CHAPTER 2

THERMAL EFFECTS IN STOKES’ SECOND PROBLEM FOR UNSTEADY MICROPOLAR FLUID FLOW THROUGH A POROUS MEDIUM

2.1 Introduction

The theory of micropolar fluids introduced by Eringen [34,35], deals with a class of fluids which exhibit certain microscopic effects arising from the local structure and micro-motions of fluid elements. These fluids can support stress moments and body moments and are influenced by the spin-inertia. A subclass of these fluids, introduced by Eringen [36], is the micro polar fluids which exhibit the micro-rotational effects and micro-rotational effects and micro-rotational inertia. The fluids containing certain additives, some polymeric fluids and animal blood are examples of micropolar fluids. The mathematical theory of equations of micropolar fluids and application of these fluids in the theory of lubrication and in the theory of porous media is presented in Lukaszewicz [49].

The flow induced by a suddenly accelerating plate on the fluid above it, usually referred to as Stokes’ first problem (Stokes, [76]), and the flow due to an oscillating flat plate, usually referred to as Stokes’ second problem (see Raleigh, [67]) are amongst a handful of unsteady flows of a Navier–Stokes fluid for which one can obtain an exact solution. Such exact solutions serve a dual purpose, that of providing an explicit solution to a problem that has physical relevance and as a means for testing the efficiency of complex numerical schemes for flows in complicated flow domains.
These two problems have been extended to the case of a host of non-Newtonian fluids. Preziosi and Joseph [60] have investigated Stokes’ first problem for viscoelastic fluid. Stokes’ first problem for a Rivlin–Ericksen fluid of second grade in a porous half-space was developed by Asghar et al. [8], have studied the Stokes’ second problem for second–grade fluid analytically. The study of a variant of Stokes’ first and second problems for fluids with pressure dependent viscosities was studied by Srinivasan and Rajagopal [75].

In classical unsteady heat transfer problems, the basic equations are derived from Fourier’s law of heat conduction, which results in a parabolic equation for the temperature field and an infinite speed of heat propagation, thus violating the principle of causality. Maxwell [50] derived the generalization of Fourier’s heat law for the dynamical theory of gases. Maxwell’s heat flux equation contains a term proportional to the time derivative of the heat flux vector multiplied by a constant relaxation time $\tau$. Had a very small magnitude in Maxwell’s work, he took it to be zero. In justification he remarked, “The first term of this equation may be neglected, as the rate of conduction will rapidly establish itself”. Ackerman et al. [3], established the second sound in solid helium, which gave a finite speed of propagation of thermal waves. Puri and Kythe [63], have studied the influence of generalized law of heat conduction, using the Maxwell-Cattaneo-Fox (MCF) model, on Stokes’ first and second problems for Rivlin-Ericksen fluids with non-classical heat conduction. Kythe and Puri [47], studied the unsteady MHD free-convection flows on a porous plate with time-dependent heating in a rotating medium. Puri and Kythe [61] have studied an unsteady flow problem which deals with non-classical heat conduction effects and the structure of waves in the Stokes’ second problem. In (MCF) model as developed
in Mc Taggart and Lindsay [51] the non-classical constitutive equation for the heat-flux vector $q$ is given by the Maxwell-Catta-neo equation in the form.

$$\tau(\dot{q}_i - \omega_j q_j) = -q_i - \chi \theta_i$$  \hspace{1cm} (2.1.1)

where $\omega_j$ is the vorticity, $\chi$ the thermal conductivity, $\theta$ the temperature, and $\tau$ the thermal relaxation time. If $\omega_j = 0$ Eq. (2.1.1) reduces to that Cattaneo model, and for $\tau = 0$ it becomes Fourier’s law (Joseph and Preziosi, [46]) While there are other good models to choose from, the Cattaneo law, as stated in Joseph and Preziosi [46] has many desirable properties, e.g., the steady heat flow may be induced by temperature gradients and gives rise to finite speeds of propagation. Based on these, Ibrahem et al. [44] have investigated the thermal effects in Stokes’ second problem for unsteady micropolar fluid flow.

