CHAPTER II

ONSET OF INSTABILITY IN THE MAGNETOHYDRODYNAMIC SIMPLE BÉNARD PROBLEM.

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CHAPTER II

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II.1 The Physical Configuration and the Initial State.

Let us consider a viscous electrically conducting Boussinesq fluid of infinite horizontal exention and finite vertical depth statically confined between two horizontal boundaries $z=0$ and $z=d$ which are maintained at constant temperatures T_0 and T_1 $(T_0>T_1)$ respectively in the presence of a uniform vertical magnetic field acting parallel and opposite to the direction of gravity. In this chapter we investigate the onset of hydrodynamic instability in this physical configuration. The initial state is then represented by,

$$(u,v,w) \equiv (0,0,0) \quad ,$$

$$T \equiv T(z) \quad ,$$

$$(H_1,H_2,H_3) = (0,0,H) \quad , \qquad \text{(II.1.1)}$$

where u , v , w are the respective components of the velocity in the x, y and z directions, T is the temperature and H_1, H_2, H_3 are the respective components of the magnetic field in the x, y and z directions.

II.2. The Governing Magnetohydrodynamical Equations and the Initial State Solution.

The magnetohydrodynamical equations that govern magnetohydrodynamic simple Bénard convective motions are as follows:

(a) Equation of Momentum.

The equation of momentum is given by ,

$$\frac{\partial u_i}{\partial t}+u_j\frac{\partial u_i}{\partial x_j}-\frac{\mu_e}{4\pi\rho_0}H_j\frac{\partial H_i}{\partial x_j}=-\frac{\partial}{\partial x_i}\left(\frac{p}{\rho_0}+\mu_e\frac{|\underline{H}|^2}{8\pi\rho_0}\right)+$$

$$[1+\alpha(T_0-T)]X_i+\nu\nabla^2u_i \;, \qquad (II.2.1)$$

where $(u_1 , u_2, u_3) \equiv (u , v , w)$, $(x_1 , x_2 , x_3) \equiv (x , y , z)$, p is the pressure , $|\underline{H}|^2 = H_j H_j$ $(j = 1 , 2 , 3)$ is the square of the modulus of the magnetic field vector (H_1, H_2, H_3), X_i $(i = 1 , 2 , 3)$ is the ith component of the external force and ρ_0 is the value of ρ at $z = 0$.

(b) Equation of Continuity.

The equation of continuity is given by ,

$$\frac{\partial u_i}{\partial x_i} = 0 . \qquad (II.2.2)$$

(c) Equation of Heat conduction .

The equation of heat conduction is given by ,

$$\frac{\partial T}{\partial t}+u_j\frac{\partial T}{\partial x_j} = \kappa\nabla^2 T. \qquad (II.2.3)$$

(d) Equation of Magnetic Induction .

The equation of magnetic induction is given by ,

$$\frac{\partial H_i}{\partial t} + u_j \frac{\partial H_i}{\partial x_j} = H_j \frac{\partial u_i}{\partial x_j} + \eta \nabla^2 H_i \quad . \qquad \text{(II.2.4)}$$

(e) Equation representing the Solenoidal Character of the Magnetic Field .

The equation representing the solenoidal character of the magnetic field is given by ,

$$\frac{\partial H_i}{\partial x_i} = 0 \quad . \qquad \text{(II.2.5)}$$

(f) Equation of State .

The equation of State is given by ,

$$\rho = \rho_0 [1 + \alpha (T_0 - T)] \quad . \qquad \text{(II.2.6)}$$

The governing equations (II.2.1) - (II.2.6) on the basis of equations (II.1.1) and with $X_i = (0, 0, -g)$ yield the initial state solution,

$$\left.\begin{aligned} (u,v,w) &= (0,0,0) \,, \\ T &= T_0 - \beta z \,, \\ (H_1, H_2, H_3) &= (0,0,H) \,, \\ \rho = \rho_0 [1 + \alpha (T_0 - T)] &= \rho_0 [1 + \alpha \beta z] \,, \end{aligned}\right|$$

$$P = p + \frac{\mu e \, |\underline{H}|^2}{8\pi} = P_o - \rho_o g[z + \alpha\beta \frac{z^2}{2}] . \qquad , \qquad (II.2.7)$$

where $\beta = \frac{T_o - T_1}{d}$ and P_o is the value of P at z = 0 .

