# CHAPTER - 7

## MODIFIED ANALYSIS OF BÉNARD CONVECTION, GENERALIZED BÉNARD CONVECTION AND THERMOHALINE CONVECTION : THE EFFECT OF ROTATION AND MAGNETIC FIELD

### 7.1. The problem and the initial state

A viscous finitely heat conducting Boussinesq fluid of infinite horizontal extension is kept rotating at a constant rate with angular velocity of rotation $\Omega$ and is statically confined between two horizontal boundaries z=0 and z=d which are respectively maintained at uniform temperatures $T_0$ and $T(< T_0)$ and uniform concentrations $S_0$ and $S_1$ $(< S_0)$ in the presence of a uniform magnetic field acting parallel and opposite to the direction of gravity. We wish to mathematically investigate the onset of linear hydrodynamic instability in the system.

The initial state is given by

$$\left.\begin{aligned} (u,v,w) &\equiv (0,0,0)\ , \\ T &\equiv T(z)\ , \\ S &\equiv S(z)\ , \\ \rho &= \rho(z)\ , \\ \text{and}\quad (H_1,H_2,H_3) &= (0,0,H). \end{aligned}\right\} \qquad (7.1.1)$$

## 7.2. The basic equations and the initial state solution

The governing equations as mentioned earlier are given by (3.2.53) - (3.2.59). These equations on the basis of (7.1.1) yield the initial state solution.

$$
\left.
\begin{array}{rcl}
(u,v,w) & = & (0,0,0), \\
T & = & T_0-\beta_1 z \, , \\
S & = & S_0-\beta_1 z \, , \\
\rho & = & \rho_0[1+\alpha(T_0-T)-\hat{\alpha}(S_0-S)] \\
 & = & [1+\alpha\beta_1 z - \hat{\alpha}_2\beta_2 z], \\
(H_1,H_2,H_3) & = & (0,0,H) \, , \\
\text{and} \qquad P_3 & = & p-\frac{1}{2}\rho_0|\Omega \times \underline{r}|^2+\frac{\mu_e|H|^2}{8\pi} \\
 & = & P_{30}-g\rho_0[z+(\alpha\beta_1-\hat{\alpha}\beta_2)\frac{z^2}{2}].
\end{array}
\right\} \qquad (7.2.1)
$$

## 7.3. The perturbation equations

Let the initial state described by (7.2.1) be slightly perturbed so that the perturbed state is given by

$$
\left.
\begin{array}{rcl}
(\bar{u},\ \bar{v},\ \bar{w}) & = & (0+u',\ 0+v',\ 0+w'), \\
\bar{T} & = & T_0-\beta_1 z+\theta' \, , \\
\bar{S} & = & S_0-\beta_2 z+\varphi' \, , \\
\bar{\rho} & = & [1+\alpha(T_0-T-\theta')-\hat{\alpha}(S_0-S-\varphi')], \\
(\bar{H}_1,\bar{H}_2,\bar{H}_3) & = & (0+h_x',\ 0+h_y',\ H+h_z') \, , \\
\text{and} \qquad \bar{P}_3 & = & P_{30}-g\rho_0[z+(\alpha\beta_1-\hat{\alpha}\beta_2)\frac{z^2}{2}]+\delta p'.
\end{array}
\right\} \qquad (7.3.1)
$$

Then the linearized equations of perturbation of continuity, momentum, heat conduction, mass diffusion, magnetic induction and magnetic field continuity are given by

$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} = 0, \qquad (7.3.2)$$

$$\rho_0 \frac{\partial u'}{\partial t} = -\frac{\partial \delta p'}{\partial x} + u_0 \nabla^2 u' + \frac{\mu_e H}{4\pi}\left(\frac{\partial h'_x}{\partial z} - \frac{\partial h'_z}{\partial x}\right) + 2\rho_0 \Omega v', \qquad (7.3.3)$$

$$\rho_0 \frac{\partial v'}{\partial t} = -\frac{\partial \delta p'}{\partial y} + u_0 \nabla^2 v' + \frac{\mu_e H}{4\pi}\left(\frac{\partial h'_y}{\partial z} - \frac{\partial h'_z}{\partial y}\right) - 2\rho_0 \Omega u', \qquad (7.3.4)$$

$$\rho_0 \frac{\partial w'}{\partial t} = -\frac{\partial \delta p'}{\partial z} + u_0 \nabla^2 w' + g\alpha \rho_0 \Theta' - g\hat{\alpha} \rho_0 \varphi', \qquad (7.3.5)$$

$$(1-T_0\alpha_2)\frac{\partial \Theta'}{\partial t} + T_0\hat{\alpha}_2 \frac{\partial \varphi'}{\partial t} - (1-T_0\alpha_2)\beta_1 w' - T_0\hat{\alpha}_2\beta_2 w' = \kappa_0 \nabla^2 \Theta', \qquad (7.3.6)$$

$$\frac{\partial \varphi'}{\partial t} - \beta_2 w' = \eta_0 \nabla^2 \varphi', \qquad (7.3.7)$$

$$\frac{\partial h'_x}{\partial t} = H \frac{\partial u'}{\partial z} + \gamma_0 \nabla^2 h'_x, \qquad (7.3.8)$$

$$\frac{\partial h'_y}{\partial t} = H \frac{\partial v'}{\partial z} + \gamma_0 \nabla^2 h'_y, \qquad (7.3.9)$$

$$\frac{\partial h'_z}{\partial t} = H \frac{\partial w'}{\partial z} + \gamma_0 \nabla^2 h'_z, \qquad (7.3.10)$$

and
$$\frac{\partial h'_x}{\partial x} + \frac{\partial h'_y}{\partial y} + \frac{\partial h'_z}{\partial z} = 0. \qquad (7.3.11)$$