The problem of micropolar fluids past through a porous media has many applications, such as, porous rocks, foams and foamed solids, aerogels, alloys, polymer blends and micro emulsions. The simultaneous effects of a fluid inertia force and boundary viscous resistance upon flow and heat transfer in a constant porosity porous medium were analyzed by Vafai and Tien [82]. Raptis [69] studied boundary layer flow of a micropolar fluid through a porous medium. Abo-Eldahab and El Gendy [1] investigated the convective heat transfer past a stretching surface embedded in non-darcian porous medium in the presence of magnetic field. Radiation effect on heat transfer of a micropolar fluid past unmoving horizontal plate through a porous medium was investigated by Abo-Eldahab and Ghonaim [2].

In view of these, we studied the non-classical heat conduction effects in Stokes’ second problem of a micropolar fluid through a porous medium. The
expressions for the velocity field, angular velocity and temperature field are obtained analytically. The effects of $\beta$, Grashof number $G$ and Hartmann number $M$ on the velocity field and Angular velocity are studied in detail.

2.2 Mathematical formulation

We consider the one-dimensional unsteady flow of a laminar, incompressible micropolar fluid through a porous medium past a vertical flat plate in the $xy$-plane and occupy the space $z > 0$, with $z$-axis in the vertical direction.

The plate initially at rest and at constant temperature $\theta_\infty$ which is the free stream temperature is moved with a velocity $U_0 e^{i\omega t}$ in its own plane along the $z$-axis, and its temperature is subjected to a periodic heating of the form $(\theta_w - \theta_\infty) e^{i\omega t}$, where $\theta_w \neq \theta_\infty$ is some constant.

The basic equations of continuity, momentum, angular momentum and energy governing such a flow, subject to the Boussinesq approximation, are

$$\mathbf{v}_{i,j}^\ast = 0, \tag{2.2.1}$$

$$\rho \mathbf{v}_i^\ast = -P_i^\ast + (\mu + \mu_r) \nabla^2 \mathbf{v}_i^\ast - \rho \left[ 1 - \alpha \left( \theta' - \theta_\infty' \right) \right] g \delta_{1i} + 2 \mu_r \mathbf{N}_i^\ast + t_{ki,k} - \frac{\mu}{k} \mathbf{v}_i \tag{2.2.2}$$

$$\rho \mathbf{j}^\ast \mathbf{N}^\ast = \gamma \nabla^2 \mathbf{N}^\ast \tag{2.2.3}$$

$$\rho \mathbf{v} = -q_{i,j} + t_{ik} d_{ik} \tag{2.2.4}$$

where the vector $\mathbf{v} = (u,0,0)$ represents the velocity, $\rho$ - the density, $j^\ast$ - the micro-inertia density, $N^\ast$ - the component of angular velocity vector, $\gamma$ - the spin gradient
viscosity, $\mu$ - the dynamic viscosity, $P$ - the pressure, $\varepsilon$ - the specific internal energy, $\alpha$ - the co-efficient of thermal expansion, $g$ - the acceleration due to gravity, $t_{ik}$ - the non-Newtonian stress tensor, $k$ - permeability of the porous medium and $d_{ik}$ - the strain tensor.

The effect of microstructure is negligible in the neighborhood of a rigid boundary since the suspended particles cannot get closer than their radius to boundary. Thus in our study we consider the only rotation is due to fluid shear as pointed in Equation (2.2.3).

Taking into account the geometry of the problem which results in the disappearance of the dissipative terms and noting that $t_{ik}=0$, equations (2.2.1) - (2.2.3) reduce to the following equations of motion:

$$u_t^* = (\nu + \nu_r)u_{zz}^* + g\alpha (\theta^* - \theta^*_\infty) + 2\nu_r N_z^* - \frac{\nu}{k} u^*$$  \hspace{1cm} (2.2.5)

$$N_i^* = \frac{1}{\eta} N_{zz}^*$$  \hspace{1cm} (2.2.6)

