CHAPTER - 6

MODIFIED ANALYSIS OF BÉNARD CONVECTION, GENERALIZED BÉNARD CONVECTION AND THERMOHALINE CONVECTION: THE EFFECT OF MAGNETIC FIELD

6.1. The problem and the initial state

A viscous finitely heat conducting Boussinesq fluid of infinite horizontal extension is statically confined between two horizontal boundaries z=0 and z=d which are respectively maintained at uniform temperatures T_0 and T_1 ($< T_0$) and uniform concentrations S_0 and S_1 ($< S_0$) in the presence of a uniform magnetic field acting parallel and opposite to the direction of gravity. We wish to mathematically investigate the onset of linear hydrodynamic instability in the system. The initial state is given by

$$\left.\begin{aligned} (u,v,w) &\equiv (0,0,0), \\ T &\equiv T(z), \\ S &\equiv S(z), \\ \rho &\equiv \rho(z), \\ \text{and}\quad (H_1,H_2,H_3) &\equiv (0,0,H). \end{aligned}\right\} \qquad (6.1.1)$$

6.2. The basic equations and the initial state solution

The governing equations as mentioned earlier are given by (3.2.46) - (3.2.52). These equations on the basis of (6.1.1) yield the initial state solution

$$
\left.\begin{aligned}
(u,v,w) &= (0,0,0)\ ,\\
T &= T_0-\beta_1 z,\\
S &= S_0-\beta_2 z\ ,\\
\rho &= \rho_0[1+\alpha(T_0-T)-\hat{\alpha}(S_0-S)]\\
&= [1+\alpha\beta_1 z - \hat{\alpha}\beta_2 z],\\
(H_1,H_2,H_3) &= (0,0,H)\\
\text{and}\qquad & \\
p_2 &= p + \frac{\mu_e|\underline{H}|^2}{8\pi}\\
&= p_{20} - g\rho_0[z + (\alpha\beta_1-\hat{\alpha}\beta_2)\ \frac{z^2}{2}\ ]
\end{aligned}\right\} \qquad (6.2.1)
$$

6.3. The perturbation equations

Let the initial state described by (6.2.1) be slightly perturbed so that the perturbed state is given by

$$
\left.\begin{aligned}
(\bar{u},\bar{v},\bar{w}) &= (0+u',\ 0+v',\ 0+w')\ ,\\
\bar{T} &= T_0-\beta_1 z + \theta'\ ,\\
\bar{S} &= S_0-\beta_2 z + \varphi'\ ,\\
\bar{\rho} &= [1+\alpha(T_0-T-\theta')-\hat{\alpha}(S_0-S-\varphi')],\\
(\bar{H}_1,\bar{H}_2,\bar{H}_3) &= (0+h_x',\ 0+h_y',\ H+h_z')\ ,\\
\text{and}\qquad & \\
\bar{p}_2 &= p_{20}-g\rho_0[z+(\alpha\beta_1-\hat{\alpha}\beta_2)\ \frac{z^2}{2}]+\delta p'.
\end{aligned}\right\} \quad (6.3.1)
$$

Then the linearized equations of perturbation of continuity, momentum, heat conduction, mass diffusion, magnetic induction and magnetic field continuity are given by

$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} = 0, \tag{6.3.2}$$

$$\rho_0 \frac{\partial u'}{\partial t} = -\frac{\partial \delta p'}{\partial x} + \mu_0 \nabla^2 u' + \frac{\mu_e H}{4\pi}\left(\frac{\partial h'_x}{\partial z} - \frac{\partial h'_z}{\partial x}\right), \tag{6.3.3}$$

$$\rho_0 \frac{\partial v'}{\partial t} = -\frac{\partial \delta p'}{\partial y} + \mu_0 \nabla^2 v' + \frac{\mu_e H}{4\pi}\left(\frac{\partial h'_y}{\partial z} - \frac{\partial h'_z}{\partial y}\right), \tag{6.3.4}$$

$$\rho_0 \frac{\partial w'}{\partial t} = -\frac{\partial \delta p'}{\partial z} + \mu_0 \nabla^2 w' + g\alpha \rho_0 \theta' - g\hat{\alpha}\rho_0 \varphi' , \tag{6.3.5}$$

$$(1-T_0\alpha_2)\frac{\partial \theta'}{\partial t} + T_0\hat{\alpha}_2 \frac{\partial \varphi'}{\partial t} - (1-T_0\alpha_2)\beta_1 w' - T_0\hat{\alpha}_2\beta_2 w' = K_0 \nabla^2 \theta' \tag{6.3.6}$$

$$\frac{\partial \varphi'}{\partial t} - \beta_2 w' = \eta_0 \nabla^2 \varphi', \tag{6.3.7}$$

$$\frac{\partial h'_x}{\partial t} = H \frac{\partial u'}{\partial z} + \gamma_0 \nabla^2 h'_x , \tag{6.3.8}$$

$$\frac{\partial h'_y}{\partial t} = H \frac{\partial v'}{\partial z} + \gamma_0 \nabla^2 h'_y , \tag{6.3.9}$$

$$\frac{\partial h'_z}{\partial t} = H \frac{\partial w'}{\partial z} + \gamma_0 \nabla^2 h'_z , \tag{6.3.10}$$

and

$$\frac{\partial h'_x}{\partial x} + \frac{\partial h'_y}{\partial y} + \frac{\partial h'_z}{\partial z} = 0 . \tag{6.3.11}$$

Further, combining (6.3.3) and (6.3.4), and (6.3.8) and (6.3.9), we derive

$$\rho_0 \frac{\partial \zeta'}{\partial t} = \mu_0 \nabla^2 \zeta' + \frac{\mu_e H}{4\pi} \frac{\partial \xi'}{\partial z} , \tag{6.3.12}$$

and

$$\frac{\partial \xi'}{\partial t} = H \frac{\partial \zeta'}{\partial z} + \gamma_o \nabla^2 \xi' , \qquad (6.3.13)$$

where ξ' is the z-component of current density .