Further, combining (7.3.3) and (7.3.4) and (7.3.8) and (7.3.9), we derive that

$$\rho_0 \frac{\partial \zeta'}{\partial t} = u_0 \nabla^2 \zeta' + \frac{\mu_e H}{4\pi}\frac{\partial \xi'}{\partial z} + 2\rho_0 \Omega \frac{\partial w'}{\partial z}, \qquad (7.3.12)$$

and

$$\frac{\partial \xi'}{\partial t} = H \frac{\partial \zeta'}{\partial z} + \gamma_o \nabla^2 \xi' , \tag{7.3.13}$$

where $\xi'$ is the z component of current density and $\zeta'$ is the z-component of vorticity.

## 7.4. The perturbation equations governing the normal modes

Making use of the normal mode analysis as given in (4.4) we obtain the following perturbation equations from (7.3.2) - (7.3.13)

$$ik_x u'' + ik_y v'' + \frac{dw''}{dz} = 0 \tag{7.4.1}$$

$$\rho_o n u'' = -ik_x \delta p'' + \mu_o \left(\frac{d^2}{dz^2} - k^2\right) u'' + \frac{\mu_e H}{4\pi}\left(\frac{\partial h_x''}{\partial z} - ik_x h_z''\right) + 2\rho_o \Omega v'', \tag{7.4.2}$$

$$\rho_o n v'' = -ik_y \delta p'' + \mu_o \left(\frac{d^2}{dz^2} - k^2\right) v'' + \frac{\mu_e H}{4\pi}\left(\frac{\partial h_y''}{\partial z} - ik_y h_z''\right) - 2\rho_o \Omega u'', \tag{7.4.3}$$

$$\rho_o n w'' = -\frac{d\delta p''}{dz} + \mu_o \left(\frac{d^2}{dz^2} - k^2\right) w'' + g\alpha \rho_o \theta'' - g\hat{\alpha} \rho_o \varphi'', \tag{7.4.4}$$

$$(1-\Gamma_o \alpha_2) n\theta'' + T_o \hat{\alpha}_2 n\varphi'' - (1-\Gamma_o \alpha_2)\beta_1 w'' - T_o \hat{\alpha}_2 \beta_2 w'' = \kappa_o \left(\frac{d^2}{dz^2} - k^2\right)\theta'', \tag{7.4.5}$$

$$n\varphi'' - \beta_2 w'' = \eta_o \left(\frac{d^2}{dz^2} - k^2\right)\varphi'', \tag{7.4.6}$$

$$n h_x'' = H \frac{du''}{dz} + \gamma_o \left(\frac{d^2}{dz^2} - k^2\right) h_x'' , \tag{7.4.7}$$

$$nh_y'' = H\frac{dv''}{dz} + \gamma_c\left(\frac{d^2}{dz^2} - k^2\right) h_y'' \quad , \tag{7.4.8}$$

$$nh_z'' = H\frac{dw''}{dz} + \gamma_c\left(\frac{d^2}{dz^2} - k^2\right) h_z'' \quad , \tag{7.4.9}$$

$$ik_x\, h_x'' + ik_y\, h_y'' + \frac{dw''}{dz} = 0, \tag{7.4.10}$$

$$\rho_c n\zeta'' = \mu_o\left(\frac{d^2}{dz^2} - k^2\right)\zeta'' + \frac{\mu_e H}{4\pi}\,\frac{d\xi''}{dz} + 2\rho_c\Omega\,\frac{dw''}{dz} \quad , \tag{7.4.11}$$

$$n\xi'' = H\frac{d\zeta''}{dz} + \gamma_c\left(\frac{d^2}{dz^2} - k^2\right)\xi'' \quad . \tag{7.4.12}$$

7.5. The boundary conditions

The boundary conditions on $w', \theta', \varphi', \xi', \zeta'$ and $h_z'$ are given by

$$\left.\begin{aligned} w' &= 0, \\ \theta' &= 0, \\ \varphi' &= 0, \\ \frac{\partial w'}{\partial z} &= 0, \\ \text{and} \quad \zeta' &= 0. \end{aligned}\right) \text{ on a rigid boundary,} \tag{7.5.1}$$

$$\left.\begin{aligned} w' &= 0, \\ \theta' &= 0, \\ \varphi' &= 0, \\ \frac{\partial^2 w'}{\partial z^2} &= 0, \\ \text{and} \quad \frac{\partial \zeta'}{\partial z} &= 0. \end{aligned}\right) \text{ on a dynamically free boundary,} \tag{7.5.2}$$

while

$$\left.\begin{aligned} h'_z &= 0, \\ \frac{\partial \xi'}{\partial z} &= 0. \end{aligned}\right\} \text{ on a perfectly conducting boundary.} \qquad (7.5.3)$$

The above boundary conditions, when analysed in terms of normal modes, respectively become

$$\left.\begin{aligned} w'' &= 0, \\ \theta'' &= 0, \\ \varphi'' &= 0, \\ \frac{dw''}{dz} &= 0, \\ \text{and} \quad \zeta'' &= 0. \end{aligned}\right\} \text{ on a rigid boundary ,} \qquad (7.5.4)$$

$$\left.\begin{aligned} w'' &= 0, \\ \theta'' &= 0, \\ \varphi'' &= 0, \\ \frac{d^2 w''}{dz^2} &= 0, \\ \text{and} \quad \frac{d\zeta''}{dz} &= 0. \end{aligned}\right\} \text{ on a dynamically free boundary,} \qquad (7.5.5)$$

$$\text{while} \quad \left.\begin{aligned} h''_z &= 0, \\ \frac{d\xi''}{dz} &= 0. \end{aligned}\right\} \text{ on a perfectly conducting boundary.} \qquad (7.5.6)$$