II.3. The Linearized Disturbance Equations .

Let the initial state represented by equations (II.2.5) be slightly disturbed and the disturbed state by represented by ,

$$\begin{aligned} (u,v,w) &= (o + u', o + v', o + w') , \\ T &= T_o - \beta z + \theta' , \\ (H_1, H_2, H_3) &= (0 + h'_x , 0 + h'_y , H + h'_z) , \\ \rho &= \rho_o [1 + \alpha (T_o - T - \theta')] \\ &= \rho_o [1 + \alpha (T_o - T)] + \rho' , \\ P &= P_o - \rho_o g [z + \alpha\beta \frac{z^2}{2}] + p' . \end{aligned} \qquad ,(II.3.1)$$

where u' , v' , w' are the respective components of disturbance velocity in the x , y and z directions , θ' is the disturbance temperature, h'_x , h'_y , h'_z are the respective components of disturbance magnetic field in the x , y , and z directions, ρ' is the disturbance density and p' is the disturbance pressure.

The linearized disturbance equations of momentum, continuity, heat conduction, magnetic induction, solenoidal character of the magnetic field and state are given by,

$$\rho_0 \frac{\partial u'}{\partial t} = -\frac{\partial p'}{\partial x} + \mu \nabla^2 u' + \frac{\mu_e H}{4\pi} \left(\frac{\partial h'_x}{\partial z} - \frac{\partial h'_z}{\partial x}\right) , \qquad (II.3.2)$$

$$\rho_o \frac{\partial v'}{\partial t} = -\frac{\partial p'}{\partial y} + \mu\nabla^2 v' + \frac{\mu_e H}{4\pi}\left(\frac{\partial h'_y}{\partial z} - \frac{\partial h'_z}{\partial y}\right), \quad \text{(II.3.3)}$$

$$\rho_o \frac{\partial w'}{\partial t} = -\frac{\partial p'}{\partial z} + \mu\nabla^2 w' - g\rho', \quad \text{(II.3.4)}$$

$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} = 0, \quad \text{(II.3.5)}$$

$$\frac{\partial \theta'}{\partial t} - \beta w' = \kappa\nabla^2\theta', \quad \text{(II.3.6)}$$

$$\frac{\partial h'_x}{\partial t} = H\frac{\partial u'}{\partial z} + \eta\nabla^2 h'_x, \quad \text{(II.3.7)}$$

$$\frac{\partial h'_y}{\partial t} = H\frac{\partial v'}{\partial z} + \eta\nabla^2 h'_y, \quad \text{(II.3.8)}$$

$$\frac{\partial h'_z}{\partial t} = H\frac{\partial w'}{\partial z} + \eta\nabla^2 h'_z, \quad \text{(II.3.9)}$$

$$\frac{\partial h'_x}{\partial x} + \frac{\partial h'_y}{\partial y} + \frac{\partial h'_z}{\partial z} = 0, \quad \text{(II.3.10)}$$

$$\rho = \rho_o[1 + \alpha(T_o - T - \theta')] = \rho_o[1 + \alpha(T_o - T)] + \rho', \quad \text{(II.3.11)}$$

where μ is the viscosity .