6.4. The perturbation equation governing the normal modes

Making use of the normal mode analysis as given in (4.4) , we obtain the following perturbation equations from (6.3.2)-(6.3.11) :

$$ik_x u'' + ik_y v'' + \frac{dw''}{dz} = 0, \qquad (6.4.1)$$

$$\rho_o n u'' = -ik_x \delta p'' + \mu_o \left(\frac{d^2}{dz^2} - k^2\right) u'' + \frac{\mu_e H}{4\pi}\left(\frac{\partial h_x''}{\partial z} - ik_x h_z''\right), \qquad (6.4.2)$$

$$\rho_o n v'' = -ik_y \delta p'' + \mu_o \left(\frac{d^2}{dz^2} - k^2\right) v'' + \frac{\mu_e H}{4\pi}\left(\frac{\partial h_y''}{\partial z} - ik_y h_z''\right), \qquad (6.4.3)$$

$$\rho_o n w'' = -\frac{d\delta p''}{dz} + \mu_o \left(\frac{d^2}{dz^2} - k^2\right) w'' + g\alpha\rho_o \theta'' - g\hat{\alpha}\rho_o \varphi'', \qquad (6.4.4)$$

$$(1-T_o\alpha_2)n\theta'' + T_o\hat{\alpha}_2 n\varphi'' - (1-T_o\alpha_2)\beta_1 w'' - T_o\hat{\alpha}_2\beta_2 w''$$

$$= \kappa_o \left(\frac{d^2}{dz^2} - k^2\right) \theta'', \qquad (6.4.5)$$

$$n\varphi'' - \beta_2 w'' = \eta_o \left(\frac{d^2}{dz^2} - k^2\right) \varphi'', \qquad (6.4.6)$$

$$n h_x'' = H \frac{du''}{dz} + \gamma_o \left(\frac{d^2}{dz^2} - k^2\right) h_x'' , \qquad (6.4.7)$$

$$nh_y'' = H\frac{dv''}{dz} + \gamma_0\left(\frac{d^2}{dz^2} - k^2\right)h_y'' , \qquad (6.4.8)$$

$$nh_z'' = H\frac{dw''}{dz} + \gamma_0\left(\frac{d^2}{dz^2} - k^2\right)h_z'' , \qquad (6.4.9)$$

and
$$ik_x h_x'' + ik_y h_y'' + \frac{dw''}{dz} = 0. \qquad (6.4.10)$$

Further, (6.3.12) and (6.3.13) respectively become

$$\rho_0 n\zeta'' = u_0\left(\frac{d^2}{dz^2} - k^2\right)\zeta'' + \frac{u_e H}{4\pi}\frac{d\xi''}{dz} , \qquad (6.4.11)$$

and

$$n\xi'' = H\frac{d\zeta''}{dz} + \gamma_0\left(\frac{d^2}{dz^2} - k^2\right)\xi'' . \qquad (6.4.12)$$

6.5. The boundary conditions

The boundary conditions on w', Θ', φ', ξ', ζ' and h_z' are given by

$$\left.\begin{aligned} w' &= 0 , \\ \Theta' &= 0 , \\ \varphi' &= 0 , \\ \frac{\partial w'}{\partial z} &= 0 , \\ \text{and} \quad \zeta' &= 0 . \end{aligned}\right\} \text{ on a rigid boundary ,} \qquad (6.5.1)$$

$$\left.\begin{aligned} w' &= 0 , \\ \Theta' &= 0 , \\ \varphi' &= 0 , \\ \frac{\partial^2 w'}{\partial z^2} &= 0 , \\ \text{and} \quad \frac{\partial \zeta'}{\partial z} &= 0 . \end{aligned}\right\} \text{ on a dynamically free boundary,} \qquad (6.5.2)$$

$$\left.\begin{aligned} \text{while} \quad h'_z &= 0, \\ \frac{\partial \xi'}{\partial z} &= 0. \end{aligned}\right\} \text{ on a perfectly conducting boundary.} \qquad (6.5.3)$$

The above boundary conditions, when analysed in terms of normal modes, respectively become

$$\left.\begin{aligned} w'' &= 0, \\ \theta'' &= 0, \\ \varphi'' &= 0, \\ \frac{dw''}{dz} &= 0, \\ \text{and} \quad \zeta'' &= 0. \end{aligned}\right\} \text{ on a rigid boundary,} \qquad (6.5.4)$$

$$\left.\begin{aligned} w'' &= 0, \\ \theta'' &= 0, \\ \varphi'' &= 0, \\ \frac{d^2w''}{dz^2} &= 0, \\ \text{and} \quad \frac{d\zeta''}{dz} &= 0. \end{aligned}\right\} \text{ on a dynamically free boundary,} \qquad (6.5.5)$$

$$\left.\begin{aligned} \text{while} \quad h''_z &= 0, \\ \frac{d\xi''}{dz} &= 0. \end{aligned}\right\} \text{ on a perfectly conducting boundary.} \qquad (6.5.6)$$

6.6. The characteristic value problem

Multiplying(6.4.2) by k_x and (6.4.3) by k_y, adding the resulting equations and making use of (6.4.1) and (6.4.10), we obtain

$$\rho_0 n \frac{dw''}{dz} = -k^2\delta p'' + \mu_0\left(\frac{d^2}{dz^2} -k^2\right) \frac{dw''}{dz} + \frac{\mu_e H}{4\pi}\left(\frac{d^2}{dz^2} -k^2\right)h_z''. \tag{6.6.1}$$

Eliminating $\delta p''$ between (6.6.1) and (6.4.4), it follows that

$$\left(\frac{d^2}{dz^2} -k^2\right)\left(\frac{d^2}{dz^2} -k^2- \frac{n}{\nu_0}\right) w'' = \frac{g\alpha k^2\theta''}{\nu_0} - \frac{g\hat{\alpha} k^2\varphi''}{\nu_0} - \frac{\mu_e H}{4\pi \rho_0\nu_0} \left(\frac{d^2}{dz^2} -k^2\right) \frac{dh_z''}{dz} . \tag{6.6.2}$$

