PART - I

CHAPTER - 1

A PAIR OF SEMICIRCLE THEOREMS IN THERMOHALINE CONVECTION

1.1. Abstract

The investigation presented in this chapter is concerned with the problem of obtaining bounds for the complex growth rate of an arbitrary oscillatory perturbation, neutral or unstable, in the domain of infinitesimal amplitude instability in thermohaline convection. A semicircle theorem for Stern's configuration and another semicircle theorem for Veronis' configuration are established in this connection. These results are new and are uniformly valid for all combinations of dynamically free and rigid boundaries.

1.2. Introduction

Stern (1960) has treated the stability of a horizontal layer of fluid which is heated from above and in which the mass concentration of a chemical dissolved is maintained at C_0 at the lower boundary and C_1 at the upper boundary ($C_1 > C_0$). The temperatures at the two boundaries are T_0 and T_1 respectively, with $T_1 > T_0$. He has shown that even if the fluid in the undisturbed condition is lighter at the top than at the bottom, instability might still occur in the configuration as exchange of stabilities provided the

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destabilizing concentration gradient is sufficiently large but compatible with the condition that the total density field is gravitationally stable. The above investigation of Stern is restricted by the assumption that the principle of exchange of stabilities is valid.

Veronis (1965) has treated the configuration in which $C_1 < C_0$ and $T_1 < T_0$ and has shown that even if the fluid in the undisturbed condition is lighter at the top than at the bottom, instability might still occur in the configuration as overstability provided the destabilizing temperature gradient is sufficiently large but compatible with the condition that the total density field is gravitationally stable. The above investigation of Veronis is restricted by the assumption that the boundaries are dynamically free.

The problem of obtaining bounds for the complex growth rate of an arbitrary oscillatory perturbation, neutral or unstable, in the above two configurations is important especially in situations when both the boundaries are not dynamically free so that exact solution in closed form is not obtainable. The present investigation is concerned precisely with this problem and a semicircle theorem is established for each of the above two configurations. These results are new and are uniformly valid for all combinations of dynamically free and rigid boundaries.

1.3. Mathematical analysis

We can treat the above two configurations by considering the same configuration but with the assumption that gravity is positive downward in one problem and positive upward in the other problem. The relevant governing equations and boundary conditions in nondimensional forms are (compare Veronis 1965).

$$(D^2-a^2)(D^2-a^2-p/\sigma)w = R_1 a^2\theta - R_2 a^2\varphi, \tag{1.3.1}$$

$$(D^2-a^2-p)\theta = -w, \tag{1.3.2}$$

$$(D^2-a^2-p/\tau)\varphi = -\frac{w}{\tau}, \tag{1.3.3}$$

and $w = 0 = \theta = \varphi = D^2w$ at $z = 0$ and $z = 1$

(both boundaries dynamically free), (1.3.4)

or $w = 0 = \theta = \varphi = Dw$ at $z = 0$ and $z = 1$

(both boundaries rigid), (1.3.5)

or $w = 0 = \theta = \varphi = D^2w$ at $z = 1$)

(upper boundary dynamically free),)

and $w = 0 = \theta = \varphi = Dw$ at $z = 0$) (1.3.6)

(lower boundary rigid),)

without loss of generality.).

In the above equations, z is the vertical coordinate and $0 \leq z \leq 1$, $D = \frac{d}{dz}$, a^2 is the square of the wave number, $p(= p_r + ip_i)$ is the complex growth rate, σ is the Prandtl number, R_1 is the usual Rayleigh number, R_2

is the concentration Rayleigh number, τ is the ratio of mass diffusivity to heat diffusivity, w is the vertical velocity, θ is the temperature and φ is the concentration.

We prove the following theorems :

Theorem 1 : (A semicircle theorm for Veronis' configuration)

For Veronis' configuration, the complex growth rate of an arbitrary oscillatory perturbation, neutral or unstable, must lie inside a semicircle in the right half of the $p_r p_i$ plane whose centre is origin and radius = $\sqrt{R_2 \sigma}$ and this result is uniformly valid for all combinations of dynamically free and rigid boundaries.

