CHAPTER I

I N T R O D U C T I O N

CHAPTER I

INTRODUCTION

I.1. MAGNETOHYDRODYNAMIC SIMPLE BÉNARD CONVECTION.

I.1.a. The Physical Problem.

Let us consider a viscous electrically conducting Boussinesq fluid, statically confined between two horizontal boundaries of infinite horizontal extension and finite vertical depth in the force field of gravity. If we impress upon it a uniform adverse tempereature gradient and a uniform vertical magnetic field then under appropriate conditions a phenomenon of convective motions, an outcome of hydrodynamic instability, is observed. This phenomenon is termed as magnetohydrodynamic simple Bénard convection [Thompson (1951), Chandrasekhar (1952,1954,1958), Nakagawa (1955,1957, 1959), Jirlow (1956), Linhert and Little (1957)]. In this section we briefly summarize the essential points of this physical problem which are relevant to the present thesis.

I.1.b. The Governing Hydrodynamical Equations and Boundary Conditions.

The magnetohydrodynamical equations and boundary conditions that govern magnetohydrodynamic simple Bénard convective motions, in their non-dimensional form, are as follows [Chandrasekhar(1961), Gibbson(1956), Banerjee et al(1985)]:

$$(D^2-a^2)(D^2-a^2-\frac{p}{\sigma})w = Ra^2\theta - QD(D^2-a^2)h_z \ , \qquad (I.1.1)$$

$$(D^2-a^2-p)\theta = -w \ , \qquad (I.1.2)$$

$$(D^2-a^2-\frac{p\sigma_1}{\sigma})h_z = -Dw \ , \qquad (I.1.3)$$

$$w = 0 = \theta \quad \text{on both the boundaries} \ , \qquad (I.1.4)$$

$DW = 0$	on a rigid boundary,	(I.1.5)
$D^2w = 0$	on a dynamically free boundary,	(I.1.6)
$h_z = 0$	on both the boundaries if the regions outside the fluid are perfectly conducting,	(I.1.7)
$Dh_z = \mp a h_z$	on both the boundaries if the regions outside the fluid are insulating.	(I.1.8)

In equations (I.1.1) - (I.1.8) we attach the following mathematical meanings to various quantities. Here z is a real independent variable such that $0 \leqslant z \leqslant 1$; D is the differential operator $\frac{d}{dz}$; a^2, R, σ, σ_1 and Q are all positive constants; $p = p_r + ip_i$ is a complex constant in general such that p_r and p_i are real constants and consequently $w(z) = w_r(z) + iw_i(z)$, $\theta(z) = \theta_r(z) + i\theta_i(z)$ and $h_z(z) = h_{zr}(z) + ih_{zi}(z)$ are complex valued functions of the real variable z such that $w_r(z)$, $w_i(z)$, $\theta_r(z)$, $\theta_i(z)$, $h_{zr}(z)$ and $h_{zi}(z)$ are real valued functions of the real variable z.

The physical meanings of the various quantities in equations (I.1.1) -(I.1.8) are as follows:

Here z is the vertical coordinate; z = 0 and z = 1 are the two horizontal boundaries; a^2 is the square of the wave number; $R = \frac{g\alpha\beta d^4}{\kappa\nu}$ is the Rayleigh number, g is the gravity, α is the coefficient of thermal expansion, β is the maintained uniform vertical adverse temperature gradient, d is the depth of the fluid layer, κ is the thermometric conductivity(coefficient of heat diffusivity), ν is the kinematic viscosity; $\sigma = \frac{\nu}{\kappa}$ is the thermal Prandtl number; $\sigma_1 = \frac{\nu}{\eta}$ is the magnetic Prandtl number, η is the electric resistivity(coefficient of magnetic diffusivity); $Q = \frac{\mu e H^2 d^2}{4\pi\rho\nu\eta}$ is the Chandrasekhar number, μ_e is the magnetic permeability, H is the impressed uniform vertical magnetic field, ρ is the density; $p = p_r + ip_i$ is the complex growth rate; w is the vertical velocity; θ is the temperature and h_z is the uniform vertical magnetic field.

I.1.c. The Mathematical Eigen Value Problem.

In this section a mathematical eigen value problem for p is formulated for the case when both the boundaries are free and perfectly conducting. The remaining cases can be dealt with on the same lines.