Further, equations (6.4.5),(6.4.6),(6.4.9),(6.4.11) and (6.4.12) can be rewritten as

$$\left[\frac{d^2}{dz^2} -k^2- \frac{n(1-T_0\alpha_2)}{\kappa_0}\right]\theta'' - \frac{T_0\hat{\alpha}_2 n\varphi''}{\kappa_0} = - \frac{(1-T_0\alpha_2)\beta_1 w''}{\kappa_0} - \frac{T_0\hat{\alpha}_2\beta_2 w''}{\kappa_0}, \tag{6.6.3}$$

$$\left(\frac{d^2}{dz^2} - k^2- \frac{n}{\eta_0}\right)\varphi'' = - \frac{\beta_2}{\eta_0} w'' , \tag{6.6.4}$$

$$\left(\frac{d^2}{dz^2} - k^2- \frac{n}{\gamma_0}\right)h_z'' = - \frac{H}{\gamma_0} \frac{dw''}{dz} , \tag{6.6.5}$$

$$\left(\frac{d^2}{dz^2} -k^2- \frac{n}{\nu_0} \right)\zeta'' = - \frac{\mu_e H}{4\pi \rho_0\nu_0} \frac{d\xi''}{dz} , \tag{6.6.6}$$

$$\left(\frac{d^2}{dz^2} -k^2- \frac{n}{\gamma_0} \right)\xi'' = - \frac{H}{\gamma_0} \frac{d\zeta''}{dz} . \tag{6.6.7}$$

We shall now introduce the non-dimensional quantities defined by

$$z_* = z/d , \quad \tau_* = \eta_0/\kappa_0 ,$$

$$\left.\begin{aligned} a_* &= kd, \quad \sigma_* = \nu_o/\kappa_o, \\ D_* &= d.\frac{d}{dz}, \quad p_* = nd^2/\kappa_o, \\ \sigma_{1*} &= \nu_o/\gamma_o, \end{aligned}\right\} \tag{6.6.8}$$

using the above non-dimensional quantities and omitting the asterisks and the double dashes for simplicity, we can reduce (6.6.2) - (6.6.7) respectively to the following partially non-dimensional forms :

$$(D^2-a^2)(D^2-a^2-p/\sigma)w = \frac{g\alpha a^2 d^2 \theta}{\nu_o} - \frac{g\hat{\alpha} a^2 d^2 \varphi}{\nu_o} - \frac{\mu_e H d}{4\pi\rho_o\nu_o}(D^2-a^2)Dh_z, \tag{6.6.9}$$

$$[D^2-a^2-p(1-T_o\alpha_2)]\theta - T_o\hat{\alpha}_2 p\varphi = -(1-T_o\alpha_2)\frac{\beta_1 d^2 w}{\kappa_o} - \frac{T_o\hat{\alpha}_2\beta_2 d^2 w}{\kappa_o}, \tag{6.6.10}$$

$$(D^2-a^2-p/\tau)\varphi = -\frac{\beta_2 d^2}{\eta_o} w, \tag{6.6.11}$$

$$(D^2-a^2-\frac{p\sigma_1}{\sigma})h_z = -\frac{Hd}{\gamma_o} Dw, \tag{6.6.12}$$

$$(D^2-a^2-p/\sigma)\zeta = -\frac{\mu_e H d}{4\pi\rho_o\nu_o} D\xi, \tag{6.6.13}$$

$$(D^2-a^2-\frac{p\sigma_1}{\sigma})\xi = -\frac{Hd}{\gamma_o} D\zeta. \tag{6.6.14}$$

If we introduce more non-dimensional quantities defined by

$$\left.\begin{array}{lll} R_{1*} = \dfrac{g\alpha\beta_1 d^4}{\kappa_0 \nu_0}, & Q_* = \dfrac{\mu_e H^2 d^2}{4\pi \rho_0 \nu_0 \gamma_0}, & \varphi_* = \varphi , \\ & & \xi_* = \xi , \\ R_{2*} = \dfrac{g\alpha\beta_2 d^4}{\kappa_0 \nu_0}, & w_* = \dfrac{\beta_1 d^2}{\kappa_0} w , & \zeta_* = \dfrac{Hd\zeta}{\gamma_0} , \\ R_{3*} = \beta_2/\beta_1 , & \theta_* = \theta , & h_{z*} = \dfrac{\gamma_0}{Hd} \dfrac{\beta_1 d}{\kappa_0} h_z \end{array}\right\} \quad (6.6.15)$$

and omit the asterisks for simplicity, we can further reduce (6.6.9), (6.6.10), (6.6.11), (6.6.12),(6.6.13) and (6.6.14) to the following non-dimensional forms

$$(D^2-a^2)(D^2-a^2-p/\sigma)w = R_1 a^2\theta - \frac{R_2 a^2}{R_3}\varphi - QD(D^2-a^2)h_z , \quad (6.6.16)$$

$$[(D^2-a^2-p(1-T_0\alpha_2)] - T_0\hat{\alpha}_2 p\varphi = -(1-T_0\alpha_2)w - T_0\hat{\alpha}_2 R_3 w, \quad (6.6.17)$$

$$(D^2-a^2-p/\tau)\varphi = -\frac{R_3}{\tau} w, \quad (6.6.18)$$

$$(D^2-a^2-p\sigma_1/\sigma)h_z = - Dw , \quad (6.6.19)$$

$$(D^2-a^2-p/\sigma)\zeta = - Q D\xi , \quad (6.6.20)$$

$$(D^2-a^2-p\sigma_1/\sigma)\xi = - D\zeta . \quad (6.6.21)$$

In the subsequent analysis we shall proceed with either (6.6.9) - (6.6.14) or (6.6.16) - (6.6.21) which are equivalent forms.