Proof : Multiplying both sides of (1.3.1) by w^* (the complex conjugate of w) and integrating the resulting equation over the vertical range of z, we obtain

$$\int_0^1 w^* (D^2-a^2)(D^2-a^2-p/\sigma)w \, dz = R_1 a^2 \int_0^1 w^* \theta \, dz - R_2 a^2 \int_0^1 w^* \varphi \, dz . \quad (1.3.7)$$

Now, taking the complex conjugate of both sides of (1.3.2), multiplying the resulting equation by θ throughout and integrating over the vertical range of z, it follows that

$$\int_0^1 w^* \theta \, dz = -\int_0^1 \theta (D^2-a^2-p^*) \theta^* dz , \quad (1.3.8)$$

where p^* is the complex conjugate of p and θ^* is the complex conjugate of θ.

Further, since $p_i \neq 0$, we get from (1.3.3) that

$$\varphi = \frac{\tau}{p}(D^2 - a^2)\varphi + \frac{w}{p}, \qquad (1.3.9)$$

which can be written in the form

$$\varphi = \frac{\tau p^*}{|p|^2}(D^2 - a^2)\varphi + \frac{p^*}{|p|^2} w. \qquad (1.3.10)$$

Multiplying both sides of (1.3.10) by w^* and integrating the resulting equation over the vertical range of z, we obtain

$$\int_0^1 w^* \varphi \, dz = \frac{\tau p^*}{|p|^2}\int_0^1 w^*(D^2-a^2)\varphi \, dz + \frac{p^*}{|p|^2}\int_0^1 |w|^2 \, dz. \qquad (1.3.11)$$

Again, taking the complex conjugate of both sides of (1.3.3), multiplying the resulting equation by $(D^2-a^2)\varphi$ throughout and integrating over the vertical range of z, it follows that

$$\int_0^1 w^*(D^2-a^2)\varphi \, dz = -\tau\int_0^1 (D^2-a^2)\varphi\,(D^2-a^2-p^*/\tau)\,\varphi^* \, dz, \qquad (1.3.12)$$

where φ^* is the complex conjugate of φ.

Combining (1.3.7), (1.3.8), (1.3.11) and (1.3.12) we obtain

$$\int_0^1 w^*(D^2-a^2)(D^2-a^2-p/\sigma)w \, dz = R_1 a^2\left[-\int_0^1 \Theta(D^2-a^2-p^*)\,\Theta^* \, dz\right]$$

$$- R_2 a^2\left[\frac{\tau p^*}{|p|^2}\left\{-\tau\int_0^1 (D^2-a^2)\varphi.(D^2-a^2-p^*/\tau)\varphi^* \, dz\right\}+\right.$$

$$\frac{p^*}{|p|^2}\int_0^1 |w|^2\, dz\ ]. \qquad (1.3.13)$$

Further, integrating by parts a suitable number of times and utilizing the boundary conditions (1.3.4) or (1.3.5) or (1.3.6), we get

$$\left.\begin{aligned}
\int_0^1 w^* D^4 w\, dz &= \int_0^1 |D^2 w|^2\, dz && > 0,\\
\int_0^1 w^* D^2 w\, dz &= -\int_0^1 |Dw|^2\, dz && < 0,\\
\int_0^1 \Theta\, D^2\Theta^*\, dz &= -\int_0^1 |D\Theta|^2\, dz && < 0,\\
\int_0^1 \varphi^* D^2\varphi\, dz &= -\int_0^1 |D\varphi|^2\, dz && < 0.
\end{aligned}\right\}. \qquad (1.3.14)$$

Making use of (1.3.14), we can write (1.3.13) as

$$\int_0^1 (|D^2 w|^2 + 2a^2|Dw|^2 + a^4|w|^2)dz + \frac{p}{\sigma}\int_0^1 (|Dw|^2 + a^2|w|^2)dz$$

$$= R_1 a^2 \int_0^1 (|D\Theta|^2 + a^2|\Theta|^2 + p^*|\Theta|^2)dz - R_2 a^2\Big\{-\frac{\tau^2 p^*}{|p|^2}$$

$$\int_0^1 |(D^2-a^2)\varphi|^2 dz - \frac{\tau p^{*2}}{|p|^2}\int_0^1 (|D\varphi|^2 + a^2|\varphi|^2)dz + \frac{p^*}{|p|^2}\int_0^1 |w|^2 dz\Big\}. \qquad (1.3.15)$$

Equating the imaginary parts of both sides of (1.3.15), we obtain

$$\frac{p_i}{\sigma}\int_0^1 (|Dw|^2 + a^2|w|^2)dz = -R_1 a^2 p_i \int_0^1 |\Theta|^2 dz - \frac{R_2 a^2 \tau^2 p_i}{|p|^2}$$