II.4. *Resolution in terms of Normal Modes .*

We analyse an arbitrary disturbance in terms of normal modes to investigate the stability of the physical configuration. Let on x , y and t of the form ,

$$\exp\,[i(k_x x + k_y y) + nt]\ , \qquad (II.4.1)$$

where $$k = \sqrt{k_x^2 + K_y^2}\ , \qquad (II.4.2)$$

is the wave number and n is the frequency, which is complex in general, of a disturbance of two dimensional wave form. Here k_x and k_y are the real constants [Stuart(1963), Drazin and Reid(1981)]. The above considerations imply that

$$u'\,(x,y,z,t) = u''\,(z)\,\exp\,[i(k_x x + k_y y) + nt]\ ,$$

$$v'\,(x,y,z,t) = v''\,(z)\,\exp\,[i(k_x x + k_y y) + nt]\ ,$$

$$w'\,(x,y,z,t) = w''\,(z)\,\exp\,[i(k_x x + k_y y) + nt]\ ,$$

$$\theta'\,(x,y,z,t) = \theta''\,(z)\,\exp\,[i(k_x x + k_y y) + nt]\ ,$$

$$h_x'\,(x,y,z,t) = h_x''\,(z)\,\exp\,[i(k_x x + k_y y) + nt]\ ,$$

$$h_y'\,(x,y,z,t) = h_y''\,(z)\,\exp\,[i(k_x x + k_y y) + nt]\ ,$$

$$h_z'\,(x,y,z,t) = h_z''\,(z)\,\exp\,[i(k_x x + k_y y) + nt]\ ,$$

$$p'\,(x,y,z,t) = p''\,(z)\,\exp\,[i(k_x x + k_y y) + nt]\ . \qquad (II.4.2)$$

For functions with the above dependence on x,y and t , it follows that

$$\frac{\partial}{\partial t} = n\ , \quad \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = -k^2 \ \text{and}\ \nabla^2 = \frac{\partial^2}{\partial z^2} - k^2\ . \qquad (II.4.4)$$

II.5 The Disturbance Equations Governing the Normal Modes.

By using the normal mode analysis given in (II.4), equations

(II.3.2) - (II.3.10) can be written in the form ,

$$\rho_o nu'' = -ik_x p'' + \mu(\frac{d^2}{dz^2}-k^2)\, u'' + \frac{\mu_e H}{4\pi}(\frac{\partial h_x''}{\partial z} - ik_x h_z''), \quad (II.5.1)$$

$$\rho_o nv'' = -ik_y p'' + \mu(\frac{d^2}{dz^2}-k^2)\, v'' + \frac{\mu_e H}{4\pi}(\frac{\partial h_y''}{\partial z} - ik_y h_z''), \quad (II.5.2)$$

$$\rho_o nw'' = -\frac{d}{dz}(p'') + \mu(\frac{d^2}{dz^2}-k^2)W'' + \rho_o g\alpha\Theta'', \quad (11.5.3)$$

$$ik_x u'' + ik_y v'' + \frac{dw''}{dz} = 0, \quad (II.5.4)$$

$$n\Theta'' - \beta w'' = \kappa(\frac{d^2}{dz^2} - k^2)\Theta'', \quad (II.5.5)$$

$$nh_x'' = H\frac{du''}{dz} + \eta(\frac{d^2}{dz^2} - k^2)h_x'', \quad (11.5.6)$$

$$nh_y'' = H\frac{dv''}{dz} + \eta(\frac{d^2}{dz^2} - k^2)h_y'', \quad (II.5.7)$$

$$nh_z'' = H\frac{dw''}{dz} + \eta(\frac{d^2}{dz^2} - k^2)h_z'', \quad (11.5.8)$$

$$ik_x h_x'' + ik_y h_y'' + \frac{d}{dz}h_z'' = 0, \quad (II.5.9)$$

where the last term on the right hand side of equation (II.5.3) is written by making use of the disturbed equation of state in the corresponding terms of equation (II.3.4).

II.6. The Governing Magnetohydrodynamical Equations in Non-dimensional form.