The boundary conditions (6.5.4), (6.5.5) and (6.5.6) for the cases when (i) both boundaries are rigid and perfectly conducting (ii) both boundaries are free and perfectly conducting (iii) one boundary is rigid and the other free while both are perfectly conducting, in the above framework respectively reduce to

$$\left.\begin{aligned} w &= 0, \\ \theta &= 0, \\ \varphi &= 0, \\ Dw &= 0, \\ \zeta &= 0, \\ h_z &= 0, \\ \text{and} \quad D\xi &= 0. \end{aligned}\right\} \quad \text{at } z = 0 \text{ and } z = 1, \qquad (6.6.22)$$

or

$$\left.\begin{aligned} w &= 0, \\ \theta &= 0, \\ \varphi &= 0, \\ D^2w &= 0, \\ D\zeta &= 0, \\ h_z &= 0, \\ \text{and} \quad D\xi &= 0. \end{aligned}\right\} \quad \text{at } z = 0 \text{ and } z = 1, \qquad (6.6.23)$$

Or

$$\left.\begin{aligned} w &= 0, \\ \theta &= 0, \\ \varphi &= 0, \\ Dw &= 0, \\ \zeta &= 0, \\ h_z &= 0, \\ \text{and} \quad D\xi &= 0. \end{aligned}\right\} \quad \text{at } z = 0 \text{ (without loss of generality)}, \qquad (6.6.24)$$

$$\left.\begin{aligned} \text{and} \quad w &= 0\,, \\ \theta &= 0\,, \\ \varphi &= 0\,, \\ D^2 w &= 0\,, \\ D\zeta &= 0\,, \\ h_z &= 0\,, \\ \text{and} \quad D\xi &= 0\,. \end{aligned}\right\} \text{ at } z = 1. \tag{6.6.25}$$

The governing perturbation equations (6.6.9) - (6.6.14) or (6.6.16) - (6.6.21) must be considered with proper boundary conditions on the flow variables. These are given by (6.6.22) or (6.6.23) or (6.6.24) and (6.6.25) according to the case under consideration. This poses a characteristic value problem for p for prescribed values of other parameters and a given normal mode is stable, marginal or unstable provided that the real part p_r of p is negative, zero or positive respectively. Further if $p_r=0$ implies p_i (the imaginary part of p) = 0 for every wave number 'a', then the 'principle of exchange of stabilities' is valid, otherwise, we will have overstability at least when instability sets in as certain modes.

6.7. Mathematical analysis

We prove the following theorems :

Theorem 1 : For $\tau = 0$, a necessary condition for the existence of nontrivial solution for w, θ, φ, h_z, ζ and ξ

satisfying (6.6.16)-(6.6.21), and (6.6.22) or (6.6.23) is that $p \neq 0$.

Proof : For $\tau = 0$, let $p = 0$ be allowed if possible.

From (6.6.18), we conclude that

$$w \equiv 0 . \tag{6.7.1}$$

Equation (6.6.19) then becomes

$$(D^2-a^2)h_z = 0. \tag{6.7.2}$$

The only solution of (6.7.2) which satisfies the relevant boundary conditions is

$$h_z \equiv 0. \tag{6.7.3}$$

Further, equations (6.6.16) and (6.6.17) respectively reduce to

$$0 = R_1 a^2 \theta - \frac{R_2 a^2}{R_3} \varphi , \tag{6.7.4}$$

and $$(D^2-a^2)\theta = 0. \tag{6.7.5}$$

The only solution of (6.7.5) which satisfies the relevant boundary conditions is

$$\theta \equiv 0 . \tag{6.7.6}$$

Making use of (6.7.6), we conclude from (6.7.4) that

$$\varphi \equiv 0 . \tag{6.7.7}$$

Equations (6.6.20) and (6.6.21) respectively reduce to

$$(D^2-a^2)\zeta = - Q D\xi , \tag{6.7.8}$$

$$(D^2-a^2)\xi = - D\zeta \tag{6.7.9}$$

When both the boundaries are rigid and perfectly conducting we have from (6.6.22)

$$\left.\begin{aligned} D\xi &= 0 , \\ \zeta &= 0 . \end{aligned}\right\} \text{ at } z = 0 \text{ and } z = 1. \tag{6.7.10}$$

Making use of (6.7.10) in (6.7.8), we get

$$D^2\zeta = 0 \text{ at } z = 0 \text{ and } z = 1. \tag{6.7.11}$$

Eliminating ξ between (6.7.8) and (6.7.9) and multiplying the resultant equation by ζ^* (the complex conjugate of ζ) on both sides and integrating over the vertical range of z, we obtain

$$\int_0^1 \zeta^*(D^2-a^2)^2\zeta \, dz = Q\int_0^1\zeta^* \, . \, D^2\zeta \, dz. \tag{6.7.12}$$

$$\int_0^1 \zeta^* D^4\zeta \, dz - 2a^2\int_0^1\zeta^*.D^2\zeta \, dz + a^4\int_0^1\zeta^*\zeta \, dz = Q\int_0^1\zeta^*.D^2\zeta \, dz. \tag{6.7.13}$$

Making use of (6.7.10) and (6.7.11), equation (6.7.13) can be written in the form

$$\int_0^1 |D^2\zeta|^2 dz + 2a^2\int_0^1 |D\zeta|^2 dz + a^4\int_0^1 |\zeta|^2 dz + Q\int_0^1 |D\zeta|^2 dz = 0 . \tag{6.7.14}$$

The only solution of (6.7.14) is

$$\zeta \equiv 0. \tag{6.7.15}$$

Equation (6.7.9) together with (6.7.10) and (6.7.15) gives

$$\xi \equiv 0 . \tag{6.7.16}$$

When, both the boundaries are dynamically free and perfectly conducting, we have from (6.6.23)

$$D\xi = 0, \quad D\zeta = 0 \quad \text{at } z = 0 \text{ and } z = 1. \tag{6.7.17}$$

Making use of (6.7.17) in (6.7.8), we get

$$(D^2-a^2)\zeta = 0, \quad \text{at } z = 0 \text{ and } z = 1. \tag{6.7.18}$$

Let $(D^2-a^2)\zeta = L$, therefore $L = 0$ at $z = 0$ and $z = 1$.