When equation(I.1.1) is operated upon by the operator $(D^2-a^2-\frac{p\sigma_1}{\sigma})(D^2-a^2-p)$, θ and h_z get eliminated and the resulting single equation in w is,

$$(D^2-a^2-p)(D^2-a^2-\frac{p\sigma_1}{\sigma})(D^2-a^2)(D^2-a^2-\frac{p}{\sigma})w =$$
$$-Ra^2(D^2-a^2-\frac{p\sigma_1}{\sigma})w + Q(D^2-a^2-p)D(D^2-a^2)Dw. \qquad (I.1.9)$$

Since the validity of the equations (I.1.1) - (I.1.3) is for $0 \leqslant z \leqslant 1$, which includes the end points z = 0 and z = 1 also, we have, by using (I.1.4), (I.1.5) and (I.1.7) together with the equations (I.1.1)-(I.1.3), the following eight boundary conditions:

$$w = 0 \quad \text{at } z = 0 \text{ and } z = 1, \qquad (I.1.10)$$

$$D^2w=0 \quad \text{at } z = 0 \text{ and } z = 1, \qquad (I.1.11)$$

$$(D^2-a^2-\frac{p\sigma_1}{\sigma})(D^2-a^2)(D^2-a^2-\frac{p}{\sigma})w-QD^4w = 0$$
$$\text{at } z = 0 \text{ and } z = 1, \qquad (I.1.12)$$

$$(D^2-a^2-p)(D^2-a^2-\frac{p\sigma_1}{\sigma})(D^2-a^2)(D^2-a^2-\frac{p}{\sigma})w-Q\{D^6w-$$
$$(2a^2+p)D^4w\} = 0 \quad \text{at } z = 0 \text{ and } z = 1. \qquad (I.1.13)$$

The general solution of the eigth order, ordinary, linear and homogeneous differential equation(I.1.9) with constant coefficients is given by,

$$w = \sum_{i=1}^{8} c_i w_i(a^2, R, \sigma, \sigma_1, Q, p; z), \qquad (I.1.14)$$

where c_i's are the constants to be determined and w_i's constitute a fundamental set of solution of equation (I.1.9). Since the above solution for w has to satisfy the eight linear and homogeneuos boundary conditions given by (I.1.10) - (I.1.13), a set of eight algebraic. linear and homogeneous equations is obtained in the eight unknown c_i's. Non-trivial solutions of this set of equations require the vanishing of the associated coefficient determinant and this leads to the characteristic equation in the form,

$$f(a^2, R, \sigma, \sigma_1, Q, p) = o, \qquad \text{(I.1.15)}$$

where f is an appropriate complex valued function in general and therefore real and imaginary parts of equation (I.1.15) give,

$$f_r(a^2, R, \sigma, \sigma_1, Q, p_r, p_i) = o, \qquad \text{(I.1.16)}$$

$$f_i(a^2, R, \sigma, \sigma_1, Q, p_r, p_i) = 0, \qquad \text{(I.1.17)}$$

where f_r and f_i are the real and imaginary parts of f.

Now if a^2, R, σ, σ_1 and Q have pre-assigned values, and p_r and p_i are treated as unknowns, then p_r and p_i can possibly be determined as,

$$p_r = p_r(a^2, R, \sigma, \sigma_1, Q), \qquad \text{(I.1.18)}$$

$$p_i = p_i(a^2, R, \sigma, \sigma_1, Q), \qquad \text{(I.1.19)}$$

which satisfy equations (I.1.16) and (I.1.17).

The eigen value problem for p is thus formulated and one of the main questions which is investigated in Chapter II of this thesis is to find sufficient conditions under which $p_r = o$ implies $p_i = o$ for all $a^2 > o$ i.e. to find the sufficient conditions under which the 'principle of exchange of stabilities'[Rayleigh(1916)] is valid.

I.1.d. The Existing Results.

In the literature the following results exist pertaining to magnetohydrodynamic simple Bénard convection which are relevant to the present thesis.

Theorem I.1.1[Thompson(1951)] : For the case of non-viscous fluid (i.e. $\nu=0$) and for a particular solution of the governing equations (I.1.1)-(I.1.3) a sufficient condition for the validity of the 'principle of exchange of stabilities' is that $\sigma_1 \geqslant \sigma$.

Theorem I.1.2[Chandrasekhar(1952)] : For the case of viscous fluid with free and perfectly conducting boudnaries and for a particular solution of the governing equations (I.1.1)-(I.1.13) a sufficient condition for the validity of the 'principle of exchange of stabilities' is that $\sigma_1 \geqslant \sigma$.