Equation (2.1.1), after substitution into (2.2.4), gives

$$\rho c_p \dot{\theta}^* = -q_{t,i}$$  \hspace{1cm} (2.2.7)

where $\varepsilon = c_p \theta$ for the MCF model. If we drop the nonlinear terms $\tau \omega_{i,j}q_j$ in (2.1.1) because $\tau$ and $\omega$ are small quantities, we get

$$\tau \dot{q}_{t,i} = -q_{t,i} - \chi \theta_{t,i}$$  \hspace{1cm} (2.2.8)
which is one-dimensional form, after dropping the convective terms (because these terms become automatically zero), leads to

$$
\tau \theta_t^* + \theta_t^* = \frac{\chi}{\rho c_p} \theta_z^* 
$$

(2.2.9)

Note that the term \( \tau \theta_t^* \) in (2.2.9) is necessary to ensure finite speed of propagation.

We shall use the non-dimensional quantities.

$$
Z^* = \frac{v}{U_0} z, \quad u^* = U_0 u, \quad t^* = \frac{v}{U_0^2} t, \quad \theta = \frac{\theta^* - \theta_0^*}{\theta_w^* - \theta_0^*}, \quad N^* = \frac{U_0^2}{v} N 
$$

(2.2.10)

Then the governing equations (2.2.5), (2.2.6) and (2.2.9) for the flow, angular velocity and heat conducting, after suppressing the primes, become

$$
u = (1 + \beta)u_{zz} + G\theta + 2\beta N_z - \frac{1}{Da} u 
$$

(2.2.11)

$$
N_1 = \frac{1}{\eta} N_{zz} 
$$

(2.2.12)

$$
\lambda p \theta_u + p \theta_t = \theta_{zz} 
$$

(2.2.13)

where \( G \) is the Grashof number and \( \beta \) viscosity ratio

$$
G = \frac{v g \alpha (\theta_w^* - \theta_0^*)}{U_0^3}, \quad p = \frac{\nu \rho c_p}{\chi}, \quad C = \frac{\tau^* U_0^2}{\nu^2 \rho c_p}, \quad \beta = \frac{v}{\nu}, 
$$

$$
\lambda = \frac{\tau U_0^2}{\nu} = C_p, \quad \eta = \frac{2}{2 + \beta}, \quad Da = \frac{k U_0^2}{\nu^2} 
$$

(2.2.14)
The boundary conditions are

\[ u(0,t) = e^{i\omega t}, \quad \theta(0,t) = e^{i\omega t}, \quad \frac{\partial N(0,t)}{\partial z} = -\frac{\partial^2 u(0,t)}{\partial z^2} \]

\[ u(\infty,t) = 0, \quad \theta(\infty,t) = 0, \quad N(\infty,t) = 0 \] (2.2.15)

2.3 Solution

To solve the nonlinear system (2.2.11) – (2.2.13) with the boundary conditions (2.2.15), we assume that

\[ u(z,t) = U(z)e^{i\omega t}, \quad \theta(z,t) = \Theta(z)e^{i\omega t}, \quad N(z,t) = N(z)e^{i\omega t} \] (2.3.1)

If we substitute by Equation (2.3.1) in Equations (2.2.11) – (2.2.13) and the boundary conditions (2.2.15), we get

\[ (1 + \beta)U'' - \left( \frac{1}{Da} + i\omega \right)U = -G\Theta - 2\beta N' \] (2.3.2)

\[ N'' - i\omega \eta N = 0 \] (2.3.3)

\[ \Theta'' + (\lambda p\omega^2 - i\omega p)\Theta = 0 \] (2.3.4)

The boundary conditions are

\[ U(0) = 1, \quad \Theta(0) = 1, \quad N'(0) = -U''(0), \]

\[ U(\infty) = 0, \quad \Theta(\infty) = 0, \quad N(\infty) = 0 \] (2.3.5)

