_b}{dx} + \frac{1}{\sigma \mu} \frac{d^2B_b}{dx^2}, \tag{6.3.2}
\]

\[
0 = \frac{d^2T_b}{dx^2}, \tag{6.3.3}
\]

\[
\rho_b = \rho_0 \left[ 1 - \alpha_r (T_b - T_e) \right]. \tag{6.3.4}
\]

Where the subscript $b$ denotes the basic state. Under these circumstance, the steady state solution are found to be

\[
T_b = T_o + \beta x, \tag{6.3.5}
\]

\[
\rho_b = \rho_0 \left[ 1 - \alpha_r \beta x \right], \tag{6.3.6}
\]

\[
w_b = \frac{\varepsilon g k \alpha_r \beta \rho_0}{\mu_f \varepsilon + B^2_b k \sigma} \left( x - h \frac{\text{Sinh} \omega x}{\text{Sinh} \omega h} \right), \tag{6.3.7}
\]

\[
B_b = \frac{\varepsilon g k \alpha_r \beta \rho_0}{\left( \mu_f \varepsilon + B^2_b k \sigma \right)} \left( \frac{h^2 - x^2}{2} \right) - \frac{h \sqrt{k \mu_c}}{\sqrt{\varepsilon \mu_f + B^2_b k \sigma}} \frac{\left( \text{Cosh} \omega h - \text{Cosh} \omega x \right)}{\text{Sinh} \omega h}, \tag{6.3.8}
\]

\[
P_b = \rho_o - \rho_0 g z, \tag{6.3.9}
\]
where \( p_o \) is a constant, \( \beta = \frac{(T_s - T)}{h} \) the horizontal temperature gradient and
\[
\omega = \sqrt{\frac{\mu_f}{k\mu_o} \frac{E + B_o^2 k^2 \sigma}{k\mu_o}}.
\]
It should be noted that \( w_b \) and \( B_b \) have been determined under the conditions that \( w_b = B_b = 0 \) at \( x = \pm h \). It should also be noted here that \( P_b \) have been determined under the condition that the total flux of flow across a horizontal plane \( z = \) constant is zero.

### 6.4 Linear Stability Analysis

To study the linear stability of fluid flow, we superimpose an infinitesimal disturbance on the base flow in the form
\[
\begin{align*}
u & = -\epsilon \frac{\partial \phi}{\partial z} (x, z, t), \\
w & = w_b(x) + \epsilon \frac{\partial \phi}{\partial x} (x, z, t), \\
P & = P_b(z) + \epsilon \dot{P}(x, z, t), \\
\dot{B}_z & = B_b \frac{\partial \psi}{\partial z} (x, z, t), \\
\dot{B}_x & = B_b(x) + \epsilon \frac{\partial \psi}{\partial x} (x, z, t), \\
\rho & = \rho_b(x) + \epsilon \dot{\rho}(x, z, t), \\
T & = T_b(x) + \epsilon \dot{T}(x, z, t),
\end{align*}
\]
(6.4.1)
where \( \phi \) the stream function, \( \psi \) is the magnetic stream function and \( \epsilon \) is a small quantity. These relations automatically satisfies the continuity Eqs. (6.2.1) and (6.2.3). Substituting Eq. (6.4.1) into Eqs. (6.2.2), (6.2.4), (6.2.5) and (6.2.6). Equating the coefficients of leading order \( (\epsilon) \) and restricting the attention to two-dimensional disturbances, we obtain
\[
\begin{align*}
\frac{1}{\epsilon} \dot{\phi}_x + \frac{1}{\epsilon^2} w_b(x) \dot{\phi}_z &= \frac{1}{\rho_o} \dot{P}_z + \frac{\mu_e}{\rho_o} \left( \dot{\phi}_{xx} + \dot{\phi}_{zz} \right) - \frac{\mu_e}{k\rho_o} \dot{\phi}_x, \\
\frac{1}{\epsilon} \dot{\phi}_z + \frac{1}{\epsilon^2} \left( w_b(x) \dot{\phi}_x - D w_b(x) \dot{\phi}_z \right) &= -\frac{1}{\rho_o} \dot{P}_z - g \alpha_z \dot{T} + \frac{\mu_e}{\rho_o} \left( \dot{\phi}_{xx} + \dot{\phi}_{zz} \right) \\
&+ \frac{1}{\mu_p} \left( B_o \psi_{xx} - DB_o \psi_z + B_o \psi_{zz} \right) - \frac{\mu_f}{k\rho_o} \dot{\phi}_x \\
\psi_{xx} + \frac{1}{\epsilon} w_b(x) \psi_{zz} &= \frac{1}{\epsilon} \left( B_o \phi_{xx} + B_o \phi_{zz} \right) + \frac{1}{\sigma \mu} \left( \psi_{xx} + \psi_{zz} \right), \\
\dot{T}_z + \frac{1}{\epsilon} \left( w_b(x) \dot{T}_x + DT_b \dot{\phi}_z \right) &= \kappa \left( \dot{T}_{xx} + \dot{T}_{zz} \right).
\end{align*}
\]
(6.4.2) – (6.4.5)
Eliminating pressure term from the Eqs. (6.4.2) and (6.4.3), we have the following