Theorem I.1.3[Chandrasekhar(1952)] : For the case of viscous fluid with more general boundaries (for which appropriate boundary conditions are given by (I.1.4)-(I.1.8)) a sufficient condition for the validity of the 'principle of exchange of stabilities' is that the total kinetic energy associated with a disturbance is greater than or equal to its total magnetic energy i.e.

$$\int_0^1 (|Dw|^2 + a^2|w|^2)dz \geqslant Q\sigma_1\int_0^1 (|Dh_z|^2 + a^2|h_z|^2)dz.$$

Theorem I.1.4[Sherman and Ostrach(1966)] : For the case of viscous fluid completely confined in an arbitrary region with a uniform magnetic field in an arbitrary direction, a sufficient condition for the validity of the 'principle of exchange of stabilities' in the limiting case $Q \to \infty$ is that $\sigma_1 \geqslant \sigma$.

Theorem I.1.5[Sherman and Ostrach(1966)] : For the case of viscous fluid completely confined in an arbitrary region with a uniform magnetic field in an arbitrary direction, a sufficient condition for the validity of the 'principle of exchange of stabilities' in the limiting case $Q \to \infty$ is that the total kinetic energy associated with a disturbance is greater than or equal to its total magnetic energy.

Theorem I.1.6[Banerjee et al(1985) : For the case of viscous fluid with more general boundaries (for which appropriate boundary conditions are given by (I.1.4)-(I.1.8)) a suifficient condition for the validity of the 'principle of exchange of stabilities' is that $Q\sigma_1 \leqslant \pi^2$.

I.1.e. A Critical Analysis.

A critical analysis of the literature which is relevant to the present thesis points out a number of limitations of the various sufficient conditions for the validity of the 'principle of exchange of stabilities' set from time to time and we present them as follows:

The investigations of Thompson [Thompson(1951)] to find a sufficient condition for the validity of the 'principle of exchange of stabilities' are confined to non-viscous fluids only. Although he derived a sufficient condition for the validity of the 'principle of exchange of stabilities' in terms of the parameters of the system alone , his results have limitations on account of the non-consideration of the effects of viscosity.

The invesigations of Chandrasekhar [Chandrasekhar(1952)], in which he extended Thompson's analysis and recovered Thompson's sufficient condition for the validity of the 'principle of exchange of stabilities' for viscous fluids, have severe limitations since his boundary conditions are incorrect [Gibson(1966), Banerjee et al(1985)] and hence his sufficient condition for the validity of the 'principle of exchange of stabilities' can not be relied upon.

Further investigations of Chandrasekhar[Chandrasekhar(1952)] in which he extended the proof of Pellew and Southwell[Pellew and Southwell(1940)] in the context of magnetohydrodynamics with more general boundaries led him to the result that a sufficient condition for the validity of the 'principle of exchange of stabilities' is that the total kinetic energy associated with a disturbance is greater than or equal to its total magnetic energy. Since it can not be ascertained a priori as to when this sufficient condition will be satisfied and since there is no rigorous mathematical derivation of this sufficient condition, it remained in the literature as a conjecture , known as Chandrasekhar's conjecture.

The investigations of Sherman and Ostrach [Sherman and Ostrach(1966)] are for a more general problem with completely confined fluid in an arbitrary region and with a uniform magnetic field in an arbitrary direction. Although they recovered Thompson - Chandrasekhar sufficient condition for the validity of the 'principle of exchange of stabilities', their analysis is limited by the fact that it is valid only for the case $Q \to \infty$.

Further investigations of Sherman and Ostrach [Sherman and Ostrach (1966)] are for a more general problem with completely confined fluid in an arbitrary region with a uniform magnetic field in an arbitrary direction. Although they gave a rigorous mathematical derivation of Chandrasekhar's conjecture, their analysis is again limited by the fact that it is valid only for the case $Q \to \infty$.

Further, through Sherman and Ostrach's analysis, it can not be ascertained as to when the Thompson - Chandrasekhar's sufficient condition for the validity of the 'principle of exchange of stabilities' is satisfied.

Banerjee et al [Banerjee et al (1985)] obtained a sufficient condition for the validity of the 'principle of exchange of stabilities' in terms of parameters of the system alone and removed one of the major obstacles which were limiting the works of Chandrasekhar and, Sherman and Ostrach. However the analysis of Banerjee et al is incomplete in the sense that it does not reveal the connection between their sufficient condition for the validity of the 'principle of exchange of stabilities' and Chandrasekhar's conjecture.

I.1.f. Aims.