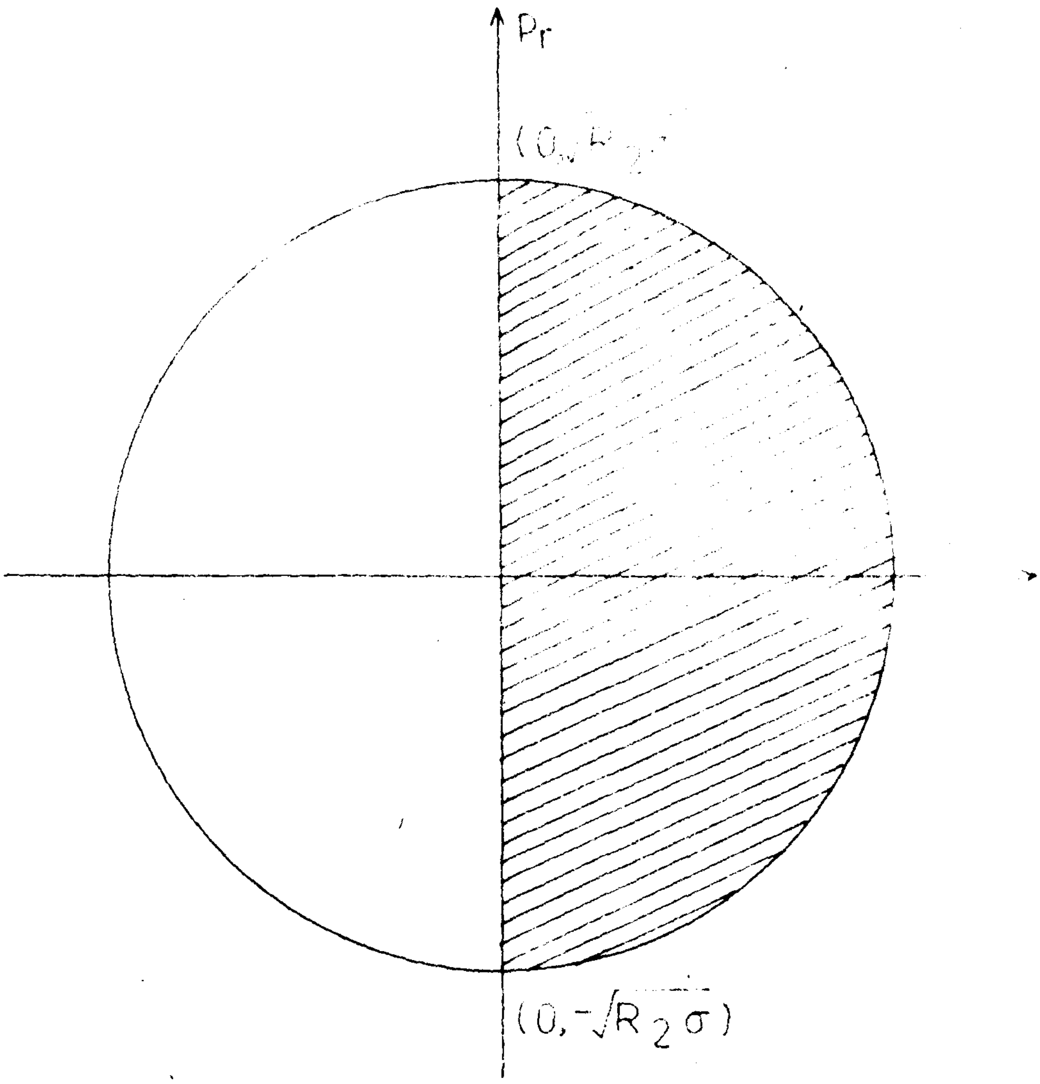


Fig.(a): Shaded area shows the region in which the complex growth rate of an arbitrary oscillatory mode, neutral or unstable, in Veronis' thermohaline configuration must lie.

$$\int_0^1 |(D^2-a^2)\varphi|^2 dz - \frac{2R_2 a^2 \tau p_r p_i}{|p|^2}\int_0^1 (|D\varphi|^2 + a^2|\varphi|^2)dz + \frac{R_2 a^2 p_i}{|p|^2}\int_0^1 |w|^2 dz. \tag{1.3.16}$$