(6.7.19)

Eliminating ξ between (6.7.8) and (6.7.9) and multiplying the resultant equation by L^* (complex conjugate of L) and integrating over the vertical range of z, we obtain

$$\int_0^1 L^* (D^2-a^2)L\, dz = Q \int_0^1 L^* D^2\zeta \, dz. \tag{6.7.20}$$

$$\int_0^1 L^* D^2L\, dz - a^2\int_0^1 L^*.L\, dz = Q[\int_0^1 D^2\zeta^* .D^2\zeta\, dz - a^2\int_0^1 \zeta^* .D^2\zeta\, dz. \tag{6.7.21}$$

Making use of (6.7.17) and (6.7.19), equation (6.7.21), can be written in the following form

$$-[\int_0^1 (|DL|^2 + a^2|L|^2)dz] = Q[\int_0^1 (|D^2\zeta|^2 + a^2|D\zeta|^2)dz]. \tag{6.7.22}$$

or

$$\int_0^1 (|DL|^2 + a^2|L|^2)dz + Q[\int_0^1 (|D^2\zeta|^2 + a^2|D\zeta|^2)dz] = 0. \tag{6.7.23}$$

The only solution of (6.7.23) is

$$\zeta \equiv 0. \tag{6.7.24}$$

Equation (6.7.9) together with (6.7.17) and (6.7.24) gives

$$\xi \equiv 0. \tag{6.7.25}$$

The above shows that p cannot be equal to zero and this proves the Theorem.

The essential content of Theorem 1, from the point of view of hydrodynamics is that the 'principle of exchange of stabilities' is not valid for the generalised Bénard problem under the effect of a uniform magnetic field acting parallel and opposite to the direction of gravity when considered in this extended framework and thus establishes the result due to Banerjee et al (1976) on a more firm basis.

Theorem 2 : An exact solution of (6.6.9)-(6.6.14) with (6.6.23) is given by

$$w = A \sin n\pi z,$$

$$\theta = \frac{[(1-T_o\alpha_2)\beta_1 + T_o\hat{\alpha}_2\beta_2]d^2 . A \sin n\pi z}{[n^2\pi^2+a^2+p(1-T_o\alpha_2)]\, K_c} - \frac{T_o\hat{\alpha}_2\beta_2 d^2 p\, A \sin n\pi z}{\eta_o(n^2\pi^2+a^2+p/\tau)[n^2\pi^2+a^2+p(1-T_o\alpha_2)]},$$

$$\varphi = \frac{\beta_2 d^2}{\eta_o} \frac{A \sin n\pi z}{n^2\pi^2+a^2+p/\tau},$$

$$h_z = \frac{Hd}{\chi_c} \frac{n\pi\, A \cos n\pi z}{n^2\pi^2+a^2+p\sigma_1/\sigma},$$

ζ and ξ are given by (6.6.13) and (6.6.14) and

$$(n^2\pi^2+a^2+p/\tau)[n^2\pi^2+a^2+p(1-T_o\alpha_2)](n^2\pi^2+a^2+p\sigma_1/\sigma)(n^2\pi^2+a^2)$$

$$(n^2\pi^2+a^2+p/\sigma)+ Q(n^2\pi^2+a^2+p/\tau)[n^2\pi^2+a^2+p(1-T_o\alpha_2)](n^2\pi^2+a^2)n^2\pi^2$$

$$= (n^2\pi^2+a^2+p\sigma_1/\sigma)[R_1a^2\{- \frac{Bp}{\tau} + (n^2\pi^2+a^2+p/\tau)(1-T_o\alpha_2) +$$

$$(n^2\pi^2+a^2p/\sigma)B\} - \frac{R_2a^2}{\tau}\{n^2\pi^2+a^2+p(1-T_o\alpha_2)\}]. \tag{6.7.26}$$

<u>Proof</u> : As before we can derive from (6.6.9), (6.6.10), (6.6.11), (6.6.12), (6.6.13) and (6.6.14) that

$$D^{2m}w = 0 \quad \text{at } z = 0 \text{ and } z = 1 \text{ and } m =1,2,... \tag{6.7.27}$$

From this it follows that the required solution for w must be $\quad w = A \sin n\pi z \qquad (n= 1,2,.....).$ (6.7.28)

Operating on (6.6.9) by $(D^2-a^2-p\sigma_1/\sigma)$ and using (6.6.12), we obtain

$$(D^2-a^2-p\sigma_1/\sigma)(D^2-a^2)(D^2-a^2-p/\sigma)w =$$

$$(D^2-a^2-p\sigma_1/\sigma)(\frac{g\alpha a^2d^2\Theta}{\nu_o} - \frac{g\hat{\alpha}a^2d^2\varphi}{\nu_o})+ Q(D^2-a^2)D^2w. \tag{6.7.29}$$

Operating on (6.6.10) by (D^2-a^2-p/τ) and using (6.6.11), we obtain

$$(D^2-a^2-p/\tau)[D^2-a^2-p(1-T_o\alpha_2)]\Theta + \frac{T_o\hat{\alpha}_2\beta_2d^2pw}{\eta_o} =$$

$$(D^2-a^2-p/\tau)[-\frac{(1-T_o\alpha_2)\beta_1 d^2 w}{K_o} - \frac{T_o\hat{\alpha}_2\beta_2 d^2 w}{K_o}]. \quad (6.7.30)$$

Now operating upon (6.7.29) by $(D^2-a^2-p/\tau)[D^2-a^2-p(1-T_o\alpha_2)]$, using (6.6.11),(6.7.30) and simplifying the resulting equation, we get

$$(D^2-a^2-p/\tau)[D^2-a^2-p(1-T_o\alpha_2)](D^2-a^2-p\sigma_1/\sigma)(D^2-a^2)(D^2-a^2-p/\sigma)w -$$

$$Q(D^2-a^2-p/\tau)[D^2-a^2-p(1-T_o\alpha_2)](D^2-a^2)D^2 w =$$

$$(D^2-a^2-p\sigma_1/\sigma)[R_1 a^2 w\{-\frac{Bp}{\tau} -(D^2-a^2-p/\tau)(1-T_o\alpha_2)-$$

$$(D^2-a^2-p/\tau)B\} + \frac{R_2 a^2}{\tau}\{D^2-a^2-p(1-T_o\alpha_2)\} w]. \quad (6.7.31)$$