The critical analysis presented in (I.1.e.) points towards some clearcut gaps in the literature on magnetohydrodynamic simple Bénard problem and we propose

(i) to complete Chandrasekhar's[Chandrasekhar(1952)] work on the validity of the 'principle of exchange of stabilities' for the problem by establishing the sufficient conditions under which his conjecture on the relationship between the kinetic and magnetic energies associated with a disturbance is valid (ii) to complete Chandrasekhar's[Chandrasekhar(1952)] work on the validity of the 'principle of exchange of stabilities' with regard to the generallity of the boundary conditions (iii) to complete Banerjee et al's[Banerjee et al(1985)] work on the validity of the 'principle of exchange of stabilities' for the problem by establishing the relationship between the desired sufficient condition as derived by them and Chandrasekhar's[Chandrasekhar(1952)] conjecture.

I.1.g. A Brief Summary of the Results Obtained.

(1) If $Q\sigma_1 \leqslant \pi^2$, then the total inetic energy associated with an unstable or marginally stable disturbance is greater than its total magnetic energy and this result is uniformly valid for quite genetic general magnetohydrodynamic boundary conditions.

(2) At the stationary marginal state, for which a sufficient condition is $Q\sigma_1 \leqslant \pi^2$, the total kinetic energy associated with a disturbance is greater than its total magnetic energy and this result is uniformly valid for quite general magnetohydrodynamic boundary conditions.

(3)A connection between Chandrasekhar's[Chandrasekhar(1952)] conjecture and Banerjee et al's [Banerjee et al(1985)] work has been established i.e. at the stationary marginal state, for which a sufficient condition is $Q\sigma_1 \leqslant \pi^2$[Banerjee et al(1985)], the total kinetic energy associated with a disturbance is greater than its total magnetic energy [Chandrasekhar(1952)].

I.2. MAGNETOHYDRODYNAMIC THERMOHALINE CONVECTION OF THE VERONIS TYPE.

I.2.a. The Physical Problem.

Let us consider a viscous electrically conducting Boussinesq fluid, statically confined between two horizontal boundaries of infinite horizontal extension and finite vertical depth in the force field of gravity. If we impress upon it a uniform vertical non-adverse concentration gradient and adverse tempereature gradient, and consider the resulting system in the presence of a uniform vertical magenetic field, then under appropriate conditions a phenomenon of convective motions,an outcome of hydrodynamic instability,is observed.This phenomenon is termed as magnetohydrodynamic — thermohaline convection of the Veronis type [Veronis(1965),Bains and Gill(1969),Turner(1974), Banerjee et al(1981),Huppert and Turner(1981)]. In this section we briefly summarize the essential points of this physical problem which are relevant to the present thesis.

I.2.b. The Governing Hydrodynamical Equations and Boundary Conditions.

The magnetohydrodynamical thermohaline equations and boundary conditions that govern magenetohydrodynamic thermohaline Veronis type convective motions, in their non-dimensional form are as follows [Gupta et al (1983))]:

$$(D^2-a^2)(D^2-a^2-\frac{p}{\sigma})w = Ra^2\theta - R_s a^2\phi - QD(D^2-a^2)h_z, \quad (I.2.1)$$

$$(D^2-a^2-p)\theta = -w, \quad (I.2.2)$$

$$(D^2-a^2-\frac{p}{\tau})\phi = -\frac{w}{\tau}, \quad (I.2.3)$$

$$(D^2-a^2-\frac{p\sigma_1}{\sigma})h_z = -Dw, \quad (I.2.4)$$

$w = 0 = \theta = \phi$ on both the boundaries, (I.2.5)

$Dw = 0$ on a rigid boundary, (I.2.6)

$D^2w = 0$ on a dynamically free boundary, (I.2.7)

$h_z = 0$ on both the boundaries if the regions outside the fluid are perfectly conducting, (I.2.8)

$Dh_z = \mp ah_z$ on both the boundaries if the regions outside the fluid are insulating. (I.2.9)

In equations (I.2.1)- (I.2.9) we attach the following mathematical meanings to various quantities:

Here R_s and τ are positive constants; $\phi(z)=\phi_r(z)+i\,\phi_i(z)$ is a complex valued function of the real variable z such that $\phi_r(z)$ and $\phi_i(z)$ are real valued functions of the real variable z. All other symbols used in the equations (I.2.1)-(I.2.9) have the same mathematical meanings as in (I.1.b).

The physical meanings of various quantities in equations (I.2.1)-(I.2.9) are as follows:

Here $R_s = \frac{g\,\gamma\delta\,d^4}{\kappa\nu}$ is the concentration Rayleigh number, γ is the coefficient of volume expansion due to concentration variation and δ is the maintained uniform vertical non-adverse concentration gradient. All other symbols used in the equations (I.2.1)-(I.2.9) have the same physical meanings as in (I.1.b).