Eliminating u'', v'', h_x'', h_y'' and p'' from equations(II.5.1)

-(II.5.9) and using the non-dimensional quantities defined by ,

$$z_* = \frac{z}{d} \;, \quad D_* = d.\frac{d}{dz} \;, \quad a_* = kd \;,$$

$$R_* = \frac{g\alpha\beta d^4}{\kappa\nu} \;, \sigma_* = \frac{\nu}{\kappa} \;, \sigma_{1*} = \frac{\nu}{\eta} \;,$$

$$Q_* = \frac{\mu e H^2 d^2}{4\pi\rho\nu\eta} \;, \quad p_* = \frac{nd^2}{\kappa} \;, \quad w_* = \frac{\beta d^2}{\kappa} W \;,$$

$$\theta_* = \theta \;, \qquad h_{z*} = \frac{\eta}{Hd} \; \frac{\beta d}{\kappa} h_z . \tag{II.6.1}$$

we obtain, upon dropping the double dashes and the asterisks for simplicity in writing, the governing magnetohydrodynamical equations in their non-dimensional form are as follows:

$$(D^2-a^2)(D^2-a^2-\frac{p}{\sigma})w = Ra^2\theta - QD(D^2-a^2)h_z , \tag{II.6.2}$$

$$(D^2-a^2-p)\theta = - w, \tag{II.6.3}$$

$$(D^2-a^2-\frac{p\sigma_1}{\sigma})h_z = - Dw \;. \tag{II.6.4}$$

II.7. The Boundary Conditions.

The boundary conditions on w', θ' and h_z' are given by,

$$\left.\begin{aligned} w' &= 0 \;, \\ \theta' &= 0 \;, \\ \frac{\partial w'}{\partial z} &= 0 \;, \end{aligned}\right| \quad \text{on a rigid boundary,} \tag{II.7.1}$$

and

$$\left.\begin{aligned} w' &= 0 \;, \\ \theta' &= 0 \;, \\ \frac{\partial^2 w'}{\partial z^2} &= 0 \;. \end{aligned}\right| \quad \text{on a dynamically free boundary,} \tag{II.7.2}$$

while

$$h'_z = o \,,$$ on both the boundaries if the regions outside the fluid are perfectly conducting, (II.7.3)

$$\frac{\partial h'_z}{\partial z} = \mp a h'_z$$ on both the boundaries if the regions outside the fluid are insulating. (II.7.4)

The above boundary conditions, when resolved in terms of normal modes , respectively become,

$$w'' = 0 \,, \quad \theta'' = 0 \,, \quad \frac{dw''}{dz} = 0 \,.$$ on a rigid boundary , (II.7.5)

and

$$w'' = 0 \,, \quad \theta'' = 0 \,, \quad \frac{d^2 w''}{dz^2} = 0 \,.$$ on a dynamically free boundary, (II.7.6)

while

$$h''_z = 0 \,,$$ on both the boundaries if the regions outside the fluid are perfectly conducting, (II.7.7)

$$\frac{dh''_z}{dz} = \mp a h''_z$$ on both the boundaries if the regions outside the fluid are insulating. (II.7.8)

II.8 The Boundary Conditions in Non-Dimensional form.

Using the non-dimensional quantitites defined by equations (II.6.1) and dropping the double dashes and the asterisks for simplicity in writing, we obtain the boundary conditions on w', θ' and h_z' in their non-dimensional form as follows:

$$w = 0 \quad \text{at } z = 0 \text{ and } z = 1 , \qquad (II.8.1)$$

$$\theta = 0 \quad \text{at } z = 0 \text{ and } z = 1 , \qquad (II.8.2)$$

$$Dw = 0 \quad \text{on a rigid boundary} , \qquad (II.8.3)$$

$$D^2w = 0 \quad \text{on a dynamically free boundary}, \qquad (II.8.4)$$

$$h_z = 0 \quad \text{on both the boundaries if the regions outside the fluid are perfectly conducting}, \qquad (II.8.5)$$

$$Dh_z = \mp a h_z \quad \text{on both the boundaries if the regions outside the fluid are insulating.} \qquad (II.8.6)$$

II.9 The Eigen Value Problem.