\[
\frac{1}{\varepsilon} \left( \hat{\phi}_{zz} + \hat{\phi}_{xx} \right) + \frac{1}{\varepsilon^2} \left( w_b(x) \left( \hat{\phi}_{zz} + \hat{\phi}_{xx} \right) - D^2 w_b \hat{\phi}_z \right) = \alpha T g \hat{T}_z + \frac{\mu_r}{\rho_0} \nabla^2 \left( \hat{\phi}_{zz} + \hat{\phi}_{xx} \right) \\
+ \frac{1}{\mu \rho_0} \left[ B_0 (\hat{\psi}_{xx} + \hat{\psi}_{zz}) + B_b (\hat{\psi}_{zz} + \hat{\psi}_{xx}) - D^2 B_0 \hat{\psi}_z \right] - \frac{\mu_r}{k \rho_0} \left( \hat{\phi}_{zz} + \hat{\phi}_{xx} \right) \\
\hat{\psi}_z + \frac{1}{\varepsilon} w_b(x) \hat{\psi}_z = \frac{1}{\varepsilon} \left( B_b \hat{\phi}_z + B_b \hat{\phi}_r \right) + \frac{1}{\sigma \mu} \left( \hat{\psi}_{xx} + \hat{\psi}_{zz} \right), \tag{6.4.7}
\]

\[
\hat{T}_i + \frac{1}{\varepsilon} \left( w_b(x) \hat{T}_z + DT_b \hat{\phi}_z \right) = \kappa \left( \hat{T}_z + \hat{T}_z \right). \tag{6.4.8}
\]

Equations (6.4.2) - (6.4.5) can be made dimensionless by choosing \( \frac{\alpha T \beta gh^3}{\nu} \), \( B_0 \), and \( \beta h \) as the units of length, time, stream function, magnetic stream function and temperature respectively, we have the following

\[
\left( \hat{\phi}_{zz} + \hat{\phi}_{xx} \right) + w_b(x) \left( \hat{\phi}_{zz} + \hat{\phi}_{xx} \right) - D^2 w_b \hat{\phi}_z = \frac{1}{G} \hat{T}_z + \frac{\Lambda}{G} \nabla^2 \left( \hat{\phi}_{zz} + \hat{\phi}_{xx} \right) \\
+ \frac{M^2}{G} \left[ \frac{1}{GPr_m} (\hat{\psi}_{xx} + \hat{\psi}_{xx}) + B_b (\hat{\psi}_{zz} + \hat{\psi}_{xx}) - D^2 B_0 \hat{\psi}_z \right] - \frac{\sigma^2}{G} \left( \hat{\phi}_{zz} + \hat{\phi}_{xx} \right), \tag{6.4.9}
\]