Now, since $p_i \neq 0$, we can cancel p_i from both sides of (1.3.16) and write the resulting equation as

$$\frac{1}{\sigma}\int_0^1 |Dw|^2 dz + a^2\left(\frac{1}{\sigma} - \frac{R_2}{|p|^2}\right)\int_0^1 |w|^2 dz + R_1 a^2\int_0^1 |\theta|^2 dz + \frac{R_2 a^2 \tau^2}{|p|^2}$$

$$\int_0^1 |(D^2-a^2)\varphi|^2 dz + \frac{2R_2 a^2 \tau p_r}{|p|^2}\int_0^1 (|D\varphi|^2 + a^2|\varphi|^2)\, dz = 0. \tag{1.3.17}$$

But, since $p_r \geq 0$, $R_1 > 0$ and $R_2 > 0$, we have from (1.3.17) that

$$|p|^2 < R_2\sigma. \tag{1.3.18}$$

In other words

$$p_r^2 + p_i^2 < R_2\sigma, \tag{1.3.19}$$

and this proves the theorem (see Fig.(a)).

Contributions of Theorem I

(i) Derives the Pellew and Southwell's (1940) result that for the simple Bénard problem 'the principle of exchange of stabilities' is valid.

(ii) Derives Banerjee's (1972, 1978a, 1978b) result that for the generalised Bénard problem, the complex growth rate of an arbitrary oscillatory mode, neutral or unstable, must lie inside a semicircle in the right half of the $p_r p_i$ plane whose centre is origin and radius = $\sqrt{R_2\sigma}$.

(iii) Derives a new result in terms of bounds for the complex growth rate of an arbitrary oscillatory mode, neutral or unstable, for all combinations of dynamically free and rigid boundaries for Veronis' (1965) thermohaline configuration.

(iv) Shows the compatibility of the exact calculations known to date (see, for example, Veronis 1965, Banerjee 1972) with the semicircle restriction of Theorem I.

(v) Shows that the smaller is the value of $R_2\sigma$ in Veronis' (1965) thermohaline configuration, the smaller is the growth rate of an arbitrary oscillatory perturbation, neutral or unstable.

(vi) Derives Banerjee and Kalthia's (1971) result on the complex growth rate of an arbitrary oscillatory perturbation in the Rayleigh-Taylor configuration of a Boussinesq fluid of constant coefficient of viscosity.

Theorem 2 : (A semicircle theorem for Stern's configuration)

For Stern's configuration, the complex growth rate of an arbitrary oscillatory perturbation, neutral or unstable must lie inside a semicircle in the right half of the $p_r p_i$ plane whose centre is origin and radius $\sqrt{-R_1\sigma}$ and this result is uniformly valid for all combinations of dynamically free and rigid boundaries.

<u>Proof</u> : Let
$$\left.\begin{aligned} R_1 &= -\hat{R}_1 , \\ R_2 &= -\hat{R}_2 . \end{aligned}\right\}, \tag{1.3.20}$$
so that $\hat{R}_1 > 0$ and $\hat{R}_2 > 0$.

Equations (1.3.1), (1.3.2) and (1.3.3) then respectively become

$$(D^2-a^2)(D^2-a^2-p/\sigma)\ w = \hat{R}_2 a^2 \varphi - \hat{R}_1 a^2 \theta\ , \tag{1.3.21}$$

$$(D^2-a^2-p)\ \theta = -\ w\ , \tag{1.3.22}$$

$$(D^2-a^2-p/\tau)\varphi = -\ \frac{w}{\tau}\ . \tag{1.3.23}$$

Multiplying both sides of (1.3.21) by w^* and integrating the resulting equation over the vertical range of z, we obtain

$$\int_0^1 w^*(D^2-a^2)(D^2-a^2-p/\sigma)w\ dz = \hat{R}_2 a^2 \int_0^1 w^* \varphi\ dz - \hat{R}_1 a^2 \int_0^1 w^* \theta\ dz. \tag{1.3.24}$$

Now, taking the complex conjugate of both sides of (1.3.23), multiplying the resulting equation by φ throughout and integrating the resulting equation over the vertical range of z, it follows that

$$\int_0^1 w^* \varphi\ dz = -\ \tau \int_0^1 \varphi(D^2-a^2-p^*/\tau)\ \varphi^*\ dz. \tag{1.3.25}$$

Further, since $p_i \neq 0$, we get from (1.3.22) that

$$\theta = \frac{1}{p}\ (D^2-a^2)\ \theta + \frac{w}{p}\ , \tag{1.3.26}$$

which can be written in the form

$$\theta = \frac{p^*}{|p|^2}(D^2-a^2)\theta + \frac{p^*}{|p|^2} w \cdot \qquad (1.3.27)$$