The requirement that non-trival solutions exist satisfying governing magnetohydrodynamical equations (II.6.2) - (II .6.4) and an appropriate number of boundary conditions given by (II.8.1) - (II.8.6) poses an eigen value problem for p for given values of a^2, R , σ , σ_1 and Q as already described in (I.1.c). One of the main questions to be investigated in this chapter is to find sufficient conditions under which $p_r = 0$ implies $p_i = 0$ for all $a^2 > 0$ i.e. to find sufficient conditions under which the 'principle of exchange of stabilities' is valid.

II.10. Mathematical Analysis.

In the following we prove four theorems that essentially settle Chandrasekhar's conjecture as mentioned earlier and also reveal the connection between Chandrasekhar's [Chandrasekhar(1952)] and Banerjee et al's [Banerjee et al(1985)] works on the validity of the 'principle of exchange of stabilities'.

It may be further remarked that our result is applicable for quite general boundary conditions.

Case (a) : Perfectly Conducting Boundaries.

We prove the following theorem.

Theorem II.10.1 : If $p_r \geqslant 0$ and $Q\sigma_1 \leqslant \pi^2$ then equation (II.6.4) and boundary conditions (II.8.1) and (II.8.5) imply that

$$\int_0^1 (|DW|^2+a^2|W|^2)dz > Q\,\sigma_1 \int_0^1 (|Dh_z|^2+a^2|h_z|^2)dz$$

Proof : Multiplying equation (II.6.4) by h_z^* (the complex conjugate of h_z) and integrating the resulting equation over the range of z, we have

$$\int_0^1 h_z^*(D^2-a^2-\frac{p\,\sigma_1}{\sigma})h_z dz = -\int_0^1 h_z^*\, DWdz. \tag{II.10.1}$$

The term $\int_0^1 h_z^*\, D^2 h_z dz$ on the left hand side of equation (II.10.1) can be put in the desired form by integrating it once and making use of boundary condition (II.8.5) as follows:

$$\int_0^1 h_z^*\, D^2 h_z dz = -\int_0^1 |Dh_z|^2 dz. \tag{II.10.2}$$

Substituting the value of $\int_0^1 h_z^* D^2 h_z dz$ from equation (II.10.2) in equation(II.10.1) and cancelling the negative sign throughout, we derive

$$\int_0^1 (|Dh_z|^2 + a^2|h_z|^2 + \frac{p\,\sigma_1}{\sigma}|h_z|^2)dz = \int_0^1 h_z^*\, Dwdz\ . \tag{II.10.3}$$

Equating the real parts of both sides of equation(II.10.3), we have

$$\int_0^1 (|Dh|^2 + a^2|h_z|^2)dz + \frac{p_r\sigma_1}{\sigma} \int_0^1 |h_z|^2 dz$$

$$= \text{Real part of } \int_0^1 h_z^* \, DW \, dz. \qquad (II.10.4)$$

Integrating the right hand side of equation (II.10.4) once by parts and making use of the boundary condition (II.8.1) or (II.8.5) we get

$$\int_0^1 h_z^* \, DW \, dz = - \int_0^1 W \, Dh_z^* \, dz. \qquad (II.10.5)$$

Substituting the value of $\int_0^1 h_z^* Dw \, dz$ from equation (II.10.5) in equation (II.10.4), we obtain

$$\int_0^1 (|Dh_z|^2 + a^2|h_z|^2)dz + \frac{p_r \sigma_1}{\sigma} \int_0^1 |h_z|^2 dz$$

$$= - \text{Real part of } \int_0^1 wDh_z^* \, dz. \qquad (II.10.6)$$

Further,

$$\begin{aligned} - \text{Real part of } \int_0^1 w \, Dh_z^* \, dz &\leqslant \left| \int_0^1 w \, Dh_z^* \, dz \right| \\ &\leqslant \int_0^1 | W \, Dh_z^* | \, dz \\ &\leqslant \int_0^1 |W| \, |Dh_z| dz \\ &\leqslant \left\{ \int_0^1 |W|^2 dz \right\}^{\frac{1}{2}} \left\{ \int_0^1 |Dh_z|^2 dz \right\}^{\frac{1}{2}} \end{aligned} \qquad (II.10.7)$$

