CHAPTER - 0

INTRODUCTION

0.1. Simple Bénard problem

It was at the turn of the last century that Bénard (1900) reported on carefully controlled experiments of convective motions in thin horizontal liquid layers heated from below. After the publication of two papers (1900,1901), Bénard and collaborators continued to work extensively on the same subject, seeking in the phenomena they studied a tentative explanation of a large number of apparently disparate problems. Bénard worked with layers thinner than about a millimeter lying on a metallic plate which are heated and maintained at a uniform temperature. The upper surface of the liquid (mostly whale supermaceti, with a melting temperature of 46°C) was in free contact with the ambient air while the temperature of the bottom plate was considerably higher, being on occasion heated to 100°C. Bénard observed two distinct phases in the convective phenomena produced under these conditions. First, when the vertical temperature drop was large enough, a random motion of the fluid resulted. Shortly thereafter, the first phase- of relatively short duration (increasing with fluid viscosity from a few seconds upto several minutes) - appeared, in which the fluid formed cells of almost regular shapes. In this phase, the cellular cross sections showed nearly regular polygons of four to seven sides. During the second stage the cells became equal and regularly spaced hexagons filling up the plane. Thus the limit of the second phase was a steady regime of prisms with vertical boundaries and hexagonal cross sections. The liquid rose in the core of the cell, moved outward at the top, descended at the outer periphery and moved inward at the bottom. Bénard made the circulation visible by pouring into the fluid a few grains of lycopod of about 20 μm diameter, whose individual motion he was able to follow in detail. He correctly characterized the spatial periodicity of the phenomenon by defining its wavelength as the distance between centres of the hexagonal parallelepipeds. In order to explain the first onset of motions in the above configuration of Bénard, Rayleigh (1916) made use of a technique known as the linear stability analysis which we now describe below.

0.2. Linear stability analysis

Linear stability analysis describes the fate of small perturbations in a given thermohydrodynamic state. It attempts, moreover, to delineate the conditions under which a system spontaneously undergoes transitions between different states that may or may not be stationary. In most cases we start with an initial motionless steady state. We assume that the field variables undergo infinitesimal perturbations, and obtain the equations governing these disturbances, from the original Boussinesq equations, neglecting all products and powers greater than first. The stability of a system can only be assessed by investigating its reaction to all possible disturbances. But if an arbitrary though infinitesimal disturbance can be decomposed into a basic set of modes, then instability to one single mode implies instability of the system, this is a sufficient condition for instability (Lin 1955, Eckhaus 1965). For example, in Rayleigh - Bénard convection with symmetry invariance under horizontal translation, the disturbances admit a Fourier decomposition. That is, if δA is one such disturbance, a typical Fourier mode is

$$(\delta A)_k \sim A_k(z,t)\ \exp(i\underline{k}.\underline{r}) \qquad (0.2.1)$$

in which k denotes a wave number, which is the inverse of a characteristic horizontal length. The system is described in rectangular Cartesian coordinates (x,y,z) with, say, the origin in the mid-plane of the layer and the z-axis directed vertically upwards. The basic set of modes (0.2.1) may be discrete or continuous, depending on the lateral boundary conditions. One says that the basic (initial) state of the system is unstable if $A_k(z,t)$ increases with time for one or more values of k.

Furthermore, one usually assumes that

$$A_k(z,t) = A_k(z) \exp(p_k t) \qquad (o.2.2)$$

and the perturbation equations will include $p_k = p(k)$ as a parameter. This is to say that the initial value problem for A(t) is solved by Laplace transformation from t to p, in the hope that stability will be decided by the poles when taking the subsequent inversion. Our stability analysis is thereby reduced to a study of the evolution of the normal mode solutions of the linear differential problem for $\delta A(t)$. Cases in which branch points and cuts exist in the complex p plane will not be considered here, as they are not common in convective instability.

In general, the vertical boundary conditions of a given problem will not allow nontrivial solutions for arbitrary p(k). A marginal state, or state of neutral stability is one for which $A_k(z,t)$ neither grows nor decays asymptotically. This situation is determined by the vanishing of the real part of the time constant p, though the system may undergo (neutral) periodic motions if the imaginary part of p(k) does not simultaneously vanish. These oscillations precede finite amplitude motions, and are called overstable modes. The term overstability seems to have been coined by Eddington in stellar theory, to describe a situation where restoring forces are large enough to make the system overshoot the state of equal displacement on the other side of the equilibrium position.