\[
\hat{\psi}_z + w_b(x) \hat{\psi}_z = \hat{\phi}_z + \frac{Pr_m G B_b \hat{\phi}_z}{\sigma \mu G} + \frac{1}{Pr_m G} \nabla^2 \hat{\psi}_z, \tag{6.4.10}
\]

\[
\hat{T}_i + w_b(x) \hat{T}_z + \hat{\phi}_z = \frac{1}{Pr G} \nabla^2 \hat{T}, \tag{6.4.11}
\]

where \( G = \frac{\alpha T \beta gh^3}{\nu^2 \varepsilon^2} \) is the Grashof number,

\( Pr = \frac{\nu}{\kappa} \) is the Prandtl number,

\( M = B_0 \sqrt{\frac{\sigma}{\rho_b \nu}} \) is the Hartmann number,

\( \sigma_p = \frac{h}{\sqrt{k}} \) is the porous parameter,

\( Pr_m = \sigma \mu \nu \) is the magnetic Prandtl number.
and $\Lambda - \frac{\mu_k}{\mu_f}$ is the ratio of viscosities.

It should be noted here that the basic velocity and basic magnetic field in dimensionless form are

$$w_b = \frac{1}{M^2 + \sigma_p^2} \left( x - \frac{\text{Sinh} \sqrt{\frac{M^2 + \sigma_p^2}{\Lambda}} - x}{\text{Sinh} \sqrt{\frac{M^2 + \sigma_p^2}{\Lambda}}} \right), \quad (6.4.12)$$

$$B_b = \frac{1}{M^2 + \sigma_p^2} \left( \frac{1-x^2}{2} - \frac{\text{Cosh} \sqrt{\frac{M^2 + \sigma_p^2}{\Lambda}} - \text{Cosh} \sqrt{\frac{M^2 + \sigma_p^2}{\Lambda}} - x}{\text{Cosh} \sqrt{\frac{M^2 + \sigma_p^2}{\Lambda}} - \text{Sinh} \sqrt{\frac{M^2 + \sigma_p^2}{\Lambda}}} \right). \quad (6.4.13)$$

It should also be noted here that $\lim_{M \to 0} w_b = \frac{1}{6} x(1-x^2)$ when $\sigma_p = 0$ and $\Lambda = 1$.

We seek normal mode solution of the linearized system of Eqs. (6.4.9) to (6.4.11) in the form

$$\begin{align*}
\hat{\phi}(x,z,t) &= \phi(x) \exp\{i\alpha(z-ct)\}, \\
\hat{\psi}(x,z,t) &= \psi(x) \exp\{i\alpha(z-ct)\}, \\
\hat{T}(x,z,t) &= \theta(x) \exp\{i\alpha(z-ct)\}
\end{align*} \quad (6.4.14)$$

Where $c = c_r + ic_i$ is the wave speed, $c_r$ is the phase velocity and $c_i$ is the growth rate and $\alpha$ is the stream wise wave number which is real and positive. If $c_i > 0$, then the system is unstable and if $c_i < 0$, then the system is stable. Substituting Eq. (6.4.14) into Eqs. (6.4.9) - (6.4.11), we obtain

$$\begin{align*}
(w_b - c) \left( D^2 - \alpha^2 \right) \phi - \left( D^2 w_b \right) \phi &= -\frac{1}{i\alpha G} D\theta + \frac{\Lambda}{i\alpha G} \left( D^2 - \alpha^2 \right) \phi \\
&\quad + M^2 \left[ \frac{1}{i\alpha G P m} D \left( D^2 - \alpha^2 \right) \psi + B_b \left( D^2 - \alpha^2 \right) \psi - D^2 B_b \psi \right] - \frac{\sigma_p^2}{i\alpha G} \left( D^2 - \alpha^2 \right) \phi \\
(w_b - c) \psi &= \frac{1}{i\alpha G} D\phi + \frac{1}{i\alpha G P m} \left( D^2 - \alpha^2 \right) \psi,
\end{align*} \quad (6.4.15)$$

$$\begin{align*}
(w_b - c) \theta &= \frac{1}{i\alpha G} D\phi + \frac{1}{i\alpha G P m} \left( D^2 - \alpha^2 \right) \psi,
\end{align*} \quad (6.4.16)$$
\[(w_b - c) \theta + \phi = \frac{1}{i \alpha Pr G} \left( D^2 - \alpha^2 \right) \theta . \quad (6.4.17)\]