(using Schwartz inequality)

[Rudin (1973)]

Making use of equation (II.10.6), inequality (II.10.7) and the fact that $p_r \geq 0$, we get

$$\int_0^1 |Dh_z|^2 \, dz < \int_0^1 |W|^2 dz. \qquad \text{(II.10.8)}$$

Further making use of equation (II.10.6), inequality (II.10.7) and inequality (II.10.8), we derive

$$\int_0^1 (|Dh_z|^2 + a^2|h_z|^2) \, dz + \frac{p_r \sigma_1}{\sigma} \int_0^1 |h_z|^2 \, dz < \int_0^1 |W|^2 \, dz$$

$$\leq \frac{1}{\pi^2} \int_0^1 |DW|^2 \, dz$$

(using Poincaré inequality) [Joseph(1976)]

$$< \frac{1}{\pi^2} \int_0^1 (|DW|^2 + a^2|W|^2) \, dz. \qquad \text{(II.10.9)}$$

Since $p_r \geq 0$, from inequality (II.10.9), we obtain

$$\int_0^1 (|Dh_z|^2 + a^2|h_z|^2) dz < \frac{1}{\pi^2} \int_0^1 (|DW|^2 + a^2|W|^2) dz. \qquad \text{(II.10.10)}$$

Therefore, if $Q\sigma_1 \leq \pi^2$ then from inequality (II.10.10), we get

$$\int_0^1 (|DW|^2 + a^2|W|^2) dz > Q\sigma_1 \int_0^1 (|Dh_z|^2 + a^2|h_z|^2) \, dz. \qquad \text{(II.10.11)}$$

and this proves the theorem.

Case (b) : Insulating Boundaries.

We prove the following theorem.

<u>Theorem II.10.2</u> : If $p_r \geq 0$ and $Q\sigma_1 \leq \pi^2$ then equation (II.6.4), boundary conditions (II.8.1) and (II.8.6) imply that

$$\int_0^1 (|DW|^2 + a^2|W|^2)\, dz > Q\sigma_1 \int_0^1 (|Dh_z|^2 + a^2|h_z|^2)\, dz.$$

<u>Proof</u> : Multiplying equation (II.6.4) by h_z^* (the complex conjugate of h_z) and integrating the resulting equation over the range of z, we have

$$\int_0^1 h_z^* \left(D^2 - a^2 - \frac{p\sigma_1}{\sigma}\right) h_z dz = - \int_0^1 h_z^* \, Dwdz. \qquad (II.10.12)$$

The term $\int_0^1 h_z^* D^2 h_z dz$ on the left hand side of equation (II.10.12) can be put in the desired form by integrating by parts and making use of boundary condition (II.8.6) as follows:

$$\int_0^1 h_z^* \, D^2 h_z dz = a\{(|h_z|^2)_0 + (|h_z|^2)_1\} - \int_0^1 |Dh_z|^2 dz. \qquad (II.10.13)$$

Substituting the value of $\int_0^1 h_z^* D^2 h_z \, dz$ from equation(II.10.13) in equation (II.10.12) and cancelling the negative sign throughout, we derive

$$a\{(|h_z|^2)_0 + (|h_z|^2)_1\} + \int_0^1 |Dh_z|^2 \, dz + a^2 \int_0^1 |h_z|^2 \, dz +$$

$$\frac{p\sigma_1}{\sigma} \int_0^1 |h_z|^2 dz = \int_0^1 h_z^* \, DW \, dz. \qquad (II.10.14)$$