Multiplying both sides of (1.3.27) by w^* and integrating the resulting equation over the vertical range of z, we obtain

$$\int_0^1 w^* \theta \, dz = \frac{p^*}{|p|^2}\int_0^1 w^*(D^2-a^2)\theta \, dz + \frac{p^*}{|p|^2}\int_0^1 |w|^2 \, dz. \qquad (1.3.28)$$

Again, taking the complex conjugate of both sides of (1.3.22) multiplying the resulting equation by $(D^2-a^2)\theta$ throughout and integrating over the vertical range of z, it follows that

$$\int_0^1 w^* (D^2-a^2)\theta \, dz = -\int_0^1 (D^2-a^2)\theta . (D^2-a^2-p^*)\theta \, dz. \qquad (1.3.29)$$

Now, combining (1.3.24), (1.3.25), (1.3.28) and (1.3.29), we obtain

$$\int_0^1 w^*(D^2-a^2)(D^2-a^2-p/\sigma)w \, dz = \hat{R}_2 a^2 \left\{-\tau\int_0^1 \varphi(D^2-a^2-p^*/\tau)\varphi^* \, dz\right\}$$

$$- \hat{R}_1 a^2 \left[\frac{p^*}{|p|^2}\left\{-\int_0^1 (D^2-a^2)\theta . (D^2-a^2-p^*)\, \theta^* \, dz\right\} + \frac{p^*}{|p|^2}\int_0^1 |w|^2 dz\right]. \qquad (1.3.30)$$

Making use of (1.3.14), we can write (1.3.30) as

$$\int_0^1 (|D^2 w|^2 + 2a^2|Dw|^2 + a^4|w|^2)dz + \frac{p}{\sigma}\int_0^1 (|Dw|^2 + a^2|w|^2)dz$$

$$= \hat{R}_2 a^2 \tau \int_0^1 (|D\varphi|^2 + a^2|\varphi|^2 + \frac{p^*}{\tau}|\varphi|^2)dz \;- \hat{R}_1 a^2 \Big\{ \frac{-p^*}{|p|^2} \int_0^1 |(D^2-a^2)\theta|^2 dz$$

$$- \frac{p^{*2}}{|p|^2} \int_0^1 (|D\theta|^2 + a^2|\theta|^2)dz + \frac{p^*}{|p|^2} \int_0^1 |w|^2 \, dz \Big\}. \qquad (1.3.31)$$

Equating the imaginary parts of both sides of (1.3.31), we obtain

$$\frac{p_i}{\sigma} \int_0^1 (|Dw|^2 + a^2|w|^2)dz = - \hat{R}_2 a^2 p_i \int_0^1 |\varphi|^2 dz -$$

$$\frac{\hat{R}_1 a^2 p_i}{|p|^2} \int_0^1 |(D^2-a^2)\theta|^2 dz - \frac{2\hat{R}_1 a^2 p_r p_i}{|p|^2} \int_0^1 (|D\theta|^2 + a^2|\theta|^2) \, dz +$$

$$\frac{\hat{R}_1 a^2 p_i}{|p|^2} \int_0^1 |w|^2 \, dz. \qquad (1.3.32)$$

Now, since $p_i \neq 0$, we can cancel p_i from both sides of (1.3.32) and write the resulting equation as