When the imaginary part of p(k) is set to zero simultaneously to the vanishing of the real part of p(k) the neutral state delinates a region where, under supercritical conditions, a steady convective regime is to be expected ($p_r > 0$, $p_i = 0$), whereas below the critical point all infinitesimal perturbations are damped. To predict such behaviour amounts to proving a theorem. Difficulties in proving the theorem, and the knowledge that a steady flow pattern may indeed show up experimentally, have led many workers in the field to adopt the pragmatic approach of considering it a 'principle'.

√0.3. Rayleigh's theory of simple Bénard convection

As mentioned earlier Rayleigh (1916) investigated the dynamical origin of Bénard cells on the basis of the linear stability theory. His analysis yielded the basic result that a top-heavy fluid layer was stable under the joint influence of viscosity and heat diffusion until the vertical temperature drop was large enough to overcome these two dissipative and stabilizing mechanisms. Rayleigh, found that the sole parameter determining stability was the temperature difference, made dimensionless by a combination of parameters that corresponds to what is now known as the Rayleigh number, which is also the product of Prandtl number times Grashof number. Thus Rayleigh discovered that convective flow sets in when the rate at which (free) energy is liberated by the uprising of the hot, less dense fluid near the base exceeds the rate at which the energy is dissipated by thermal conduction and viscous damping. This argument was later made use of in the development of a variational principle that governs the (linear)mathematical stability problem. [For historical remarks and details see Chandrasekhar (1961), Jeffreys (1956) Pellew and Southwell (1940), Sani (1963) and Gershuni and Zhukovitskii (1976)].

On account of the importance of the Bénard configuration in the problems of meteorology and oceanography and various other fields of practical interest, its meaningful extensions in the framework of various external force fields has been the main centre of activity in the recent past of many research workers in the field of hydrodynamic and hydromagnetic stability. The effect of uniform magnetic field or rotation acting individually and parallel to gravity, on the Bénard problem has been investigated by Chandrasekhar (1952, 53, 54, 55, 57), and others and it is shown that in some respect the effects are remarkably alike namely, they both inhibit the onset of instability and elongate the cells which appear at the marginal stability for certain ranges of the concerned parameters. Chandrasekhar traces the origin of this similar behaviour in the

Taylor-Proudman theorem in the case of rotation and in its exact analogue in the case of a magnetic field. Another interesting point predicted by Chandrasekhar's analysis and in general qualitative agreement with the experimental results of Nakagawa (1955, 57, 59), Fultz-Nakagawa and Frezen (1954,55), and others is that, in both the problems the marginal state could either be stationary or oscillatory in character for which sufficient conditions are obtained. To put the matter more explicitly it is shown that, when magnetic field alone is present, and the magnetic Prandtl number σ_1 is less than the thermal Prandtl number σ which is a requirement met by a large margin under most terrestrial conditions, overstability cannot occur and the principle of exchange of stabilities is valid. Moreover, for $\sigma_1 > \sigma$, there exists a value of the Chandrasekhar number Q, say $Q^{(\sigma,\sigma_1)}$, depending on σ and σ_1 such that when (i) $Q \leq Q^{(\sigma,\sigma_1)}$ the onset of instability will be as stationary convection, and (ii) for $Q > Q^{(\sigma,\sigma_1)}$ it will be as overstability. Likewise, when rotation alone is present, it is the principle of exchange of stabilities which is valid so long as σ exceeds a certain critical value σ_c, where σ_c depends on the nature of bounding surface, while the onset of instability will be as overstable oscillations if $\sigma < \sigma_c$ and the Taylor number T exceeds a certain critical value $T^{(\sigma)}$ depending on σ, and for $T \leq T^{(\sigma)}$ the onset of instability will be as stationary convection. However, there are certain basic differences in the character of motions which manifest when rotation alone is present and when a magnetic field alone is present. More explicitly, rotation causes a component of vorticity in the direction of the rotation vector $\underline{\Omega}$ and this has predominant effects. For large Taylor number T, the motions are principally confined to planes transverse to $\underline{\Omega}$. On the other hand, magnetic field does not cause any such component of vorticity and thus there are not corresponding effects. On the other hand, for large values of the Chandrasekhar number Q, the motions along the magnetic field lines become predominant, while the motions transverse to the magnetic field vector $\underline{H}$ are much reduced. Further, there is another point to consider, namely, that viscosity encourages the onset of instability in the presence of rotation, while a magnetic field imparts to the liquid certain amount of rigidity. Therefore even though the two acting individually postpone the development of instability, they might have opposing tendencies when acting together. Chandrasekhar (1954,55) examined the problem where $\underline{\Omega}$ and $\underline{H}$ are both uniform and in the vertical direction and the boundaries are free and non-conducting. He has shown that the manner of the onset of instability, must in general, depend in an extremely complicated manner on the relevant parameters involved. Assuming the principle of exchange of stabilities to be valid the numerical results indicated that, if T was kept fixed and Q increased from zero, the critical Rayleigh number remained roughly constant until Q reached a certain critical value when it started to decrease. When Q was increased further, the critical Rayleigh number continued to decrease, reaching a minimum before starting to increase again.