7.6. The characteristic value problem

Multiplying (7.4.2) by $k_x$ and (7.4.3) by $k_y$, adding the resulting equations and making use of (7.4.1) and (7.4.10) we obtain

$$\rho_0 n \frac{dw''}{dz} = -k^2\delta p'' + \mu_0\left(\frac{d^2}{dz^2} - k^2\right)\frac{dw''}{dz} + \frac{\mu_e H}{4\pi}\left(\frac{d^2}{dz^2} - k^2\right)h_z'' - 2\rho_0\Omega\zeta'' \tag{7.6.1}$$

Eliminating $\delta p''$ between (7.6.1) and (7.4.4), it follows that

$$\left(\frac{d^2}{dz^2} - k^2\right)\left(\frac{d^2}{dz^2} - k^2 - \frac{n}{\nu_0}\right)w'' = \frac{g\alpha k^2\theta''}{\nu_0} - \frac{g\hat{\alpha}k^2\varphi''}{\nu_0} - \frac{\mu_e H}{4\pi\rho_0\nu_0}\left(\frac{d^2}{dz^2} - k^2\right)\frac{dh_z''}{dz} + \frac{2\Omega}{\nu_0}\frac{d\zeta''}{dz}. \tag{7.6.2}$$

Further, equation (7.4.5), (7.4.6), (7.4.9), (7.4.11) and (7.4.12) can be rewritten as

$$\left[\frac{d^2}{dz^2} - k^2 - n(1-T_0\alpha_2)/\kappa_0\right]\theta'' - \frac{T_0\hat{\alpha}_2 n\varphi''}{\kappa_0} = -\frac{(1-T_0\alpha_2)\beta_1 . w''}{\kappa_0} - \frac{T_0\hat{\alpha}_2\beta_2 w''}{\kappa_0}, \tag{7.6.3}$$

$$\left(\frac{d^2}{dz^2} - k^2 - \frac{n}{\eta_0}\right)\varphi'' = -\frac{\beta_2}{\eta_0}w'', \tag{7.6.4}$$

$$\left(\frac{d^2}{dz^2} - k^2 - \frac{n}{\gamma_0}\right)h_z'' = -\frac{H}{\gamma_0}\frac{dw''}{dz}, \tag{7.6.5}$$

$$\left(\frac{d^2}{dz^2} - k^2 - \frac{n}{\nu_0}\right)\zeta'' = -\frac{\mu_e H}{4\pi\rho_0\nu_0}\frac{d\xi''}{dz} - 2\Omega\frac{dw''}{dz}, \tag{7.6.6}$$

$$\left(\frac{d^2}{dz^2} - k^2 - \frac{n}{\gamma_0}\right)\xi'' = -\frac{H}{\gamma_0}\frac{d\zeta''}{dz}, \tag{7.6.7}$$

We shall now introduce the non-dimensional quantities defined by

$$
\left.
\begin{array}{llll}
z_* = z/d , & \tau_* = \eta_0/\kappa_0 , \\
a_* = kd , & \sigma_* = \nu_0/\kappa_0 , \\
D_* = d.\dfrac{d}{dz} , & p_* = \dfrac{nd^2}{\kappa_0} , \\
\sigma_{1*} = \nu_0/\gamma_0 , &
\end{array}
\right) \qquad (7.6.8)
$$

using the above non-dimensional quantities and omitting the asterisks and the double dashes for simplicity, we can reduce (7.6.2), (7.6.3), (7.6.4), (7.6.5), (7.6.6) and (7.6.7) to the following partially non-dimensional forms :

$$
(D^2-a^2)(D^2-a^2-p/\sigma) = \frac{g\alpha a^2 d^2 \theta}{\nu_0} - \frac{g\hat{\alpha} a^2 d^2 \varphi}{\nu_0} - \frac{\mu_e H d}{4\pi\rho_0 \nu_0}(D^2-a^2)Dh_z + \frac{2\Omega d^3}{\nu_0} D\zeta , \qquad (7.6.9)
$$

$$
[D^2-a^2-p(1-T_0\alpha_2)]\theta - T_0\hat{\alpha}_2 p\varphi = -(1-T_0\alpha_2)\frac{\beta_1 d^2 w}{\kappa_0} - \frac{T_0\hat{\alpha}_2\beta_2 d^2 w}{\kappa_0} , \qquad (7.6.10)
$$

$$
(D^2-a^2-p/\tau)\varphi = -\frac{\beta_2 d^2}{\eta_0}.w , \qquad (7.6.11)
$$

$$
(D^2-a^2-p\sigma_1/\sigma)h_z = -\frac{Hd}{\gamma_0} Dw , \qquad (7.6.12)
$$

$$
(D^2-a^2-p/\sigma)\zeta = -\frac{\mu_e H d \, D\xi}{4\pi\rho_0 \nu_0} - \frac{2\Omega d \, Dw}{\nu_0} , \qquad (7.6.13)
$$

$$
(D^2-a^2-p\sigma_1/\sigma)\xi = -\frac{Hd}{\gamma_0} D\zeta . \qquad (7.6.14)
$$