Substituting for w in (6.7.31) from (6.7.28), we get

$$(n^2\pi^2+a^2+p/\tau)[n^2\pi^2+a^2+p(1-T_o\alpha_2)](n^2\pi^2+a^2+p\sigma_1/\sigma)(n^2\pi^2+a^2)$$

$$(n^2\pi^2+a^2+p/\sigma) + Q(n^2\pi^2+a^2+p/\tau)[n^2\pi^2+a^2+p(1-T_o\alpha_2)]$$

$$(n^2\pi^2+a^2)n^2\pi^2 = (n^2\pi^2+a^2+p\sigma_1/\sigma)[R_1 a^2\{-\frac{Bp}{\tau} +(n^2\pi^2+a^2+p/\tau)$$

$$(1-T_o\alpha_2) + (n^2\pi^2+a^2+p/\tau)B\} - \frac{R_2 a^2}{\tau}\{n^2\pi^2+a^2+p(1-T_o\alpha_2)\}]. \quad (6.7.32)$$

Substituting for w from (6.7.28) in (6.7.30),(6.6.11) and (6.6.12) respectively, we obtain after simplification the following equations :

$$\theta = \frac{[(1-T_0\alpha_2)\beta_1+T_0\hat{\alpha}_2\beta_2]d^2.A\ \sin n\pi z}{n^2\pi^2+a^2+p(1-T_0\alpha_2)\quad K_0} -$$

$$\frac{T_0\hat{\alpha}_2\beta_2 d^2 p\ A\ \sin n\pi z}{\eta_0(n^2\pi^2+a^2+p/\tau)[n^2\pi^2+a^2+p(1-T_0\alpha_2)]} . \qquad (6.7.33)$$

$$\varphi = \frac{\beta_2 d^2}{\eta_0}\ \frac{A\ \sin n\pi z}{n^2\pi^2+a^2+p/\tau} , \qquad (6.7.34)$$

$$h_z = \frac{Hd}{\gamma_c}\ \frac{n\pi\ A\ \cos n\pi z}{n^2\pi^2+a^2+p\sigma_1/\sigma} , \qquad (6.7.35)$$

ζ and ξ are given by (6.6.13) and (6.6.14). This proves the theorem.

The essential content of Theorem 2 from hydrodynamic instability point of view is this that it provides us with the exact solutions for the vertical component of perturbation velocity, perturbation temperature, perturbation concentration, vertical component of perturbation magnetic field, vertical component of perturbation vorticity and vertical component of current density and complex frequency for the case when both the bounding surfaces are dynamically free and perfectly conducting.

Theorem 3 : For $p_r = 0$, and $p_i \neq 0$, the exact solution of Theorem 2 implies

$$R_1a^2 = \frac{1}{(\sigma+\tau\sigma_1)(1-T_o\alpha_2)+B\tau\sigma_1}\Big[(n^2\pi^2+a^2)\Big\{Qn^2\pi^2+(n^2\pi^2+a^2)^2\Big\}$$

$$\Big\{1+\tau(1-T_o\alpha_2)\Big\}\sigma + (1+\sigma_1)\tau(n^2\pi^2+a^2)^3+R_2a^2\Big\{\sigma_1+\sigma(1-T_o\alpha_2)\Big\}$$

$$-\frac{p_i^2(n^2\pi^2+a^2)}{\sigma}\Big\{\sigma_1+\tau\sigma_1(1-T_o\alpha_2)+\sigma(1+\sigma_1)(1-T_o\alpha_2)\Big\}\Big], \qquad (6.7.36)$$

and

$$p_i^4(n^2\pi^2+a^2)(1-T_o\alpha_2)\;\frac{\sigma_1^2}{\sigma^2}\Big[\frac{-\tau B+1+\sigma(1-T_o\alpha_2)}{(\sigma+\tau\sigma_1)(1-T_o\alpha_2)+B\tau\sigma_1}\Big] +$$

$$p_i^2\Bigg[\frac{\tau\sigma_1}{\sigma^2}(n^2\pi^2+a^2)^3 + \frac{(n^2\pi^2+a^2)^3}{\sigma}(1+\sigma_1)\Big\{1+\tau(1-T_o\alpha_2)\Big\} +$$

$$(n^2\pi^2+a^2)(1-T_o\alpha_2)\Big\{Qn^2\pi^2+(n^2\pi^2+a^2)\Big\} + R_2a^2\sigma_1(1-T_o\alpha_2)/\sigma +$$

$$\frac{\tau(n^2\pi^2+a^2)^3\Big\{-B-(1-T_o\alpha_2)\Big\}(\tau\sigma_1+\sigma+\sigma.\sigma_1)}{\sigma\,[(\sigma+\tau\sigma_1)(1-T_o\alpha_2)+B\tau\sigma_1]} -$$

$$\frac{\sigma_1(1-T_o\alpha_2)}{\sigma[(\sigma+\tau\sigma_1)(1-T_o\alpha_2)+B\tau\sigma_1]}\Big[(n^2\pi^2+a^2)\Big\{Qn^2\pi^2+(n^2\pi^2+a^2)^2\Big\}$$

$$\Big\{1+\tau(1-T_o\alpha_2)\Big\}\sigma + (1+\sigma_1)\tau(n^2\pi^2+a^2)^3+R_2a^2\Big\{\sigma_1+\sigma(1-T_o\alpha_2)\Big\}\Big]\Bigg]$$

$$-\tau(n^2\pi^2+a^2)^3\Big\{Qn^2\pi^2+(n^2\pi^2+a^2)^2\Big\} - R_2a^2(n^2\pi^2+a^2)^2 -$$

$$\frac{\tau(n^2\pi^2+a^2)^2\Big\{-B-(1-T_o\alpha_2)\Big\}}{[(\sigma+\tau\sigma_1)(1-T_o\alpha_2)+B\tau\sigma_1]}\Big[(n^2\pi^2+a^2)\Big\{Qn^2\pi^2+(n^2\pi^2+a^2)^2\Big\}$$