The work of Eltayeb (1972) is also concerned with the combined effect of rotation and magnetic field on the simple Bénard problem when both T and Q are large i.e. in the double limit $T \to \infty$, $Q \to \infty$. The analysis is carried for three different configurations classified by the orientations of the magnetic field and rotation axes under a variety of different surface conditions. It is shown that irrespective of the nature of the boundaries the asymptotic dependence of the critical Rayleigh number on T and Q is the same, apart from constants of proportionality of order unity. More specifically, assuming the principle of exchange of stabilities to be valid, $R_{1c} = O(T^{1/2})$ when $T = O(Q^2)$ in all the configurations which, of course, implies that, for large values of T, the presence of a magnetic field facilitates convection (Eltayeb and Roberts 1970) .

Further work in this direction has been done by Eltayeb (1975); Busse (1975, 78a, 78b); Fearn (1979);

Moore (1978); Roberts and Loper (1979); Roberts and Soward (1972); Roberts and Stewartson (1974,75); Soward (1979) and Roberts (1978).

For latest developments concerning the problem of simple Bénard convection and some of its extensions, one is referred to Hopfinger et al (1979); Busse (1978,80); Normand et al (1977); Rogers (1976); Spiegel (1971); Joseph (1976,80); Banerjee (1978,81) and Haken (1979).

0.4 Thermohaline (or thermosolutal) convection or double-diffusive phenomena

As mentioned earlier the study of convective motions produced by unstable density distributions in a fluid is now highly developed. A striking feature of many systems of interest is that instabilities can develop even when the net density decreases upwards. Diffusion, which is generally stabilizing in a fluid containing a single solute, can now act so as to allow the release of the potential energy in the component that is heavy at the top. Much of this work was initiated with an application to the ocean in mind, and because heat and salt (or some other dissolved substance) are then important, the process has been called thermohaline (or thermosolutal) convection. Related effects have now been observed in other contexts and the name double-diffusive convection has been used to encompass this wider range of phenomena. The minimum requirements for the occurrence of double-diffusive convection, in the sense implied here, are the following :

(i) The fluid must contain two or more components having different molecular diffusivities. It is the differential diffusion that produces the density differences required to derive the motion.

(ii) The components must make opposing contributions to the vertical density gradient.

It is assumed throughout that the fluids are completely miscible, so that surface-tension effects do not arise.