$$
\left.
\begin{aligned}
R_{1*} &= \frac{g\alpha\beta_1 d^4}{\kappa_0 \nu_0}, \quad Q_* = \frac{\mu_e H^2 d^2}{4\pi \rho_0 \nu_0 \gamma_0}, \quad T_0 = \frac{4\Omega^2 d^4}{\nu_0^2}, \\
R_{2*} &= \frac{g\hat{\alpha}\beta_2 d^4}{\kappa_0 \nu_0}, \quad w_* = \frac{\beta_1 d^2 w}{\kappa_0}, \quad h_{z*} = \frac{\gamma_0}{Hd} \cdot \frac{\beta_1 d^2}{\kappa_0} h_z, \\
R_{3*} &= \beta_2/\beta_1, \quad \theta_* = \theta, \quad \zeta_* = \frac{\beta_1 d \nu_0 \zeta}{2\Omega \kappa_0}, \\
& \qquad \varphi_* = \varphi, \quad \xi_* = \frac{\gamma_0 \nu_0 \beta_1 \xi}{2H\Omega \kappa_0}.
\end{aligned}
\right)
\tag{7.6.15}
$$

and omit the asterisks for simplicity, we can further reduce (7.6.9), (7.6.10), (7.6.11), (7.6.12), (7.6.13) and (7.6.14) to the following non-dimensional forms

$$(D^2-a^2)(D^2-a^2-p/\sigma)w = R_1 a^2\theta - \frac{R_2 a^2}{R_3}\varphi - QD(D^2-a^2)h_z + TD\zeta , \tag{7.6.16}$$

$$[D^2-a^2-p(1-T_0\alpha_2)]\theta - T_0\hat{\alpha}_2 p\varphi = -(1-T_0\alpha_2)w - T_0\hat{\alpha}_2 R_3 w, \tag{7.6.17}$$

$$(D^2-a^2-p/\tau)\varphi = -\frac{R_3}{\tau} w, \tag{7.6.18}$$

$$(D^2-a^2-p\sigma_1/\sigma)h_z = -Dw , \tag{7.6.19}$$

$$(D^2-a^2-p/\sigma)\zeta = -QD\xi - Dw , \tag{7.6.20}$$

$$(D^2-a^2-p\sigma_1/\sigma)\xi = -D\zeta . \tag{7.6.21}$$

In the subsequent analysis we shall proceed with either (7.6.9)-(7.6.14) or (7.6.16) - (7.6.21) which are equivalent forms.

The boundary conditions (7.6.4),(7.5.5) and (7.5.6) for the cases when (i) both boundaries are rigid and perfectly conducting (ii) both boundaries are dynamically free and perfectly conducting (iii) one boundary is rigid and the other dynamically free while both are perfectly conducting, in the above framework respectively reduce to

$$\left.\begin{array}{l} w = 0, \\ \theta = 0, \\ \varphi = 0, \\ Dw = 0, \\ \zeta = 0, \\ h_z = 0, \\ \text{and} \quad D\xi = 0. \end{array}\right\} \text{ at } z = 0 \text{ and } z = 1, \qquad (7.6.22)$$

$$\text{or} \quad \left.\begin{array}{l} w = 0, \\ \theta = 0, \\ \varphi = 0, \\ D^2 w = 0, \\ D\zeta = 0, \\ h_z = 0, \\ \text{and} \quad D\xi = 0. \end{array}\right\} \text{ at } z = 0 \text{ and } z = 1, \qquad (7.6.23)$$

$$\text{or} \quad \left.\begin{array}{l} w = 0, \\ \theta = 0, \\ \varphi = 0, \\ Dw = 0, \end{array}\right\} \text{ at } z = 0 \text{ (without loss of generality)}, \qquad (7.6.24)$$

$$
\left.\begin{array}{ll}
 & \zeta = 0, \\
 & h_z = 0, \\
\text{and} & D\xi = 0 .
\end{array}\right)
$$

$$
\text{and} \quad \left.\begin{array}{l}
w = 0 , \\
\theta = 0 , \\
\varphi = 0 , \\
D^2 w = 0 , \\
D\zeta = 0 , \\
h_z = 0 , \\
D\xi = 0 .
\end{array}\right) \quad \text{at } z = 1. \qquad (7.6.25)
$$

The governing perturbation equations (7.6.9)-(7.6.14) or (7.6.16)-(7.6.21) must be considered with proper boundary conditions on the flow variables. These are given by (7.6.22) or (7.6.23) or (7.6.24) and (7.6.25) according to the case under consideration. This poses a characteristic value problem for p for prescribed values of other parameters and a given normal mode is stable, marginal or unstable provided that the real part $p_r$ or p is negative, zero or positive respectively. Further if $p_r = 0$ implies $p_i$(the imaginary part of p) = 0 for every wave number 'a', then the 'principle of exchange of stabilities' in valid, otherwise, we shall have overstability at least when instability sets in as certain modes.

7.7. Mathematical analysis

We prove the following theorems :

Theorem 1 : For $\tau = 0$, a necessary condition for the existence of nontrivial solution for $w$, $\Theta$, $\varphi$, $h_z$, $\zeta$ and $\xi$ satisfying (7.6.16)-(7.6.21) and (7.6.22) or (7.6.23) is that $p \neq 0$.

Proof : For $\tau = 0$, let $p = 0$ be allowed if possible.