Solving the equations (2.3.2) - (2.3.4) using the boundary conditions Eq. (2.3.5), we get
\[ U(Z) = c_6 e^{-(r_1 + ir_2)Z} - k_1 e^{-(r_1 + ir_2)Z} + k_2 c_1 e^{-m_2Z} \]  \hspace{1cm} (2.3.6)

\[ N(Z) = c_4 e^{-m_2Z} \]  \hspace{1cm} (2.3.7)

\[ \Theta(Z) = e^{-(r_1 + ir_2)Z} \]  \hspace{1cm} (2.3.8)

where

\[ s_1 = \sqrt{\frac{\left(\frac{1}{Da}\right)^2 + \omega^2 + \frac{1}{Da}}{2(1+\beta)}} \]  \hspace{1cm} \quad \text{and} \quad  \hspace{1cm} \[ s_2 = \sqrt{\frac{\left(\frac{1}{Da}\right)^2 + \omega^2 - \frac{1}{Da}}{2(1+\beta)}} \]

\[ r_1 = \omega P \left( \frac{\sqrt{\omega^2 \lambda^2 + 1 - \lambda \omega}}{2} \right) \]  \hspace{1cm} \quad \text{and} \quad  \hspace{1cm} \[ r_2 = \omega P \left( \frac{\sqrt{\omega^2 \lambda^2 + 1 + \lambda \omega}}{2} \right) \]

\[ m_1 = (1+i) \frac{\omega}{2+\beta} \]  \hspace{1cm} \quad \text{and} \quad  \hspace{1cm} \[ k_1 = \frac{G}{(1+\beta)(r_1 + ir_2)^2 - \left( \frac{1}{Da} + i\omega \right)} \]

\[ k_2 = \frac{2\beta m_1}{m_1^2(1+\beta) - \left( \frac{1}{Da} + i\omega \right)} \]  \hspace{1cm} \quad \text{and} \quad  \hspace{1cm} \[ G_2 = \left( 1 + \frac{K_2(s_1 + is_2)}{(K_2-1)m_1} \right) \]

\[ G_2 = \frac{\left(1 + K_1 - \frac{K_2(s_1 + is_2)}{(K_2-1)m_1}\right)}{G_1} \]  \hspace{1cm} \quad \text{and} \quad  \hspace{1cm} \[ C = \frac{(s_1 + is_2)}{(K_2-1)m_1} G_2 + \frac{K_1(r_1 + ir_2)}{(K_2-1)m_1} \]

2.4 Results and Discussions

2.4.1 Velocity field

Figs.2.1 - 2.10 show the effects of various values of the emerging parameters \( \beta, G, \omega \) and \( Da \) on the velocity (Re \( u \) and \( |u| \)) profiles.
Fig. 2.1 shows the effect of $\beta$ on the velocity profile $Reu$ for $Da = 0.1$, $p = 1$, $\omega = 1000$, $t = 0.1$, $\lambda = 0.005$ and $G = 5$. It is found that the momentum boundary layer thickness increase with an increase in $\beta$. Also it is observed that as $\beta$ increases $Reu$ increases. The same trend is observed in Fig. 2.2 for $|u|$. 

In order to study the effect of $G$ on velocity profiles ($Reu$ and $|u|$) we have plotted the graphs of $Reu$ and $|u|$ in Figs. 2.3 and 2.4 for $Da = 0.1$, $p = 1$, $\omega = 10$, $t = 0.1$, $\lambda = 0.005$ and $\beta = 0.2$. It is found that from these figures as $G$ increases, both $Reu$ and $|u|$ increases and there is no change in fluid boundary layer. Also it is found that the peak velocity decreases as $G$ increases. For large $\omega(=1000)$ the effect of $G$ is negligible as shown in Figs. 2.5 and 2.6.

Figs. 2.7 and 2.8 show the effect of the Darcy number $Da$ on $Reu$ and $|u|$, for $G = 5$, $p = 1$, $\omega = 10$, $t = 0.1$, $\lambda = 0.005$ and $\beta = 0.2$. It is found that, initially $Reu$ and $|u|$, both increase as $Da$ increases and then decreases with $Da$. For layer $\omega(=1000)$ the effect of $Da$ is negligible as shown in Figs. 2.9 and 2.10.