$$\Big\{1+\tau(1-T_o\alpha_2)\Big\}\sigma+(1+\sigma_1)\tau(n^2\pi^2+a^2)^3+R_2a^2\Big\{\sigma_1+\sigma(1-T_o\alpha_2)\Big\}\Big]$$

$$= 0. \qquad (6.7.37)$$

Proof : For $p_r = 0$ and $p_i \neq 0$, equation (6.7.32) becomes

$$(n^2\pi^2+a^2+ip_i/\tau)[n^2\pi^2+a^2+ip_i(1-T_o\alpha_2)](n^2\pi^2+a^2 ip_i\sigma_1/\sigma)$$
$$(n^2\pi^2+a^2)(n^2\pi^2+a^2+ip_i/\sigma) + Q(n^2\pi^2+a^2+ip_i/\tau)[n^2\pi^2+a^2+ip_i(1-T_o\alpha_2)$$
$$(n^2\pi^2+a^2)n^2\pi^2 = (n^2\pi^2+a^2+ip_i\sigma_1/\sigma)[R_1 a^2\left\{-\frac{B}{\tau}ip_i+\right.$$
$$\left.(n^2\pi^2+a^2+ip_i/\tau)(1-T_o\alpha_2)+(n^2\pi^2+a^2+ip_i/\tau)B\right\}] -$$
$$\frac{R_2 a^2}{\tau}(n^2\pi^2+a^2+ip_i\sigma_1/\sigma)\,[n^2\pi^2+a^2+ip_i(1-T_o\alpha_2)]. \qquad (6.7.38)$$

Equation (6.7.38) after simplification can be written as

$$[\tau(n^2\pi^2+a^2)^3-p_i^2(n^2\pi^2+a^2)(1-T_o\alpha_2)][(n^2\pi^2+a^2)^2 -p_i^2\sigma_1/\sigma^2] +$$
$$i[p_i(n^2\pi^2+a^2)^2\left\{1+\tau(1-T_o\alpha_2)\right\}\left\{(n^2\pi^2+a^2)^2-p_i^2\sigma_1/\sigma^2\right\} +$$
$$\frac{p_i(n^2\pi^2+a^2)(1+\sigma_1)}{\sigma}\left\{\tau(n^2\pi^2+a^2)^3-p_i^2(n^2\pi^2+a^2)(1-T_o\alpha_2)\right\}] -$$
$$\frac{p_i^2(n^2\pi^2+a^2)^3}{\sigma}(1+\sigma_1)\left\{1+\tau(1-T_o\alpha_2)\right\} + Qn^2\pi^2[\tau(n^2\pi^2+a^2)^3 -$$
$$p_i^2(n^2\pi^2+a^2)(1-T_o\alpha_2) \quad +ip_i(n^2\pi^2+a^2)^2\left\{1+\tau(1-T_o\alpha_2)\right\}] =$$
$$R_1 a^2[(n^2\pi^2+a^2)^2\left\{(1-T_o\alpha_2)\tau+B\tau\right\} - \frac{p_i^2\sigma_1}{\sigma}(1-T_o\alpha_2) +$$
$$\frac{ip_i\sigma_1}{\sigma}(n^2\pi^2+a^2)\tau\left\{(1-T_o\alpha_2)+B\right\} +ip_i(n^2\pi^2+a^2)(1-T_o\alpha_2)\ ]-$$

$$R_2 a^2[(n^2\pi^2+a^2)^2 - \frac{p_i^2\sigma_1}{\sigma}(1-T_o\alpha_2)+ip_i(n^2\pi^2+a^2)\{(1-T_o\alpha_2)+\frac{\sigma_1}{\sigma}\}] \quad (6.7.39)$$

Equating real and imaginary parts on both sides of (6.7.39) we obtain after rearrangement of terms the following equations:

$$R_1 a^2 = \frac{1}{(\sigma+\tau\sigma_1)(1-T_o\alpha_2)+B\tau\sigma_1}[(n^2\pi^2+a^2)\{Qn^2\pi^2+(n^2\pi^2+a^2)^2\}$$

$$\{1+\tau(1-T_o\alpha_2)\}\sigma + (1+\sigma_1)\tau(n^2\pi^2+a^2)^3+ R_2a^2\{\sigma_1+\sigma(1-T_o\alpha_2)\} -$$

$$p_i^2\frac{(n^2\pi^2+a^2)}{\sigma}\{\sigma_1+\tau\sigma_1(1-T_1\alpha_2)+\sigma(1+\sigma_1)(1-T_o\alpha_2)\}], \quad (6.7.40)$$

and

$$p_i^4\frac{\sigma_1}{\sigma^2}(n^2\pi^2+a^2)(1-T_o\alpha_2) - p_i^2[\frac{\tau(n^2\pi^2+a^2)^3\sigma_1}{\sigma^2}+$$

$$(n^2\pi^2+a^2)^3(1-T_o\alpha_2)+\frac{(n^2\pi^2+a^2)^3}{\sigma}(1+\sigma_1)\{1+\tau(1-T_o\alpha_2)\}+$$

$$Qn^2\pi^2(n^2\pi^2+a^2)(1-T_o\alpha_2) - \frac{R_1a^2\sigma_1(1-T_o\alpha_2)}{\sigma}+R_2a^2\frac{\sigma_1}{\sigma}(1-T_o\alpha_2)]+$$

$$\tau(n^2\pi^2+a^2)^5+Qn^2\pi^2\tau(n^2\pi^2+a^2)^3+R_1a^2.\tau(n^2\pi^2+a^2)[-B-(1-T_o\alpha_2)]+$$

$$R_2a^2(n^2\pi^2+a^2)^2 = 0. \quad (6.7.41)$$

Substituting the value of R_1a^2 from (6.7.40) in (6.7.41), we obtain

$$p_i^4(n^2\pi^2+a^2)(1-T_o\alpha_2)\frac{\sigma_1^2}{\sigma^2}[\frac{-\tau B+1+\sigma(1-T_o\alpha_2)}{(\sigma+\tau\sigma_1)(1-T_o\alpha_2)+B\tau\sigma_1}]+$$