Equating the real parts of both sides of equation (II.10.14), we have

$$a\{(|h_z|^2)_0 + (|h_z|^2)_1\} + \int_0^1 (|Dh_z|^2 + a^2|h_z|^2)dz +$$

$$\frac{p_r\sigma_1}{\sigma} \int_0^1 |h_z|^2 dz = \text{Real part of } \int_0^1 h_z^* \, Dwdz. \quad (II.10.15)$$

Integrating the right hand side of equation (II.10.15) once by parts and making use of the boundary condition (II.8.1), we get

$$\int_0^1 h_z^* \, Dw \, dz = - \int_0^1 W \, Dh_z^* \, dz. \qquad (II.10.16)$$

Substituting the value of $\int_0^1 h_z^* Dw \, dz$ from equation (II.10.16) in equation (II.10.15), we obtain

$$a\{(|h_z|^2)_0 + (|h_z|^2)_1\} + \int_0^1 (|Dh_z|^2 + a^2|h_z|^2)dz +$$

$$\frac{p_r \sigma_1}{\sigma} \int_0^1 |h_z|^2 dz = - \text{Real part of} \int_0^1 wDh_z^* \, dz. \qquad (II.10.17)$$

Further,

$$\begin{aligned} - \text{Real part of} \int_0^1 wDh_z^* dz &\leqslant \left| \int_0^1 wDh_z^* \, dz \right| \\ &\leqslant \int_0^1 |w \, Dh_z^*| \, dz \\ &\leqslant \int_0^1 |W| \, |Dh_z| dz \\ &\leqslant \left\{ \int_0^1 |W|^2 dz \right\}^{\frac{1}{2}} \left\{ \int_0^1 |Dh_z|^2 dz \right\}^{\frac{1}{2}}. \end{aligned}$$

(II.10.18)

(using Schwartz inequality)

Making use of equation (II.10.16), inequality (II.10.17) and the fact that $p_r \geqslant 0$, we get

$$\int_0^1 |Dh_z|^2 \, dz < \int_0^1 |W|^2 \, dz. \qquad (II.10.19)$$

Further making use of equation (II.10.17), inequality (II.10.18) and inequality (II.10.19), we derive

$$a\{(|h_z|^2)_0 + (|h_z|^2)_1\} + \int_0^1 (|Dh_z|^2 + a^2 |h_z|^2)dz + \frac{p_r \sigma_1}{\sigma} \int_0^1 |h_z|^2 dz$$

$$< \int_0^1 |W|^2 dz$$

$$< \frac{1}{\pi^2} \int_0^1 |DW|^2 dz \text{ (Using Poincaré inequality)}$$

$$< \frac{1}{\pi^2} \int_0^1 (|DW|^2 + a^2|W|^2)dz. \qquad \text{(II.10.20)}$$

Since $p_r \geq o$, from inequality (II.10.20), we obtain

$$\int_0^1 (|Dh_z|^2 + a^2|h_z|^2)\, dz < \frac{1}{\pi^2} \int_0^1 (|DW|^2+a^2|W|^2)\, dz. \qquad \text{(II.10.21)}$$

Therefore, if $Q\sigma_1 \leqslant \pi^2$, then from inequality (II.10.21) we get

$$\int_0^1 (|DW|^2+a^2|W|^2)dz > Q\,\sigma_1 \int_0^1 (|Dh_z|^2+a^2|h_z|^2)dz. \qquad \text{(II.10.22)}$$

and this proves the theorem.

Theorem II.10.3 : If $p_r = 0$ and $Q\sigma_1 \leqslant \pi^2$, then equations (II.6.2) -(II.6.4) and the boundary conditions (II.8.1), (II.8.2), (II.8.3) or (II.8.4), and (II.8.5) imply that the total kinetic energy associated with a disturbance is greater than its magnetic energy.

Proof : Since $p_r = 0$ and $Q\sigma_1 \leqslant \pi^2$ it follows from the result of Banerjee et al [Banerjee et al(1985)] that $p_i = 0$.