I.2.c. The Mathematical Eigen Value Problem.

In this section a mathematical eigen value problem for p is formulated for the case when both the boundaries are free and perfectly conducting. The remaining cases can be dealt with on the same lines.

When equation (I.2.1) is operated upon by the operator $(D^2-a^2-p)(D^2-a^2-\frac{p}{\tau})(D^2-a^2-\frac{p\sigma_1}{\sigma})$ then θ,ϕ and h_z get eliminated and the resulting single equation in w is,

$$(D^2-a^2-p)(D^2-a^2-\frac{p}{\tau})(D^2-a^2-\frac{p\sigma_1}{\sigma})(D^2-a^2)(D^2-a^2-\frac{p}{\sigma})w =$$
$$-Ra^2(D^2-a^2-\frac{p}{\tau})(D^2-a^2-\frac{p\sigma_1}{\sigma})\,w+\frac{Ra^2}{\tau}\,(D^2-a^2-p)(D^2-a^2-\frac{p\sigma_1}{\sigma})w$$
$$+\,QD(D^2-a^2-p)(D^2-a^2-\frac{p}{\tau})D^2(D^2-a^2)w. \qquad (I.2.10)$$

Since the validity of the equations (I.2.1)-(I.2.4) is for $0 \leqslant z \leqslant 1$, which includes the end points z = 0 and z = 1 also, we have by using (I.2.5),(I.2.7) and (I.2.8) together with the equations (I.2.1)-(I.2.4), the following ten boundary conditions:

$$w = 0 \quad \text{at} \quad z = 0 \text{ and } z = 1, \qquad (I.2.11)$$

$$D^2w = 0 \quad \text{at } z = 0 \text{ and } z = 1, \tag{I.2.12}$$

$$(D^2-a^2)(D^2-a^2-\frac{p}{\sigma})(D^2-a^2-\frac{p\sigma_1}{\sigma})\, w - QD^4w=0 \text{ at } z=0 \text{ and } z=1, \tag{I.2.13}$$

$$(D^2-a^2)(D^2-a^2-\frac{p}{\sigma})(D^2-a^2-\frac{p}{\tau})(D^2-a^2-\frac{p\sigma_1}{\sigma})w$$
$$+ QD^2(D^2-a^2-\frac{p}{\tau})w = 0 \text{ at } z = 0 \text{ and } z = 1, \tag{I.2.14}$$

$$(D^2-a^2)(D^2-a^2-\frac{p}{\sigma})(D^2-a^2-p)(D^2-a^2-\frac{p}{\tau})(D^2-a^2-\frac{p\sigma_1}{\sigma})w$$
$$+(Ra^2-\frac{R_s a^2}{\tau})D^4w - Q\{D^8w - (3a^2+p+\frac{p}{\tau})D^6w$$
$$+(3a^4+2a^2p+2a^2\frac{p}{\tau}+\frac{p^2}{\tau})D^4w\} \text{ at } z=0 \text{ and } z = 1. \tag{I.2.15}$$

The general solution of the tenth order ordinary linear homogenous differential equation (I.2.10) with constant coefficients is given by ,

$$w = \sum_{i=1}^{10} c_i w_i (a^2,R,R_s,\sigma,\tau,\sigma_1,Q,p;z) \tag{I.2.16}$$

where c_i 's are the constants to be determined and w_i 's constitute a fundamental set of solutions of equation (I.2.10). Since the above solution for w has to satisfy the ten linear and homogenous boundary conditions given by (I.2.11)-(I.2.15), a set of ten algebraic linear and homogeneous equations is obtained in the ten unknown c_i 's. Non-trivial solutions of this set of equations require the vanishing of the associated coefficient determinant and this leads to the characteristic equation in the form,

$$f\ (a^2,R,R_s,\sigma,\tau,\sigma_1,Q,p) = 0 \tag{I.2.17}$$

where f is an appropriate complex valued function in general and therefore real and imaginary parts of equation (I.2.17) give ,

$$f_r(a^2,R,R_s,\sigma,\tau,\sigma_1,Q,p_r,p_i) = 0, \tag{I.2.18}$$

$$f_i(a^2,R,R_s,\sigma,\tau,\sigma_1,Q,p_r,p_i) = 0, \tag{I.2.19}$$

where f_r and f_i are the real and imaginary parts of f respectively,

Now if $a^2, R, R_s, \sigma, \tau, \sigma_1$, and Q have pre-assigned values, and p_r and p_i are treated as unknowns, then p_r and p_i can possibly be determined as,

$$p_r = p_r(a^2, R, R_s, \sigma, \tau, \sigma_1, Q), \qquad (I.2.20)$$

$$p_i = p_i(a^2, R, R_s, \sigma, \tau, \sigma_1, Q), \qquad (I.2.21)$$

which satisfy equations(I.2.18) and (I.2.19).