From (7.6.18), we conclude that

$$w \equiv 0 . \tag{7.7.1}$$

Equation (7.6.19) then becomes

$$(D^2-a^2)h_z = 0 . \tag{7.7.2}$$

The only solution of (7.7.2) which satisfies the relevant boundary conditions is

$$h_z \equiv 0. \tag{7.7.3}$$

Equation (7.6.17) reduces to

$$(D^2-a^2)\,\Theta = 0 . \tag{7.7.4}$$

The only solution of (7.7.4) satisfying the relevant boundary conditions is

$$\Theta \equiv 0. \tag{7.7.5}$$

Equation (7.6.16) reduces to

$$0 = -\frac{R_2}{R_3}\, a^2\, \varphi + T\, D\zeta . \tag{7.7.6}$$

Equations (7.6.20) and (7.6.21) respectively reduce to

$$(D^2-a^2)\zeta = - Q\, D\xi , \tag{7.7.7}$$

$$(D^2-a^2)\xi = -D\zeta \quad . \tag{7.7.8}$$

<u>When both the boundaries are rigid and perfectly conducting</u>, we have from (7.6.22)

$$\left.\begin{aligned} D\xi &= 0 , \\ \zeta &= 0 . \end{aligned}\right\} \text{ at } z = 0 \text{ and } z = 1. \tag{7.7.9}$$

Making use of (7.7.9) in (7.7.7) we get

$$D^2\zeta = 0 \text{ at } z = 0 \text{ and } z = 1. \tag{7.7.10}$$

Eliminating $\xi$ between (7.7.7) and (7.7.8) and multiplying the resultant equation by $\zeta^*$ (the complex conjugate of $\zeta$ ) on both sides and integrating over the vertical range of z, we obtain

$$\int_0^1 \zeta^*(D^2-a^2)^2\zeta \, dz = Q\int_0^1 \zeta^* D^2\zeta \, dz . \tag{7.7.11}$$

$$\int_0^1 \zeta^* D^4\zeta \, .dz - 2a^2 \int_0^1 \zeta^* D^2\zeta \, dz + a^4 \int_0^1 \zeta^*.\zeta \, dz = Q \int_0^1 \zeta^*.D^2\zeta \, dz. \tag{7.7.12}$$

Making use of (7.7.9) and (7.7.10), equation (7.7.12) can be written in the form

$$\int_0^1 |D^2\zeta|^2 dz + 2a^2 \int_0^1 |D\zeta|^2 dz + a^4\int_0^1 |\zeta|^2 dz + Q \int_0^1 |D\zeta|^2 dz = 0. \tag{7.7.13}$$

The only solution of (7,7.13) is

$$\zeta \equiv 0. \tag{7.7.14}$$

Equation (7.7.8) together with (7.7.9) and (7.7.14) gives

$$\xi \equiv 0. \tag{7.7.15}$$

Also making use of (7.7.14) in (7.7.6), we obtain

$$\varphi \equiv 0 . \qquad (7.7.16)$$

When both the boundaries are dynamically free and perfectly conducting, we have from (7.6.23)

$$\left.\begin{aligned} D\xi &= 0 , \\ D\zeta &= 0 . \end{aligned}\right\} \text{ at } z = 0 \text{ and } z = 1. \qquad (7.7.17)$$

Making use of (7.7.17) in (7.7.7), we get

$$(D^2-a^2)\zeta = 0 \text{ at } z = 0 \text{ and } z = 1. \qquad (7.7.18)$$

Let $(D^2-a^2)\zeta = L$, therefore (7.7.18) gives

$$L = 0 \text{ at } z = 0 \text{ and } z = 1. \qquad (7.7.19)$$

Eliminating $\xi$ between (7.7.7) and (7.7.8) and multiplying the resultant equation by $L^*$ (complex conjugate of L) and integrating over the vertical range of z, we obtain

$$\int_0^1 L^*(D^2-a^2)L\, dz = Q\int_0^1 L^*.\, D^2\zeta\, dz. \qquad (7.7.20)$$

$$\int_0^1 L^*\, D^2L\, dz - a^2\int_0^1 L^*.Ldz = Q[\int_0^1 D^2\zeta^*.D^2\zeta\, dz - a^2\int_0^1 \zeta^*.D^2\zeta\, dz.] \qquad (7.7.21)$$

Making use of (7.7.17) and (7.7.19), equation (7.7.21) can be written in the form

$$-[\int_0^1 (|DL|^2 + a^2|L|^2)dz] = Q[\int_0^1 (|D^2\zeta|^2 + a^2|D\zeta|^2)dz] \qquad (7.7.22)$$

or

$$\int_0^1 (|DL|^2+a^2|L|^2)dz + Q[\int_0^1(|D^2\zeta|^2 + a^2|D\zeta|^2)dz] = 0 \qquad (7.7.23)$$

The only solution of (7.7.23) is

$$\zeta \equiv 0. \tag{7.7.24}$$

Equation (7.7.8) together with (7.7.17) and (7.7.24) gives

$$\xi \equiv 0. \tag{7.7.25}$$

Also making use of (7.7.24) in (7.7.6), we obtain

$$\varphi \equiv 0. \tag{7.7.26}$$

The above shows that p cannot be equal to zero and this proves the theorem.

The essential content of <u>Theorem 1</u>, from the point of view of hydrodynamics is that the 'principle of exchange of stabilities' is not valid for the generalized Bénard problem under the effect of uniform rotation and magnetic field when considered in this extended framework and thus establishes the result due to Banerjee et al (1976) on a more firm basis.