Since the magnetic Prandtl number \( Pr_m \) is very small for most of the electrically conducting fluids, therefore Eqs. (6.4.15) and (6.4.16) may be approximated as

\[
(w_b - c) \left( D^2 - \alpha^2 \right) \phi - \left( D^2 w_b \right) \phi = \frac{1}{i \alpha G} D \theta + \frac{\Lambda}{i \alpha G} \left( D^2 - \alpha^2 \right)^2 \phi
\]

\[
+ \frac{M^2}{i \alpha G Pr_m} D \left( D^2 - \alpha^2 \right) \psi - \frac{\sigma_p^2}{i \alpha G} \left( D^2 - \alpha^2 \right) \phi
\]

\[-Pr_m D \phi = \left( D^2 - \alpha^2 \right) \psi . \quad (6.4.19)\]

Operating \( D \) on both sides of the Eq. (6.4.19) and substituting in Eq. (6.4.18), we get the following stability equations

\[
(w_b - c) \left( D^2 - \alpha^2 \right) \phi - \left( D^2 w_b \right) \phi = \frac{1}{i \alpha G} \begin{bmatrix}
-D \theta + \Lambda \left( D^2 - \alpha^2 \right)^2 \phi \\
-M^2 D^2 \phi - \sigma_p^2 \left( D^2 - \alpha^2 \right) \phi
\end{bmatrix}, \quad (6.4.20)
\]

\[
(w_b - c) \theta + \phi = \frac{1}{i \alpha Pr G} \left( D^2 - \alpha^2 \right) \theta . \quad (6.4.21)
\]

The boundaries are rigid and the appropriate boundary conditions are:

\[\phi = D \phi = \theta = 0 \text{ at } x = \pm 1. \quad (6.4.22)\]

### 6.5 Numerical solution

Equations (6.4.20) and (6.4.21) together with the boundary conditions (6.4.22) constitute an eigenvalue problem which has to be solved numerically. The resulting eigenvalue problem is solved using Galerkin method. Accordingly, \( \phi(x) \) and \( \theta(x) \) are expanded in terms of Legendre polynomials in the form

\[\phi(x) = \sum_{n=0}^{N} a_n \xi_n(x), \quad \theta(x) = \sum_{n=0}^{N} b_n \xi_n(x), \quad (6.5.1)\]

with the corresponding base functions

\[\xi_n(x) = (1-x^2)^2 P_n(x), \quad \zeta_n(x) = (1-x^2) P_n(x), \quad (6.5.2)\]
where, \( P_n(x) \) is the Legendre polynomial of degree \( n \) and \( a_n \) and \( b_n \) are constants. It may be noted that \( \phi(x) \) and \( \theta(x) \) satisfies the boundary conditions. Eq. (6.5.1) is substituted into Eqs. (6.4.20) and (6.4.21) and the resulting error is required to be orthogonal to \( \xi_m(x) \) and \( \zeta_m(x) \) for \( m = 0,1,2,\ldots,N \). This gives