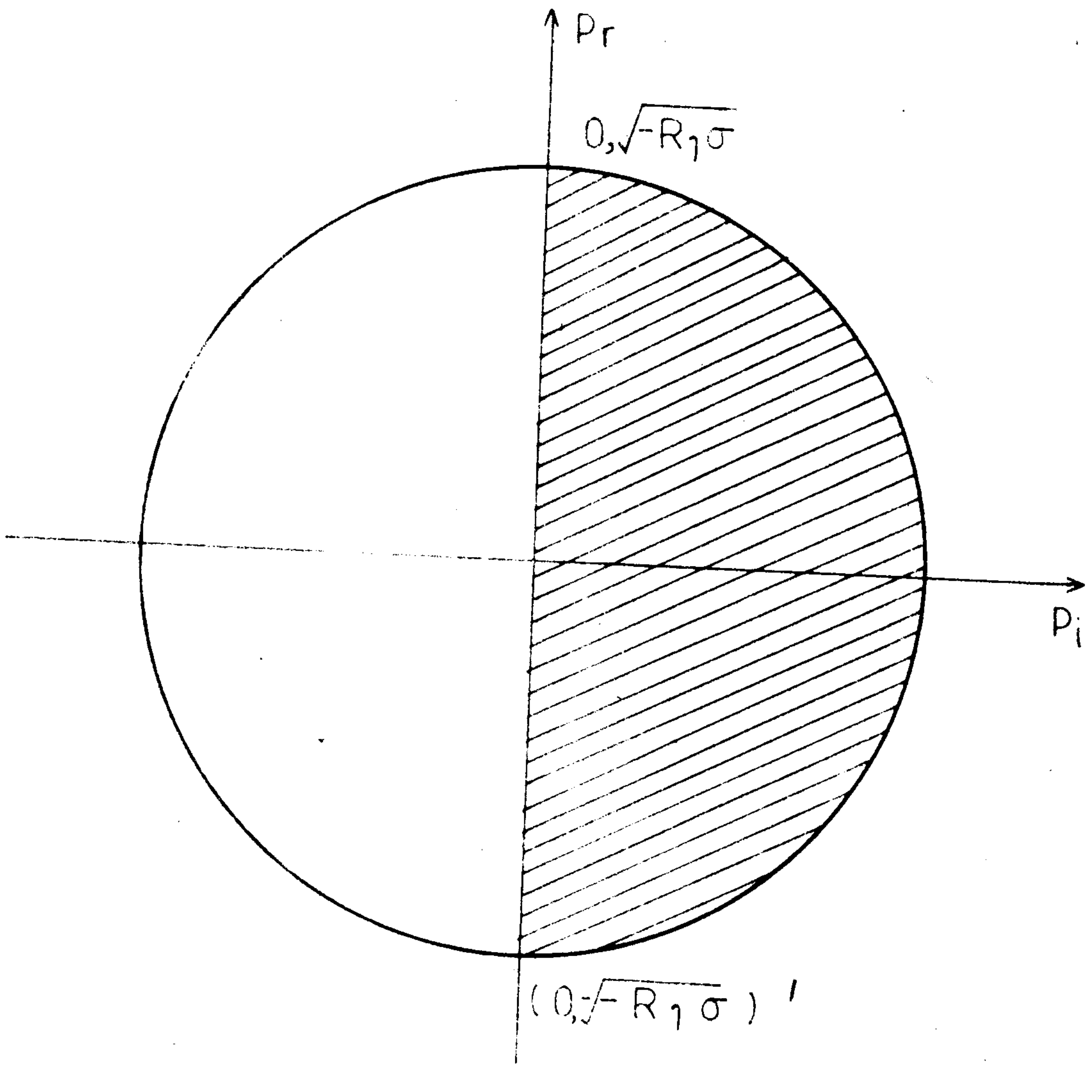
$$\frac{1}{\sigma} \int_0^1 |Dw|^2 dz + a^2\left(\frac{1}{\sigma} - \frac{\hat{R}_1}{|p|^2}\right) \int_0^1 |w|^2 dz + \hat{R}_2 a^2 \int_0^1 |\varphi|^2 dz \;+$$

$$\frac{\hat{R}_1 a^2}{|p|^2} \int_0^1 |(D^2-a^2)\theta|^2 dz + \frac{2\hat{R}_1 a^2 p_r}{|p|^2} \int_0^1 (|D\theta|^2 + a^2|\theta|^2)dz = 0. \qquad (1.3.33)$$

But since $p_r \geq 0$, $\hat{R}_1 > 0$ and $\hat{R}_2 > 0$, we have from (1.3.33) that

$$|p|^2 < \hat{R}_1 \sigma. \qquad (1.3.34)$$

In other words, since $R_1 = - \hat{R}_1$, we obtain from (1.3.34) that



[illegible] Shaded area shows the region in which the complex growth rate of an arbitrary oscillatory mode, neutral or unstable, in Stern's tr[illegible]line configuration must lie

$$p_r^2 + p_i^2 < -R_1\sigma, \qquad (1.3.35)$$

and this proves the theorem (see Fig. (b)).

Contributions of Theorem 2

(i) Derives a new result in terms of bounds for the complex growth rate of an arbitrary oscillatory mode, neutral or unstable, for all combinations of dynamically free and rigid boundaries for Stern's (1960) thermohaline configuration.

(ii) Shows that the smaller is the value of $-R_1\sigma$ in Stern's (1960) thermohaline configuration, the smaller is the growth rate of an arbitrary oscillatory perturbation, neutral or unstable.