**Theorem 2** : An exact solution of (7.6.9)-(7.6.14) with (7.6.23) is given by

$$w = A \sin n\pi z ,$$

$$\Theta = \frac{[(1-T_0\alpha_2)\beta_1+T_0\hat{\alpha}_2\beta_2]d^2.A \sin n\pi z}{[n^2\pi^2+a^2+p(1-T_0\alpha_2)]\kappa_0} - \frac{T_0\hat{\alpha}_2\beta_2 d^2 p\ A \sin n\pi z}{\eta_0(n^2\pi^2+a^2+p/\tau)[n^2\pi^2+a^2+p(1-T_0\alpha_2)]},$$

$$\varphi = \frac{\beta_2 d^2}{\eta_0} \frac{A \sin n\pi z}{n^2\pi^2+a^2+p/\tau},$$

$$h_z = \frac{Hd}{\gamma_c} \frac{n\pi A \cos n\pi z}{n^2\pi^2+a^2+p\sigma_1/\sigma},$$

$$\zeta = \frac{2\Omega d}{\nu_c} \frac{(n^2\pi^2+a^2+p\sigma_1/\sigma)\, n\pi\, A \cos n\pi z}{(n^2\pi^2+a^2+p/\sigma)(n^2\pi^2+a^2+p\sigma_1/\sigma)+Qn^2\pi^2},$$

$$\xi = -\frac{2H\Omega d^2}{\gamma_c \nu_c} \frac{n^2\pi^2 A \sin n\pi z}{(n^2\pi^2+a^2+p/\sigma)(n^2\pi^2+a^2+p\sigma_1/\sigma) + Qn^2\pi^2}. \tag{7.7.27}$$

and

$$[(n^2\pi^2+a^2+p/\sigma)(n^2\pi^2+a^2+p\sigma_1/\sigma) + Qn^2\pi^2]^2 (n^2\pi^2+a^2)$$

$$[n^2\pi^2+a^2+p(1-T_0\alpha_2)](n^2\pi^2+a^2+p/\tau)+ Tn^2\pi^2(n^2\pi^2+a^2+p\sigma_1/\sigma)^2$$

$$[n^2\pi^2+a^2+p(1-T_0\alpha_2)](n^2\pi^2+a^2+p/\tau) =[(n^2\pi^2+a^2+p/\sigma)(n^2\pi^2+a^2+p\sigma_1/\sigma$$

$$+ Qn^2\pi^2](n^2\pi^2+a^2+p\sigma_1/\sigma)[R_1 a^2\Big\{ - \frac{Bp}{\tau} +(n^2\pi^2+a^2+p/\tau)$$

$$[B+(1-T_0\alpha_2)]\Big\} - R_2 \frac{a^2}{\tau}\Big\{ n^2\pi^2+a^2+p(1-T_0\alpha_2)\Big\} ]. \tag{7.7.28}$$

<u>Proof</u> : As before we can derive from (7.6.9),(7.6.10), (7.6.11),(7.6.12), (7.6.13) and (7.6.14) that

$$D^{2m}w = 0 \text{ at } z = 0 \text{ and } z = 1 \text{ and } m =1,2.... \tag{7.7.29}$$

From this it follows that the required solution for w must be

$$w = A \sin n\pi z \ (n = 1,2,......). \tag{7.7.30}$$

Operating on (7.6.9) by $(D^2-a^2-p\sigma_1/\sigma)$ and using (7.6.12), we obtain

$$(D^2-a^2-p\sigma_1/\sigma)(D^2-a^2)(D^2-a^2-p/\sigma)w - Q(D^2-a^2)D^2w -$$

$$\frac{2\Omega d^3}{\nu_0}(D^2-a^2-p\sigma_1/\sigma)D\zeta = (D^2-a^2-p\sigma_1/\sigma)\left(\frac{g\alpha a^2d^2\theta}{\nu_0} - \frac{g\hat{\alpha}a^2d^2\varphi}{\nu_0}\right). \tag{7.7.31}$$

Operating on (7.6.10) by $(D^2-a^2-p/\tau)$ and using (7.6.11), we get

$$(D^2-a^2-p/\tau)[D^2-a^2-p(1-T_0\alpha_2)]\theta + \frac{T_0\hat{\alpha}_2\beta_2d^2pw}{\eta_0} =$$

$$(D^2-a^2-p/\tau)\left[-\frac{(1-T_0\alpha_2)\beta_1d^2w}{K_0} - \frac{T_0\hat{\alpha}_2\beta_2d^2w}{K_0}\right]. \tag{7.7.32}$$

Eliminating $\xi$ between (7.6.13) and (7.6.14), we obtain

$$(D^2-a^2-p/\sigma)(D^2-a^2-p\sigma_1/\sigma)\zeta = -\frac{2\Omega d}{\nu_0}D(D^2-a^2-p\sigma_1/\sigma)w + QD^2\zeta. \tag{7.7.33}$$

Equation (7.7.33) can be rewritten as

$$[(D^2-a^2-p/\sigma)(D^2-a^2-p\sigma_1/\sigma) - QD^2]\zeta = -\frac{2\Omega d}{\nu_0}D(D^2-a^2-p\sigma_1/\sigma)w. \tag{7.7.34}$$