\[
\sum_{n=0}^{N} a_n \int_{-1}^{1} \left( \xi''_{m} \xi''_{n} + 2\alpha^2 \xi'_{m} \xi'_{n} + \alpha^4 \xi_{m} \xi_{n} \right) dx + i\alpha G \sum_{n=0}^{N} a_n \int_{-1}^{1} \left( D^2 w \xi_{m} \xi_{n} + \alpha^2 \xi_{m} \xi_{n} - w \xi_{m} \xi_{n} \right) dx \\
+ M \sum_{n=0}^{N} a_n \int_{-1}^{1} \xi'_{m} \xi'_{n} dx + \sigma_p \sum_{n=0}^{N} a_n \int_{-1}^{1} \left( \xi''_{m} \xi_{n} + \alpha^2 \xi_{m} \xi_{n} \right) dx - \sum_{n=0}^{N} b_n \int_{-1}^{1} \xi'_{m} \xi_{n} dx \quad (6.5.3)
\]

\[
- i\alpha G c \sum_{n=0}^{N} a_n \int_{-1}^{1} \left( \xi''_{m} \xi_{n} + \alpha^2 \xi_{m} \xi_{n} \right) dx
\]

\[
i\alpha Pr G \sum_{n=0}^{N} a_n \int_{-1}^{1} \xi_{m} \xi_{n} dx + \sum_{n=0}^{N} b_n \int_{-1}^{1} \left( \xi''_{m} \xi_{n} + \alpha^2 \xi_{m} \xi_{n} \right) dx +
\]

\[
i\alpha Pr G \sum_{n=0}^{N} b_n \int_{-1}^{1} w \xi_{m} \xi_{n} dx = c i\alpha Pr G \sum_{n=0}^{N} b_n \int_{-1}^{1} \xi_{m} \xi_{n} dx \quad (6.5.4)
\]

in which the primed quantities denote differentiation with respect to \( x \).

The above equations form the following system of linear algebraic equations

\[
AX = c BX, \quad (6.5.5)
\]

where \( A \) and \( B \) are the complex matrices, \( c \) is the eigenvalue and \( X \) is the eigenvector. For fixed values of \( M, \sigma_p, Pr, G \) and \( \Lambda \), the values of \( c \) which ensure a non-trivial solution of Eq. (6.5.5) are obtained as the eigenvalues. The above presented system can be solved by standard routine. Here, the DGVLCG of the IMSL library was employed. The routine is based on the QZ algorithm due to Molar and Stewart. The first step of this algorithm is to simultaneously reduce \( A \) to upper Heisenberg form and \( B \) to upper triangular form. Then, orthogonal transformations are used to reduce \( A \) to quasi-upper-triangular form while keeping \( B \) upper triangular. The generalized Eigen values for the reduced problem are then computed. The critical wave speed \( c_c \) and the corresponding critical Grashof number \( G_c \) and the critical wave number \( \alpha_c \) are determined for various values of \( M, \sigma_p \) and \( \Lambda \).
6.6 Results and discussion

The effect of transverse magnetic field on the stability of buoyancy driven vertical porous layer is investigated. The resulting eigenvalue problem similar to Orr-Sommerfeld equation is solved numerically. The parameters involved in the present study are the porous parameter $\sigma_p$, the ratio of effective and fluid viscosities $\Lambda$, the Prandtl number $Pr$, Grashof number $G$ and the Hartmann number $M$.

The convergence of the numerical method employed is tested for different sets of parametric values by varying the order of base polynomial $N$ and the results obtained are tabulated in Table 6.1. From this table it is evident that, as the number of terms increased in Eq. (6.5.1), the results are found to remain consistent and accuracy improved up to 6 digits for $N = 30$. Solutions of up to $8^{th}$ digit accuracy could be reached by taking 41 terms in the Galerkin method and hence the results are obtained for $N = 40$ in general. For code validation, we compared with published results for the natural convection in a vertical fluid layer of electrically conducting fluid in the presence of a transverse magnetic field (Takashima(1994)), the results so obtained for different values of Hartmann number are compared in Table 6.2 and the results are found to be in good agreement.