### 2.4.2 Angular velocity

Figs. 2.11 - 2.20 show the effects of various values of the emerging parameters $\beta$, $G$, $\omega$, $\lambda$ and $Da$ on the angular velocity ($ReN$ and $|N|$) profiles.

Fig. 2.11 depicts effects the variation of $ReN$ with $\beta$ for $Da = 0.1$, $p = 1$, $\omega = 1000$, $t = 0.1$, $\lambda = 0.005$ and $G = 5$. It is noticed that $ReN$ initially increases and then decreases with an increase in $\beta$. 


In order to study the effect of \( \beta \) on \( |Nu| \), we have plotted Fig. 2.12 for \( Da = 0.1, \ p = 1, \ \omega = 1000, \ t = 0.1, \ \lambda = 0.005 \) and \( G = 5 \). It is found that as \( \beta \) increases the amplitude of the angular velocity decreases.

Fig. 2.13 shows the effect of \( G \) on \( ReN \) for \( Da = 0.1, \ p = 1, \ \omega = 10, \ t = 0.1, \ \lambda = 0.005 \) and \( \beta = 0.2 \). It is observed that, \( ReN \) first increases and then decreases with increasing \( G \) across the boundary layer.

The variation of \( |N| \) with \( G \) for \( M = 1, \ p = 1, \ \omega = 10, \ t = 0.1, \ \lambda = 0.005 \) and \( \beta = 0.2 \) is presented in Fig 2.14. It is observed that, as \( G \) increases \( |N| \) decreases across the boundary layer. For large \( \omega = 1000 \) the effect of \( G \) is negligible in both \( ReN \) and \( |N| \) as shown in Figs. 2.15 and 2.16.

Fig. 2.17 depicts the effect of \( Da \) on \( ReN \) for \( G = 5, \ p = 1, \ \omega = 10, \ t = 0.1, \ \lambda = 0.005 \) and \( \beta = 0.2 \). It is observed that, \( ReN \) first increases and then with increasing \( Da \).

The variation of \( |N| \) with \( Da \) for \( G = 5, \ p = 1, \ \omega = 10, \ t = 0.1, \ \lambda = 0.005 \) and \( \beta = 0.2 \). Is displayed Fig.2.18. It is observed that the amplitude of the angular velocity decreases with increasing \( Da \). For large \( \omega = 1000 \) the influence of \( G \) is negligible in both \( ReN \) and \( |N| \) as shown in Figs. 2.19 and 2.20.

2.4.3 Temperature field

Typical variations of the temperature profiles along the spanwise coordinate are the same that presented by Puri and Kythe [61], therefore it omitted here.
2.5 Conclusions

We studied the non-classical heat conduction effects in Stokes’ second problem of a micropolar fluid through a porous medium. The expressions for the velocity field, angular velocity and temperature field are obtained analytically. It is found that, the \( \text{Re } u \) first increases and then decreases with increasing \( \beta \), \( G \) or \( Da \), whereas the \( |u| \) increases with increasing \( \beta \), \( G \) or \( Da \). Also it is observed that, the \( \text{Re } N \) first decreases and then increases with increasing \( \beta \), \( G \) or \( Da \), whereas \( |N| \) decreases with increasing \( \beta \), \( G \) or \( Da \).
Fig. 2.1. Effects of $\beta$ on $Re \ u$ for $Da=0.1$, $p=1$, $\omega=1000$, $t=0.1$, $\lambda=0.005$ and $G=5$. 
Fig. 2.2. Effects of $\beta$ on $|u|$ for $Da = 0.1$, $p = 1$, $\omega = 1000$, $t = 0.1$, $\lambda = 0.005$ and $G = 5$. 
Fig. 2.3. Effects of $G$ on $\text{Re } u$ for $Da = 0.1$, $p = 1$, $\omega = 10$, $t = 0.1$, $\lambda = 0.005$ and $\beta = 0.2$. 
Fig. 2.4. Effects of $G$ on $|u|$ for $Da = 0.1$, $p = 1$, $\omega = 10$, $t = 0.1$, $\lambda = 0.005$ and $\beta = 0.2$. 
Fig. 2.5. Effects of $G$ on $Re u$ for $Da = 0.1$, $p = 1$, $\omega = 1000$, $t = 0.1$, $\lambda = 0.005$ and $\beta = 0.2$. 