0.5. The fundamental mechanisms

The first of the two fundamental mechanisms of thermohaline (or thermosolutal) convection occurs in a fluid for which the temperature and salinity both decrease with depth, while the (overall) density increases with depth. In this statically stable situation, the dynamic instability that arises can be examined by considering a parcel of fluid displaced vertically downward. Initially warmer and saltier than its surroundings, the parcel comes to thermal equilibrium before its excess salinity can be diffused. It is thus heavier than its surroundings and continues to descend. The ensuing motion consists of adjacently rising and falling cells, interchanging their heat, and to much smaller extent their salt, much like a heat exchanger. The kinetic energy of the motion is extracted from the potential energy stored in the salt field. Experiments indicate that in typical conditions, the plan form of the cells, called salt-fingers, is squarish with a horizontal length scale of $[\{\alpha g/\kappa\nu\}(\frac{d\bar{T}}{dz})]^{-1/4}$, where α is the coefficient of thermal expansion, g is the acceleration due to gravity, κ is the coefficient of thermal diffiusivity, ν is the kinematic viscosity and $(\frac{d\bar{T}}{dz})$ is the mean (positive) vertical temperature gradient. This length scale, represents a balance between dissipative effects acting preferentially on small scale motions and the increasing inefficiency of diffusing heat over ever larger horizontal distances. The second fundamental mechanism occurs in a fluid whose temperature, salinity and (as before) overall density increases with depth. Displacement of the typical fluid particle vertically downwards now places it in a warmer, saltier and more dense environment. As before, the thermal field of the parcel begins to equilibrate with its surroundings more rapidly than does the salt field. The parcel is then lighter than its surroundings and rises. But due to the finite value of the thermal diffusion coefficient, the temperature field of the parcel lags the displacement field and the parcel returns to its original position lighter than it was at the outset. It thus rises through a distance greater than the original displacement, whereupon the above process continues and leads to a series of oscillations, or overstability, which is resisted only by the effects of viscosity.

0.6. The fundamental works of Stern, Veronis, Banerjee and others

Stern (1960) has treated the stability of a horizontal layer of fluid which is heated from above and in which the mass concentration of a chemical dissolved is maintained at C_0 at the lower boundary and C_1 at the upper boundary ($C_1 > C_0$). The temperature at the two boundaries are T_0 and T_1 respectively, with $T_1 > T_0$. He has shown that even if the fluid in the undisturbed condition is lighter at the top than at the bottom, instability might still occur in the configuration as exchange of stabilities provided the destabilizing concentration gradient is sufficiently large but compatible with the condition that the total density field is gravitationally stable. The above investigation of Stern is restricted by the assumption that the principle of exchange of stabilities is valid but more general in the sense that this investigation holds for all combinations of dynamically free and rigid boundaries. Veronis (1965) has treated the configuration in which $C_1 < C_0$ and $T_1 < T_0$ and has shown that even if the fluid in the undisturbed condition is lighter at the top than at the bottom, instability might still occur in the configuration as overstability provided the destabilizing temperature gradient is sufficiently large but compatible with the condition that the total density field is gravitationally stable. The investigation of Veronis is restricted by the assumption that the boundaries are dynamically free.

The effect of an initial non-homogeneity on the simple Bénard problem has been analysed by Banerjee (1969, 71, 72, 73). This work is based on the rather plausible hypothesis that a real fluid in general is initially non-homogeneous and one may not always be justified in neglecting the effects of this initial non-homogeneity everywhere from the governing equations as compared to other effects that are retained. Thus, taking the initial fluid non-homogeneity of the form $\rho = \rho_0 e^{-\delta z}$ with δ = constant > 0 he sets up, what we shall subsequently refer to, a generalized Bénard problem. It is interesting to note here that the governing equations of the problem of infinitesimal amplitude instability in thermohaline convection wherein one investigates the instability of a horizontally infinite layer of fluid subjected to a destabilizing salt gradient and to a stabilizing temperature gradient (Stern 1960) or the gravitationally opposite configuration (Veronis 1965), coincide exactly, when the mass diffusivity of the dissolved solute is neglected, with the governing equations of the generalized Bénard problem with $\delta < 0$ or $\delta > 0$ which **were** derived independently and with a basically different outlook namely, to present a unified treatment of the well known Bénard and Rayleigh-Taylor instability problems.

Banerjee has shown that the generalized Bénard system should become unstable to only overstable perturbations when the disturbance is infinitesimal.

He has also evaluated the critical Rayleigh number and the corresponding wave length and frequency for these overstable oscillations at the marginal state. The above investigation of Banerjee is valid for all combinations of dynamically free and rigid boundaries.

This oscillatory form of motion as predicted by the analysis of Veronis (1965) and Banerjee (1969) has been experimentally documented (Shirtcliffe, 1969). For sufficiently large temperature gradients, steady motions can occur because a large temperature field can overcome the restoring tendency of the salinity field.