Before analyze the linear stability, we have investigated the influence of Hartmann number $M$, porous parameter $\sigma_p$, and the ratio of viscosities $\Lambda$ on the basic velocity profiles $w_b$, which are shown in Figs. 6.2(a-c). From these figures it is observed that the velocity profiles for various values of the physical parameters does not have symmetry about vertical line $x = 0$, which is may be due to the fixed direction of the gravitational field. Also the base flow consists of three streams, one is upstream near to $x = 1/2$, other one is downstream near to $x = -1/2$ and zero at $x = 0$ and at the wall. Fig. 6.2(a) shows the influence of Hartmann number $M$ on the base flow and which indicate that increase in the value of $M$ is to suppress the fluid flow. Similar behavior can be observed for both the porous parameter $\sigma_p$ (Fig. 6.2b) and the ratio of viscosities $\Lambda$ (Fig. 6.2c). Fig. 6.3(a-c), respectively shows the basic magnetic field profiles for different values of the Hartmann number, porous parameter and ratio of viscosities. All these
profiles are symmetrically distributed on either side of the line $x=0$. From these figures it is observed that the basic magnetic field becomes suppresses when $M, \sigma_p$ and $\Lambda$ are increased.

The shapes of the marginal stability curves in the $(G, \alpha)$–plane are illustrated in Fig. 6.4(a) – 6.4(d) for various values of $Pr, M, \sigma_p$ and $\Lambda$. In these figures, curves for neutral stability for the stationary and travelling wave modes are shown for different values of the above mentioned parameters. Also these neutral curves exhibit single but different minima with respect to wave number for various parameters involved in the problem. In all these figures, the portion below each neutral curve is corresponds to stable region represented by $c_i(G, \alpha)<0$ and the region above neutral stability curve $c_i(G, \alpha)>0$ corresponds to unstable region. It may noted that increase in $Pr$ leads to increase in the region of stability in travelling wave mode while opposite trend is observed via stationary mode (Fig. 6.4a). Fig. 6.4(b) exhibits that increasing $M$ is to increase the region of stability and similar effect could be seen with increasing porous parameter (Fig. 6.4c) and ratio of viscosities (Fig. 6.4d).

Fig. 6.5(a) shows the variation of $G_c$ with $Pr$ for different values of Hartmann number $M$. There are four distinct curves corresponds $M = 0.1, 1, 3$ and $5$ are considered here for the sake of observation. In all the figures the dashed curves represents stationary modes and solid curves represent travelling wave modes. The curves for higher values of $M$ emanate from lower values of $G_c$. All the curves follow a similar pattern of constant values of $G_c$ for increasing values of $Pr$. The curve for $M = 0.1$ plummets as the value of $Pr$ equals 7.4 where the $G_c$ value reduces from 23193 to 12731. Subsequently, it experiences a gradual fall in $G_c$ values as the value of $Pr$ increases. The curves for $M = 1, 3, 5$ also experience a fall in $G_c$ values as the curve progresses further. $M = 0.1, 1$ have a very slight variation and they follow a similar pattern of change in trend where they almost coalesce at higher values of $Pr$. In other words that dependence of $G_c$ on $Pr$ is very meager for the case of stationary modes but strongly decreasing function of $Pr$ in the case of travelling wave mode. It is further seen that increase in $M$ is to increase the value of $G_c$ and hence magnetic field has a stabilizing effect on the fluid flow in both travelling and stationary wave disturbances. Also, depending on the value of $M$,
there exists a threshold value of $Pr$ at which the instability changes from stationary to travelling wave mode. This value of $Pr$ is found to decrease with increasing $M$. The vertical lines represent the discontinuous changes in $a_c$ due to the transition from stationary to travelling-wave mode (Fig. 6.5b). It is evident from the figure that the dependence of $a_c$ at stationary mode upon $Pr$ is weak, whereas $a_c$ at travelling-wave mode depends strongly upon $Pr$ and $a_c$ for stationary mode decreases with increasing $M$ while an opposite behavior is noticed when the disturbances are travelling waves (Fig. 6.5b). The results of travelling-wave instability summarized in Fig. 6.5(c) indeed confirm the above observed behavior more evidently, which shows the variation of positive $c_c$ with $Pr$ for various values of $\Lambda_c$. The discontinuous changes in $c_c$ due to the transition from stationary ($c_c = 0$) to travelling-wave ($c_c \neq 0$) mode are represented by the vertical lines. Form this figure it is seen that $c_c$ decreases with increase in $M$.