$G = 5, -5$
Fig. 2.6. Effects of $G$ on $|u|$ for $Da = 0.1$, $p = 1$, $\omega = 1000$, $t = 0.1$, $\lambda = 0.005$ and $\beta = 0.2$. 
Fig. 2.7. Effects of Darcy number $Da$ on $Re\ u$ for $G = 5$, $p = 1$, $\omega = 10$, $t = 0.1$, $\lambda = 0.005$ and $\beta = 0.2$. 
Fig. 2.8. Effects of Darcy number $Da$ on $|u|$ for $G = 5$, $p = 1$, $\omega = 10$, $t = 0.1$, $\lambda = 0.005$ and $\beta = 0.2$. 
Fig. 2.9. Effects of Darcy number $Da$ on $Re\ u$ for $G = 5$, $p = 1$, $\omega = 1000$, $t = 0.1$, $\lambda = 0.005$ and $\beta = 0.2$. 
Fig. 2.10. Effects of Darcy number $Da$ on $|u|$ for $G = 5$, $p = 1$, $\omega = 1000$, $t = 0.1$, $\lambda = 0.005$ and $\beta = 0.2$. 

$Da = \infty, 10, 0.1, 0.01$
Fig. 2.11. Behavior of Re Nu vs. \( z \) for different values of \( \beta \) with \( Da = 0.1, \ p = 1, \ \omega = 1000, \ t = 0.1, \ \lambda = 0.005 \) and \( G = 5 \).
Fig. 2.12. Behavior of $|Nu|$ vs. $z$ for different values of $\beta$ with $Da = 0.1$, $p = 1$, $\omega = 1000$, $t = 0.1$, $\lambda = 0.005$ and $G = 5$. 
Fig. 2.13. Behavior of $\text{Re} \ Nu$ vs. $z$ for different values of $G$ with $Da = 0.1$, $p = 1$, $\omega = 10$, $t = 0.1$, $\lambda = 0.005$ and $\beta = 0.2$. 
Fig. 2.14. Behavior of $|Nu|$ vs. $z$ for different values of $G$ with $Da = 0.1$, $p = 1$, $\omega = 10$, $t = 0.1$, $\lambda = 0.005$ and $\beta = 0.2$. 
Fig. 2.15. Behavior of $Re\ Nu$ vs. $y$ for different values of $G$ with $Da = 0.1$, $p = 1$, $\omega = 1000$, $t = 0.1$, $\lambda = 0.005$ and $\beta = 0.2$. 
Fig. 2.16. Behavior of $|Nu|$ vs. $z$ for different values of $G$ with $M = 1$, $p = 1$, $\omega = 1000$, $t = 0.1$, $\lambda = 0.005$ and $\beta = 0.2$. 
Fig. 2.17. Behavior of $\text{Re} \ Nu$ vs. $z$ for different values of $Da$ with $G = 5$, $p = 1$, $\omega = 10$, $t = 0.1$, $\lambda = 0.005$ and $\beta = 0.2$. 
Fig. 2.18. Behavior of $|Nu|$ vs. $z$ for different values of $Da$ with $G = 5$, $p = 1$, $\omega = 10$, $t = 0.1$, $\lambda = 0.005$ and $\beta = 0.2$. 

$Da = \infty, 10, 0.1$
Fig. 2.19. Behavior of $Re$ vs. $z$ for different values of $Da$ with $G = 5$, $p = 1$, $\omega = 10$, $t = 0.1$, $\lambda = 0.005$ and $\beta = 0.2$. 
Fig. 2.20. Behavior of $|Nu|$ vs. $z$ for different values of $Da$ with $G = 5$, $p = 1$, $\omega = 1000$, $t = 0.1$, $\lambda = 0.005$ and $\beta = 0.2$. 