The eigen value problem for p is thus formulated and one of the main questions which is investigated in Chapter III of this thesis is to find sufficient conditions under which $p_r = 0$ implies $p_i = 0$ for all $a^2 > 0$ i.e. to find the sufficient conditions under which the 'principle of exchange of stabilities' is valid.

1.2.d. The Existing Results.

In the literature the following result exists pertaining to magnetohydrodynamic thermohaline convection of the Veronis type which is relevant to the present thesis.

<u>Theorem I.2.1</u> [Gupta et al (1986)] : For the case of viscous fluid with more general boundaries (for which appropriate boundary conditions are given by (I.2.5)-(I.2.9)) a sufficient condition for the validity of the 'principle of exchange of stabilities' is that $\frac{Q\sigma_1}{\pi^2} + \frac{R_s\sigma}{\tau^2\pi^4} \leq 1$.

I.2.e.A Critical Analysis.

The work of Gupta et al[Gupta et al(1986)] establishes a sufficient condition for the validity of the 'principle of exchange of stabilities' for magnetohydrodynamic thermohaline convection of the Veronis type but they neither have made any attempt to find out in terms of physical concepts the meanings of various integral inequalities that they have obtained nor have they tried on the lines of Chandrasekhar's[Chandrasekhar(1952)] work to propound an extended sufficient condition for the validity of the 'principle of exchange of stabilities' for the problem.

I.2.f. Aims.

We propose to complete the gap pointed out in (I.2.e) by discovering an extended form of Chandrasekhar's [Chandrasekhar (1952)] conjecture for the validity of the 'principle of exchange of stabilities' for the present problem and showing its connection with Gupta et al's [Gupta et al(1986)] work.

I.2.g. A Brief Summary of the Results Obtained.

(1) If $\frac{Q\sigma_1}{\pi^2} + \frac{R_s\sigma}{\tau^2\pi^4} \leqslant 1$, then the total kinetic energy associated with an unstable or marginally stable disturbance is greater than the sum of its total magnetic and concentration energies and this result is uniformly valid for quite general magnetohydrodynamic boundary conditions.

(2) At the stationary marginal state, for which a sufficient condition is $\frac{Q\sigma_1}{\pi^2} + \frac{R_s\sigma}{\tau^2\pi^4} \leqslant 1$, the total kinetic energy associated with a disturbance is greater than the sum of its total magnetic and concentration energies and this result is uniformly valid for quite general magnetohydrodynamic boundary conditions.

I.3. MAGNETOHYDRODYNAMIC THERMOHALINE CONVECTION OF THE STERN TYPE.

I.3.a. The Physical Problem.

Let us consider a viscous electrically conducting Boussinesq fluid, statically confined between two horizontal boundaries of infinite horizontal extension and finite vertical depth in the force field of gravity. If we impress upon it a uniform vertical non-adverse temperature gradient and adverse concentration gradient, and consider the resulting system in the presence of a uniform vertical magnetic field, then under appropriate conditions, a phenomenon of convective motions, an outcome of hydrodynamic instability, is observed. This phenomenon is termed as magnetohydrodynamic thermohaline convection of the Stern type [Stern(1960),Bains and Gill(1969), Turner(1974),Yih (1980), Banerjee et al(1981)]. In this section we briefly summarize the essential points of this physical problem which are relevant to the present thesis.

I.3.b. The Governing Hydrodynamical Equations and Boundary Conditions.

The Governing magnetohydrodynamical equations and boundary conditions that govern magnetohydrodynamic thermohaline convective motions of the Stern type, in their non-dimensional form are as follows[Gupta et al(1983)]:

$$(D^2-a^2)(D^2-a^2-\frac{p}{\sigma})w = Ra^2\theta - R_s a^2\phi - QD(D^2-a^2)h_z, \qquad (I.3.1)$$

$$(D^2-a^2-p)\theta = -w, \qquad (I.3.2)$$

$$(D^2-a^2-\frac{p}{\tau})\phi = -\frac{w}{\tau}, \qquad (I.3.3)$$

$$(D^2-a^2-\frac{p\sigma_1}{\sigma})h_z = - Dw, \qquad (I.3.4)$$

$$w = 0 = \theta = \phi \qquad \text{on both the boundaries}, \qquad (I.3.5)$$

$DW = 0$, on a rigid boundary, (I.3.6)