Now operating upon (7.7.31) by

$$(D^2-a^2-p/\tau)[D^2-a^2-p(1-T_0\alpha_2)][(D^2-a^2-p/\sigma)(D^2-a^2-p\sigma_1/\sigma)-QD^2]$$

and simplifying the resulting equation, we obtain

$$[(D^2-a^2-p/\sigma)(D^2-a^2-p\sigma_1/\sigma)-QD^2]^2(D^2-a^2)(D^2-a^2-p/\tau)$$

$$[D^2-a^2-p(1-T_o\alpha_2)]w + TD^2(D^2-a^2-p\sigma_1/\sigma)^2(D^2-a^2-p/\tau)$$

$$[D^2-a^2-p(1-T_o\alpha_2)]w = \Big\langle [(D^2-a^2-p/\sigma)(D^2-a^2-p\sigma_1/\sigma)-QD^2]\times$$

$$(D^2-a^2-p\sigma_1/\sigma)[R_1a^2\Big\{-\frac{Bp}{\tau}-(D^2-a^2-p/\tau)(B+1-T_o\alpha_2)\Big\}+$$

$$R_2\frac{a^2}{\tau}\Big\{D^2-a^2-p(1-T_o\alpha_2)\Big\}]\Big\rangle w. \qquad (7.7.35)$$

Substituting for w in (7.7.35) from (7.7.30), we obtain

$$[(n^2\pi^2+a^2+p/\sigma)(n^2\pi^2+a^2+p\sigma_1/\sigma)+Qn^2\pi^2]^2(n^2\pi^2+a^2)(n^2\pi^2+a^2+p/\tau)$$

$$[n^2\pi^2+a^2+p(1-T_o\alpha_2)]+Tn^2\pi^2(n^2\pi^2+a^2+p\sigma_1/\sigma)^2(n^2\pi^2+a^2+p/\tau)$$

$$[n^2\pi^2+a^2+p(1-T_o\alpha_2)] = [(n^2\pi^2+a^2+p/\sigma)(n^2\pi^2+a^2+p\sigma_1/\sigma)+Qn^2\pi^2]$$

$$(n^2\pi^2+a^2+p\sigma_1/\sigma)[R_1a^2\Big\{-\frac{Bp}{\tau}+(n^2\pi^2+a^2+p/\tau)(B+1-T_o\alpha_2)\Big\}-$$

$$\frac{R_2a^2}{\tau}\Big\{n^2\pi^2+a^2+p(1-T_o\alpha_2)\Big\}]. \qquad (7.7.36)$$

Substituting for w from (7.7.30) in (7.7.32), (7.6.11), (7.6.12) and (7.7.34) respectively, we obtain after simplificiation the following equations

$$\Theta = \frac{[(1-T_o\alpha_2)\beta_1+T_o\hat{\alpha}_2\beta_2]d^2 A \sin n\pi z}{K_o[n^2\pi^2+a^2+p(1-T_o\alpha_2)]} -$$

$$\frac{T_o\hat{\alpha}_2\beta_2d^2p A \sin n\pi z}{\eta_o(n^2\pi^2+a^2+p/\tau)[n^2\pi^2+a^2+p(1-T_o\alpha_2)]}, \qquad (7.7.37)$$

$$\varphi = \frac{\beta_2 d^2}{\eta_0} \frac{A \sin n\pi z}{n^2\pi^2+a^2+p/\tau}, \tag{7.7.38}$$

$$h_z = \frac{Hd}{\chi_0} \frac{n\pi A \cos n\pi z}{n^2\pi^2+a^2+p\sigma_1/\sigma}, \tag{7.7.39}$$

$$\zeta = \frac{2\Omega d}{\nu_0} \frac{(n^2\pi^2+a^2+p\sigma_1/\sigma)n\pi A \cos n\pi z}{(n^2\pi^2+a^2+p/\sigma)(n^2\pi^2+a^2+p\sigma_1/\sigma)+Qn^2\pi^2}. \tag{7.7.40}$$

Using (7.7.40) in (7.6.14), we get

$$\xi = -\frac{2H\Omega d^2}{\chi_0 \nu_0} \frac{n^2\pi^2 A \sin n\pi z}{(n^2\pi^2+a^2+p/\sigma)(n^2\pi^2+a^2p\sigma_1/\sigma)+Qn^2\pi^2}. \tag{7.7.41}$$

This proves the theorem.

The essential content of Theorem 2 from the hydrodynamic instability point of view is this that it provides us with the exact solutions for the vertical component of perturbation velocity, perturbation temperature, perturbation concentration, vertical component of perturbation magnetic field, vertical component of perturbation vorticity and vertical component of current density and complex frequency for the case when both the bounding surfaces are dynamically free and perfectly conducting.

Further, for $p_r=0$ and $p_i \neq 0$, equation (7.7.36) becomes

$$[(n^2\pi^2+a^2+ip_i/\sigma)(n^2\pi^2+a^2+ip_i\sigma_1/\sigma)+Qn^2\pi^2]^2(n^2\pi^2+a^2)$$
$$(n^2\pi^2+a^2+ip_i/\tau)[n^2\pi^2+a^2+ip_i(1-T_0\alpha_2)]+ Tn^2\pi^2(n^2\pi^2+a^2+ip_i\sigma_1/\sigma)^2$$

$$(n^2\pi^2+a^2+ip_i/\tau)[n^2\pi^2+a^2+ip_i(1-T_o\alpha_2)] = [(n^2\pi^2+a^2+ip_i/\sigma)$$

$$(n^2\pi^2+a^2+ip_i\sigma_1/\sigma)+Qn^2\pi^2](n^2\pi^2+a^2+ip_i\sigma_1/\sigma)[R_1a^2\left\{-\frac{Bip_i}{\tau}+\right.$$

$$\left.(n^2\pi^2+a^2+ip_i/\tau)(B+1-T_o\alpha_2)\right\} - \frac{R_2a^2}{\tau}\left\{n^2\pi^2+a^2ip_i(1-T_o\alpha_2)\right\}]. \qquad (7.7.42)$$