Fig. 6.6(a) illustrates $G_c$ as a function of $Pr$ for different values of $\sigma_p$ when $M = 2$ and $\Lambda = 1$. The curve for $\sigma_p = 5$ progresses with constant values of $G_c$ for increasing values of $Pr$. It then experiences an exponential fall in $G_c$ values as $Pr$ increases. The curves for $\sigma_p = 0.1, 1, 3$ experience a similar trend. Subsequently, they experience a fall in $G_c$ values as the value of $Pr$ increases. Increasing the value of $\sigma_p$ leads to suppress the system due to decrease in the permeability of the porous media. It is seen from Fig. 6.6(b) that as $\sigma_p$ increases $a_c$ decreases at stationary mode, whereas it exhibits opposite trend at travelling wave mode. The corresponding positive critical wave speed $c_c$ are illustrated in Fig. 6.6(c) as a function of $Pr$. It is noted that $c_c$ decreases with increase in $\sigma_p$.

The variation of $G_c$ with $P$ for different values of $\Lambda$ is illustrated in Fig. 6.7(a). The curves for $\Lambda = 10, 8, 6, 1$ follow a very similar trend where they experience a increase in $Pr$ values with constant $G_c$ values at the stationary mode, forthwith the curve experiences negative gradient with fall in $G_c$ values for an increase in $Pr$ via travelling wave mode. Moreover, the threshold value of $Pr$ (transition mode) decreases considerably with increasing $\Lambda$. It is also
observed that decreasing $\Lambda$ will lead to the decrease in $G_c$ and thus it has a stabilizing effect on the system. This is because, increasing $\Lambda$ amounts to increase in the viscous effect, which in turn retards the fluid flow. From Fig. 6.7(b), it is seen that $a_c$ drops suddenly at the transition mode and increasing $Pr$ is to increase the value of $a_c$ for all values of $\Lambda$ at the travelling wave mode while at stationary mode the variation in the $a_c$ is found to be insignificant. Figure 6.7(c) shows that increasing $\Lambda$ is to decrease critical wave speed.

![Diagram](image.png)

**Fig. 6.1:** Physical configuration of the system.
Fig. 6.2: Basic velocity profiles.
Fig. 6.3: Basic magnetic field profiles.
Fig. 6.4: Neutral stability curves. (………. ) stationary modes, ( ———— ) travelling wave modes.
Fig. 6.5: Variation of (a) critical Grashof number $G_c$, (b) critical wave number $\alpha_c$ and (c) critical wave speed $c_c$ with the Prandtl number $Pr$ for a fixed values of porous parameter $\sigma_p = 5$, ratio of effective to fluid viscosity $\Lambda = 1$ and for various values of Hartmann number $M$. (---------) stationary modes, (-----) travelling wave modes.
Fig. 6.6: Variation of (a) critical Grashof number $G_c$, (b) critical wave number $\alpha_c$ and (c) critical wave speed $c_c$ with the Prandtl number $Pr$ for fixed values of Hartmann number $M = 2$, ratio of effective to fluid viscosity $\Lambda = 1$ and for various values of porous parameter $\sigma_p$. (…………) stationary modes, (———) travelling wave modes.
Fig. 6.7: Variation of (a) critical Grashof number $G_c$, (b) critical wave number $\alpha_c$, and (c) critical wave speed $c_c$ with the Prandtl number $Pr$ for fixed values of Hartmann number $M = 2$, porous parameter $\sigma_p = 2$ and for various values of ratio of effective to fluid viscosity $\Lambda$. (..........) stationary modes, (-----) travelling wave modes.
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Table 6.1: Order of polynomial independence

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<th>$\alpha_c$</th>
<th>$\gamma_c$</th>
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Table 6.2: Comparison between published results and present results.