$D^2w = 0$, on a dynamically free boundary, (I.3.7)

$h_z = 0$, on both the boundaries if the regions outside the fluid are perfectly conducting, (I.3.8)

$Dh_z = \mp ah_z$ on both the boundaries if the regions outside the fluid are insulating. (I.3.9)

In equations (I.3.1)-(I.3.9) we attach the following mathematical meanings to various quantities:

Here R and R_s are negative constants. All other symbols used in the above equations have the same mathematical meanings as in (I.1.b) and (I.2.b).

The physical meanings of various quantities in equations (I.3.1)-(I.3.9) are as follows:

Here $R = \frac{g\alpha\beta d^4}{\kappa\nu}$ and $R_s = \frac{g\gamma\delta d^4}{\kappa\nu}$ where β is the maintained uniform vertical non-adverse temperature gradient and δ is the maintained uniform vertical adverse concentration gradient. All other symbols used in the above equations have the same physical meanings as in (I.1.b) and (I.2.b).

I.3.c. The Mathematical Eigen Value Problem.

A mathematical eigen value problem for p is formulated on the same lines as in (I.2.c).

I.3.d. The Existing Results.

In the literature the following result exists pertaining to magnetohydrodynamic thermohaline convection of the Stern type which is relevant to the present thesis.

Theorem I.3.1 [Gupta et al (1986)]: For the case of viscous fluid with more general boundaries (for which appropriate boundary conditions are given by (I.3.5)-(I.3.9)) a sufficient condition for the validity of the 'principle of exchange of stabilities' is that

$$\frac{Q\sigma_1}{\pi^2} + \frac{|R|\sigma}{\pi^4} \leqslant 1.$$

I.3.e. A Critical Analysis.

The work of Gupta et al [Gupta et al (1986)] establishes a sufficient condition for the validity of the 'principle of exchange of stabilities' for magnetohydrodynamic thermohaline convection of the Stern type but they neither have made any attempt to find out in terms of physical concepts the meanings of various integral inequalities that they have obtained nor have they tried on the lines of Chandrasekhar's[Chandrasekhar (1952)] work to propound an extended sufficient condition for the validity of the 'principle of exchange of stabilities' for the problem.

I.3.f. Aims.

We propose to complete the gap pointed out in (I.3.e) by discovering an extended form of Chandrasekhar's[Chandrasekhar (1952)] conjecture for the validity of the 'principle of exchange of stabilities' for the present problem and showing its connection with Gupta et al's [Gupta et al (1986) work.

I.3.g. A Brief Summary of the Results obtained,

(1) If $\frac{Q\sigma_1}{\pi^2} + \frac{|R|\sigma}{\pi^4} \leqslant 1$, then the total kinetic energy associated with an unstable or marginally stable disturbance is greater than the sum of its total magnetic and thermal energies and this result is uniformly valid for quite general magnetohydrodynamic boundary conditions.

(2) At the stationary marginal state, for which a sufficient condition is $\frac{Q\sigma_1}{\pi^2}+\frac{|R|\sigma}{\pi^4} \leqslant 1$, the total kinetic energy associated with a disturbance is greater than the sum of its total magnetic and thermal energies and this result is uniformly valid for quite general magnetohydrodynamic boundary conditions.

I.4. MAGNETOHYDRODYNAMIC SIMPLE BÉNARD CONVECTION: A NECESSARY CONDITION FOR THE VALIDITY OF THE 'PRINCIPLE OF EXCHANGE OF STABILITIES'.

I.4.a. The Physical Problem.

Let us consider a viscous electrically conducting Boussinesq fluid, statically confined between two horizontal boundaries of inifinite horizontal extension and finite vertical depth in the force field of gravity. If we impress upon it a uniform adverse temperature gradient and a uniform vertical magnetic field then under appropriate conditions a phenomenon of convective motions, an outcome of hydrodynamic instability, is observed. This phenomenon is termed as magnetohydrodynamic simple Bénard convection and in this section we briefly summarize the essential points of this physical problem that are relevant with regard to the validity of Chandrasekhar's prediction for the case of general boundaries which imples that when the 'principle of exchange of stabilities' is valid, $R > \pi^2 Q$ for all finite values of Q.

I.4.b. The Governing Hydrodynamical Equations and Boundary Conditions.