On account of the importance of the thermohaline (or thermosolutal)/generalised Bénard configuration in the problems of meteorology, oceanography and stellar convection and various other fields of practical interest, its meaningful extensions in the frame-work of various external force fields has been one of the main centre of activity in the recent past of many research workers in the field of hydrodynamic and hydromagnetic stability. Banerjee et al (1976a, 76b, 76c) have shown that for the case of generalized Bénard configuration with non-conducting boundaries which may be rigid or free the principle of exchange of stabilities is not valid irrespective of whether the magnetic field and rotation are acting individually or together. The overstable solutions at the marginal state, when the boundaries are free and non-conducting and magnetic field alone is acting, show that the onset of instability is postponed further and further as Q increases whenever $\sigma_1 < \sigma$ and $R_2 > 0$. The stabilizing character of the magnetic field is thus established in the above situation. Further, the above solutions at the marginal state when applied to liquid metals show that for increasing values of $R_2 > 0$ i.e. the initial distribution of density becoming more and more bottom heavy the system becomes more and more stable. On the other hand the system becomes unstable through non-oscillatory modes for $\sigma_1 < \sigma$ and $R_2 < 0$. The analytical solutions at the marginal state when the boundaries are free and rotation alone is acting establish the stabilizing character of rotation as T increases, for all admissible values of σ and $R_2 > 0$, while for $\sigma > 1$ and $R_2 < 0$ the system becomes unstable through non-oscillatory modes as in the case with the magnetic field. It is noted here that while for $R_2 = 0$ the marginal state, in both the problems wherein magnetic field or rotation are individually included, could either be stationary or oscillatory in character depending on certain values of the parameters involved (Chandrasekhar, 1961), it is definitely oscillatory in character for $R_2 > 0$. The investigation of the situation when magnetic field and rotation are simultaneously present and the boundaries are free and non-conducting with $R_2 > 0$ leads to an interesting result. The overstable solutions at the marginal state when applied to liquid metals show an important effect. Thus, while in the presence of a uniform magnetic field acting separately, an increasing stable initial density stratification i.e. increasing values of $R_2 > 0$ stabilizes the system more and more, it plays a dual role, i.e. it first destabilizes the system and then stabilizes it when magnetic field and rotation are simultaneously present.

Further work on the various extensions of thermohaline (or thermosolutal) convection model has been done by Palliwal and Chen (1980a, 1980b); Masuda (1978); Roberts and Stewartson (1977); Soward (1979); Griffiths (1979); Acheson (1978a, 78b, 79a).

For latest developments concerning the problem of thermohaline convection and its various extensions one is referred to Huppert (1977); Hopfinger et al (1979); Roberts (1978); Pearlstein (1977); Turner (1973, 74) and Acheson (1980).

0.7. Contributions of Part I

The problem of obtaining bounds for the complex growth rate of an arbitrary oscillatory perturbation, neutral or unstable, in the configurations of Stern(1960) and Veronis (1965) is important especially in situations when both the boundaries are not dynamically free so that exact solution in closed form is not obtainable.

If a constraint such as rotation, is present in the configuration, instability may occur first as an over-stable, i.e. time-dependent, maintained perturbation. This type of instability arises because a steady type of motion may be too restrictive in the sense that it cannot take advantage of sources of potential energy that are available to a time-dependent motion. A further fact, observed both theoretically and experimentally, is that these overstable motions, when they do occur, are generally less efficient in transporting heat and mass and in altering the mean gradients than are steady convective motions. Hence, one can conjecture that, when the determining parameter (usually a Rayleigh number) exceeds the critical eigenvalue, over-stable motions will occur first ; as the parameter is increased further, so that steady convective instability can occur, the observed motions will be the latter.

The problem of obtaining bounds for the complex growth rate of an arbitrary oscillatory perturbation, neutral or unstable, in rotatory thermohaline configuration is thus important especially in situations when both the boundaries are not dynamically free so that exact solution in closed form is not available.

We prove the following four theorems in this part. The importance of these theorems in the context of thermal/thermohaline instability of a liquid layer is also emphasized.

Theorem 1

For Veronis' configuration, the complex growth rate of an arbitrary oscillatory perturbation, neutral or unstable, must lie inside a semicircle in the right half of the $p_r p_i$ plane whose centre is origin and radius = $\sqrt{R_2\sigma}$ and this result is uniformly valid for all combinations of dynamically free and rigid boundaries.