As a consequence the total kinetic and magnetic energies associated with a disturbance at the marginal state are respectively given by [Chandrasekhar (1952), Sherman and Ostrach (1966)]

$$\frac{1}{a^2}\int_0^1 (|DW|^2 + a^2|W|^2)\,dz \qquad \text{and} \qquad \frac{Q\sigma_1}{a^2}\int_0^1 (|Dh_z|^2 + a^2|h_z|^2)\,dz \,.$$

But since [Theorem II.10.1],

$$\int_0^1 (|DW|^2 + a^2|W|^2)\,dz > Q\sigma_1 \int_0^1 (|Dh_z|^2 + a^2\,|h_z|^2)\,dz \,, \tag{II.10.23}$$

it follows that at the stationary marginal state, for which a sufficient condition is $Q\sigma_1 \leqslant \pi^2$, the total kinetic energy associated with a disturbance is greater than its total magnetic energy and this settles Chandrasekhar's conjecture for the case when the regions outside the fluid are perfectly conducting.

<u>Theorem II.10.4</u> : If $p_r = 0$ and $Q\sigma_1 \leqslant \pi^2$, then equations (II.6.2)-(II.6.4) and the boundary conditions (II.8.1), (II.8.2) , (II.8.3) or (II.8.4), and (II.8.6) imply that the total kinetic energy associated with a disturbance is greater than its total magnetic energy.

<u>Proof</u> : Since $p_r = 0$ and $Q\sigma_1 \leqslant \pi^2$ it follows from the result of Banerjee et al [Banerjee et al (1985)] that $p_i = 0$.

As a consequence the total kinetic and magnetic energies associated with a disturbance at the marginal state are respectively given by [Chandrasekhar (1952) , Sherman and Ostrach (1966)]

$$\frac{1}{a^2}\int_0^1 (|Dw|^2 + a^2|w|^2)\,dz \quad \text{and} \quad \frac{Q\sigma_1}{a^2}\int_0^1 (|Dh_z|^2 + a^2|h_z|^2)\,dz.$$

But since [Theorem II.10.1],

$$\int_0^1 (|Dw|^2 + a^2|w|^2)\,dz > Q\sigma_1 \int_0^1 (|Dh_z|^2 + a^2|h_z|^2)dz, \tag{II.10.23}$$

it follows that at the stationary marginal state, for which a sufficient condition is $Q\sigma_1 \leqslant \pi^2$, the total kinetic energy associated with a disturbance is greater than its total magnetic energy and this settles Chandrasekhar's conjecture for the case when the regions outside the fluid are insulating.

II.11. Conclusions.

1. If $Q\sigma_1 \leqslant \pi^2$, then the total kinetic energy associated with an unstable or marginally stable disturbance is greater than its total magnetic energy and this result is uniformly valid for quite general magnetohydrodynamic boundary conditions.

2. At the stationary marginal state, for which a sufficient condition is $Q\sigma_1 \leqslant \pi^2$, the total kinetic energy associated with a disturbance is greater than its total magnetic energy and this result is uniformly valid for quite general magnetohydrodynamic boundary conditions.

3. It reveals for the first time the connection between Chandrasekhar's conjecture [Chandrasekhar(1952) and Banerjee et al's [Banerjee et al(1985)] work on the validity of the 'principle of exchange of stabilities' for the problem. More explicitly, it is shown here for the first time that at the stationary marginal state, for which a sufficient condition is $Q\sigma_1 \leqslant \pi^2$[Banerjee et al(1985)], the total kinetic energy associated with a disturbance is greater than its total magnetic energy.

4. The integral inequality (II.10.11) is a consequence of the magnetic induction equation (II.6.4) and the boundary conditions (II.8.1), and (II.8.5) or (II.8.6) only and does not depend either upon the equations of conservation of mass/momentum/energy or the dynamically free/rigid character of the boundaries.

5. The mathematical analysis, although conducted in the context of parallel plates and a relatively simpler physical problem, has the quality of being generalised in a much wider context as will be shown in the latter chapters.

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