The magnetohydrodynamical equations and boundary conditions that govern magnetohydrodynamic simple Bénard convective motions, in their non-dimensional form are given by (I.1.1)-(I.1.8)(Chapter I). However, since we are interested in the present context in the onset of hydrodynamic instability through stationary convective motions, which implies that the 'principle of exchange of stabilities' is valid, we must take p= 0 in the equations (I.1.1)-(I.1.3). The governing equations thus reduce to:

$$(D^2-a^2)^2 w = Ra^2\theta - QD(D^2-a^2)h_z, \qquad (I.4.1)$$

$$(D^2-a^2)\theta = -w, \qquad (I.4.2)$$

$$(D^2-a^2)h_z = -Dw, \qquad (I.4.3)$$

while the boundary conditions remain unaltered.

I.4.c. The Mathematical Eigen Value Problem.

In this section a mathematical eigen value problem for R is formulated for the case when both the boundaries are free and perfectly conducting. The remaining cases can be dealt with on the same lines.

When equation (I.4.1) is operated upon by the operator (D^2-a^2), θ and h_z get eliminated and the resulting single equation in w is,

$$(D^2-a^2)^3 w = -Ra^2 w + QD^2(D^2-a^2)w. \qquad (I.4.4)$$

Since the validity of the equations (I.4.1)-(I.4.3) is for $0 \leqslant z \leqslant 1$, which includes the end points z = 0 and z = 1 also, we have, by using (1.1.4), (1.1.5) and (I.1.7) together with the equations (I.4.1)-(I.4.3), the following six boundary conditions:

$$w = 0 \text{ at } z = 0 \text{ and } z = 1, \qquad (I.4.5)$$

$$D^2 w = 0 \text{ at } z = 0 \text{ and } z = 1, \qquad (I.4.6)$$

$$(D^2-a^2)^3 w = 0 \text{ at } z = 0 \text{ and } z = 1. \qquad (I.4.7)$$

The general solution of sixth order ordinary, linear, homogeneous equation (I.4.4) with constant coefficients is given by,

$$w = \sum_{i=1}^{6} C_i w_i (a^2, R, Q; z), \qquad (I.4.8)$$

where C_i's are the constants to be determined and w_i's constitute a fundamental set of solutions of equation(I.4.4). Since the above solution for w has to satisfy the six linear and homogeneous boundary condtiions given by (I.4.5)-(I.4.7), a set of six algebraic linear and homogeneous equations is obtained in the six unknown C_i's. Non-trivial solutions of this set of equations require the vanishing of the associated co-efficient determinant and this leads to the Characteristic equation in the form,

$$q(a^2,R,Q) = 0, \qquad (I.4.9)$$

where q is an appropriate real valued function.

Now if a^2 and Q have pre-assigned values, then R can possibly be determined as

$$R = R(a^2,Q) , \qquad (I.4.10)$$

which satisfy equation (I.4.9)

The eigen value problem for R is thus formulated and one of the main questions, in Chapter V, is to investigate the validity of Chandrasekhar's prediction.

I.4.d. The Existing Results.

In the literature the following results exist pertaining to magnetohydrodynamic simple Bènard convection which are relevant to the present thesis.

Theorem I.4.1[Chandrasekhar(1952)]:

For the case of viscous fluid with free boundaries and for a particular solution of the governing equations(I.1.1)-(I.1.3) and boundary conditions(I.1.4),(I.1.6) and (I.1.7), a necessary condition for the validity of the 'principle of exchange of stabilities' is that $R > \pi^2 Q$ for all finite values of Q.

Theorem I.4.2 [Chandrasekhar(1952)]:

For the case of viscous fluids with rigid boundaries or any combinations of free and rigid boundaries, it appears that the general features of the problem for the case of free boundaries are preserved.

I.4.e. A Critical Analysis.

A critical analysis of the literature which is relevant to the present thesis points out that the investigations of Chandrasekhar[Chandrasekhar(1952)] are valid only for the case of free boundaries and for a particular solution of the governing equations. Thus the validity of his prediction can not be taken to be proved in general.

I.4.f. Aims.

In the light of the critical analysis presented in (I.4.e) we propose to investigate mathematically the validity of Chandrasekhar's prediction for all combinations of dynamically free and rigid boundaries which may be perfectly conducting or insulating.

I.4.g. A Brief Summary of the Results Obtained.

(1) A necessary condition for the validity of the 'principle of exchange of stabilities' for any combination of dynamically free or rigid boundaries that may be perfectly conducting or insulating is that $R > \pi^2 Q$ for all finite values of Q.

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