This equation after simplification gives two equations when real and imaginary parts are equated on both sides. Therefore, we get

$$[\tau(n^2\pi^2+a^2)^2-p_i^2(1-T_o\alpha_2)][\left\{(n^2\pi^2+a^2)^2+Qn^2\pi^2\right\}^2(n^2\pi^2+a^2) +$$

$$\frac{p_i^4\sigma_1^2}{\sigma^4}(n^2\pi^2+a^2) - 2\left\{(n^2\pi^2+a^2)^2+Qn^2\pi^2\right\}\frac{p_i^2\sigma_1}{\sigma^2}(n^2\pi^2+a^2) -$$

$$\frac{(n^2\pi^2+a^2)^3}{\sigma^2} p_i^2(1+\sigma_1)^2 + Tn^2\pi^2\left\{(n^2\pi^2+a^2)^2 - \frac{p_i^2\sigma_1^2}{\sigma^2}\right\}] -$$

$$p_i(n^2\pi^2+a^2)\left\{1+\tau(1-T_o\alpha_2)\right\}[2Tn^2\pi^2\frac{p_i\sigma_1}{\sigma}(n^2\pi^2+a^2)+$$

$$\frac{2(n^2\pi^2+a^2)^2p_i(1+\sigma_1)}{\sigma}\left\{(n^2\pi^2+a^2)^2+Qn^2\pi^2-p_i^2\frac{\sigma_1}{\sigma^2}\right\}] =$$

$$[(n^2\pi^2+a^2)\left\{(n^2\pi^2+a^2)^2+Qn^2\pi^2\right\} - \frac{p_i^2\sigma_1}{\sigma^2}(n^2\pi^2+a^2) -$$

$$\frac{p_i^2\sigma_1}{\sigma^2}(1+\sigma_1)(n^2\pi^2+a^2)][R_1a^2\tau(n^2\pi^2+a^2)\left\{B+1-T_o\alpha_2\right\} - R_2a^2(n^2\pi^2+a^2)]-$$

$$p_i(1-T_o\alpha_2)(R_1-R_2)a^2[\frac{p_i\sigma_1}{\sigma}\left\{(n^2\pi^2+a^2)^2+Qn^2\pi^2\right\} - \frac{p_i^3\sigma_1^2}{\sigma^3}+$$

$$\frac{(n^2\pi^2+a^2)^2}{\sigma} p_i(1+\sigma_1) \;\Big],\tag{7.7.43}$$

and

$$\Big\{1+\tau(1-T_o\alpha_2)\Big\}\Big[\Big\{(n^2\pi^2+a^2)^2+Qn^2\pi^2\Big\}^2 (n^2\pi^2+a^2) + \frac{p_i^4\sigma_1^2}{\sigma^4}(n^2\pi^2+a^2) -$$

$$2\Big\{(n^2\pi^2+a^2)^2+Qn^2\pi^2\Big\}\frac{p_i^2\sigma_1}{\sigma^2}(n^2\pi^2+a^2) - \frac{(n^2\pi^2+a^2)^3}{\sigma^2}p_i^2 \;(1+\sigma_1)^2 +$$

$$Tn^2\pi^2(n^2\pi^2+a^2)^2 - Tn^2\pi^2 p_i^2 \frac{\sigma_1^2}{\sigma^2}\Big] + \Big\{\tau(n^2\pi^2+a^2)^2 - p_i^2(1-T_o\alpha_2)\Big\}$$

$$\Big[\frac{2(n^2\pi^2+a^2)}{\sigma}(1+\sigma_1)\Big\{(n^2\pi^2+a^2)^2+Qn^2\pi^2-p_i^2\frac{\sigma_1}{\sigma^2}\Big\} + 2Tn^2\pi^2\frac{\sigma_1}{\sigma}\Big] =$$

$$\Big[(n^2\pi^2+a^2)^2 + Qn^2\pi^2 - \frac{p_i^2\sigma_1}{\sigma^2} - \frac{p_i^2\sigma_1}{\sigma^2}(1+\sigma_1)\Big](1-T_o\alpha_2)(R_1-R_2)a^2 +$$

$$\Big[\frac{\sigma_1}{\sigma}\Big\{(n^2\pi^2+a^2)^2+Qn^2\pi^2\Big\} - \frac{p_i^2\sigma_1^2}{\sigma^3} + \frac{(n^2\pi^2+a^2)}{\sigma}(1+\sigma_1)\Big]\times$$

$$\Big[R_1 a^2\tau\Big\{B+(1-T_o\alpha_2)\Big\} - R_2a^2\Big].\tag{7.7.44}$$

Equations (7.7.43) and (7.7.44) determine the critical Rayleigh number and the frequency of oscillations of the overstable motions, at the marginal state, when both the boundary surfaces are dynamically free.

7.8. Contributions of Chapter - 7

(i) Shows, for the first time to the best our knowledge, that a more consistent and relatively more accurate application of Boussinesq approximation leads to a formulation of thermal and thermohaline instability problems under the joint effect of rotation and magnetic field which does significantly depend upon whether the fluid layer is hotter or cooler and this is on account of the variations in the specific heat at constant volume due to variations in temperature and/or concentration.

(ii) Establishes, on a more sound basis, the results of Banerjee et al (1976) regarding the character of the marginal state in the generalised Bénard problem under the joint effect of rotation and magnetic field.

(iii) Exactly solves the problem of generalized Bénard convection and thermohaline convection under the joint effect of rotation and magnetic field with dynamically free and perfectly conducting boundaries for oscillatory modes and distinguishes between a hotter and cooler layer as regards the onset of instability.