Theorem 2

For Stern's configuration, the complex growth rate of an arbitrary oscillatory perturbation, neutral or unstable, must lie inside a semicircle in the right half of the $p_r p_i$ plane whose centre is origin and radius = $\sqrt{-R_1\sigma}$ and this result is uniformly valid for all combinations of dynamically free and rigid boundaries.

Theorem 3

For rotatory Veronis' configuration, the complex growth rate of an arbitrary oscillatory perturbation, neutral or unstable, must lie inside a semicircle in the right half of the $p_r p_i$ plane whose centre is origin and radius $= \sqrt{\text{greater of } (T\sigma^2, R_2\sigma)}$ and this result is uniformly valid for all combinations of dynamically free and rigid boundaries.

Theorem 4

For rotatory Stern's configuration, the complex growth rate of an arbitrary oscillatory perturbation, neutral or unstable, must lie inside a semicircle in the right half of the $p_r p_i$ plane whose centre is the origin and radius $= \sqrt{\text{greater of } (T\sigma^2, -R_1\sigma)}$ and this result is uniformly valid for all combinations of dynamically free and rigid boundaries.

0.8. Contributions of Part II

The point that we wish to emphasise and utilize in this part is this that linear theoretical explanation of the phenomenon of gravity dominated thermal instability in a liquid layer heated underside (Bénard convection) should depend not only upon the Rayleigh number which is porportional to the uniform temperature difference maintained across the layer but also upon another parameter so that a provision could be made in the theory to recognise the fact that a relatively hotter layer with its heat diffusivity apparently increased/decreased as a consequence of an actual decrease/increase (depending upon the fluid) in its specific heat at constant volume must exhibit Bénard convection at a higher/lower temperature difference across the layer and hence at a higher/lower Rayleigh number than a cooler layer under identical conditions otherwise and further that this qualitative effect is not quantitatively insignificant.

To make the argument more explicit consider two static liquid layers L_1 and L_2 of the same liquid and same depth with respective maintained uniform temperatures of lower and upper boundaries as $T_o^{L_1}$ and $T_o^{L_2}$ and $T_1^{L_1}$ and $T_1^{L_2}$ where $T_o^{L_1} > T_1^{L_1}$ and $T_o^{L_2} > T_1^{L_2}$ so that uniform adverse temperature differences

$$\left.\begin{aligned} \Delta T^{L_1} &= T_o^{L_1} - T_1^{L_1}, \\ \Delta T^{L_2} &= T_o^{L_2} - T_1^{L_2}. \end{aligned}\right\}$$

are respectively acting on L_1 and L_2. An application of the consequences of Rayleigh's theory implies that by raising ΔT^{L_1} and ΔT^{L_2} within limits and thereby keeping all other parameters fixed in a suitable framework we can bring about thermal instability to manifest itself in L_1 and L_2 and further that when it occurs we must have $\Delta T^{L_1} = \Delta T^{L_2}$ irrespective of whether $T_o^{L_1} > T_o^{L_2}$ or $T_o^{L_1} < T_o^{L_2}$. In other words Rayleigh's theory of Bénard convection does not distinguish whether the layer is hotter or cooler and predicts the same Rayleigh number at the onset of instability in both the cases which is contrary to physical intuition and not in very good agreement with the experimental observations as mentioned earlier. The origin of the above conclusion lies in the Boussinesq approximation, which Rayleigh utilizes wherein the variability in the density and in other coefficients is due to variations in temperature of only moderate amounts. Since the coefficient of volume expansion of liquids and gases such as we are mostly concerned with is in the range 10^{-3} to 10^{-4} it follows that for variations in temperature not exceeding 10^{0}(say), the variations in the density are at most one percent and as a consequence the variations in the other coefficients must be of the same order. In the usual application of the Boussinesq approximation the variations of this small amount are, in general, ignored in all the terms of the governing equations with the exception of the term that contains the external force and the theory that results does not either give a complete qualitative picture of Bénard convection or can be said to be in very good quantitative agreement with the experimental observations (Schmidt and Milverton 1935 ; Silveston 1958).

The individual/simultaneous effects of a uniform rotation/a uniform magnetic field in the above modified framework are investigated in the context of thermal/

thermohaline instability of a liquid layer heated underside and the results are compared with those obtained in the classical framework.