$$p_i^2 \Big[\frac{\tau\sigma_1}{\sigma^2} (n^2\pi^2+a^2)^3 + \frac{(n^2\pi^2+a^2)^3(1+\sigma_1)}{\sigma} \{1+\tau(1-T_o\alpha_2)\} +$$

$$(n^2\pi^2+a^2)(1-T_o\alpha_2) \{Qn^2\pi^2+(n^2\pi^2+a^2)^2\} + \frac{R_2a^2\sigma_1(1-T_o\alpha_2)}{\sigma} +$$

$$\frac{\tau(n^2\pi^2+a^2)^3 [-B-(1-T_o\alpha_2)] (\tau\sigma_1+\sigma+\sigma\sigma_1)}{\sigma [(\sigma+\tau\sigma_1)(1-T_o\alpha_2) + B\tau\sigma_1]} -$$

$$\frac{\sigma_1(1-T_o\alpha_2)}{\sigma[(\sigma+\tau\sigma_1)(1-T_o\alpha_2)+B\tau\sigma_1]} [(n^2\pi^2+a^2) \{Qn^2\pi^2+(n^2\pi^2+a^2)^2\}$$

$$\{1+\tau(1-T_o\alpha_2)\}\sigma + (1+\sigma_1)\tau(n^2\pi^2+a^2)^3+R_2a^2\{\sigma_1+\sigma(1-T_o\alpha_2)\}]\Big] -$$

$$\tau(n^2\pi^2+a^2)^3 \{Qn^2\pi^2+(n^2\pi^2+a^2)^2\} - R_2a^2(n^2\pi^2+a^2)^2 -$$

$$\frac{\tau(n^2\pi^2+a^2)^2[-B-(1-T_o\alpha_2)]}{(\sigma+\tau\sigma_1)(1-T_o\alpha_2)+B\tau\sigma_1} [(n^2\pi^2+a^2) \{Qn^2\pi^2+(n^2\pi^2+a^2)^2\}$$

$$\{1+\tau(1-T_o\alpha_2)\}\sigma + (1+\sigma_1)\tau(n^2\pi^2+a^2)^3+R_2a^2\{\sigma_1+\sigma(1-T_o\alpha_2)\}]$$

$$= 0. \qquad (6.7.42)$$

This establishes the validity of theorem 3.

The essential content of Theorem 3 from the point of view of hydrodynamic instability is this that overstable solutions do exist when both the bounding surfaces are dynamically free and perfectly conducting and provides us with the exact calculations for the critical Rayleigh number and the frequency of oscillations of the overstable motions, at the marginal state with respect to them. It is important to note in this connection that in Banerjee's generalized Bénard model under magnetic effects, it is only the overstable motions that manifest at the marginal state while in simple Bénard model under magnetic effects or in Veronis' thermohaline model under magnetic effects, the possibility of both, the stationary as well as the overstable motions exist at the marginal state. However, Veronis' work gives an ample support to the proposition that overstable motions at the marginal state are the more likely ones. It is on the basis of the above works of Banerjee, and Veronis which are carried out for the case when both the bounding surfaces are free that we have taken $p_i \neq 0$ and $p_r = 0$ in Theorem 3 although solutions with $p_i = 0$ when $p_r = 0$ also exist as they exist in simple Bénard model under magnetic effects or in Veronis' thermohaline model under magnetic effects.

For the simple Bénard problem under magnetic effects $R_2 = \tau = 0 = B = \alpha_2$. If, further, we confine our attention on stationary modes only then we also have $p_i=0$. Equation (6.7.40) then gives for the lowest mode (n=1) that

$$R_1 = \frac{(\pi^2+a^2)}{a^2} [(\pi^2+a^2)^2 + Q\pi^2]$$

and this is Chandrasekhar's (1953) result.

Similarly, we can easily recover the results from (6.7.40) and (6.7.42) concerning the critical Rayleigh number and the corresponding frequency of oscillations at the marginal state of overstable motions in (i) simple Bénard problem under magnetic effects (ii) generalized Bénard problem under magnetic effects (iii) thermohaline problem under magnetic effects. We can also derive from here the critical Rayleigh number that corresponds to stationary motions in thermohaline problem under magnetic effects. In the following we give below the tables based on (6.7.40) and (6.7.42) which show the variations of the square of the critical frequency and the critical Rayleigh number with $T_0\alpha_2$ ($\alpha_2 > 0$) and Q for hydromagnetic generalized Bénard problem and hydromagnetic thermohaline convection problem in the present framework when the values of the other parameters are prescribed.

It is clear from Table 1 and Table 2 that for the hydromagnetic generalised Bénard problem for which $\tau = 0$, p_i^2 increases for a given value of Q with increasing values of $T_0\alpha_2$ while R_1 increases for values of Q from 0 to 100 and decreases for values of Q from 1000 to 10,000 provided α_2 is positive. On the otherhand p_i^2 and R_1 both increase for a given value of $T_0\alpha_2$ for increasing values of Q and further that these values of p_i^2 and R_1 are significantly larger than the classical values. A similar behaviour is observed for the thermohaline convection problem also for

TABLE - 1

Values of p_1^2 for $\tau=0, B=T_o\alpha_2$, $\sigma=7, \sigma_1=10, R_2=500, a^2=4.9348$ and

$T_o\alpha_2$	Q = 0	Q = 10	Q =100	Q=1000	Q=10,000
0	145.8341	156.5917	276.4475	1860.3982	18174.700
.3	197.7409	226.0819	547.1924	4275.6756	41894.483
.31	200.1152	229.3230	559.628	4386.2432	42979.445
.32	202.5471	232.6474	573.0813	4499.5606	44090.778

TABLE - 2

Values of R_1 for $\tau=0, B=T_o\alpha_2, \sigma=7, \sigma_1=10, R_2=500, a^2=4.9348$ and

$T_o\alpha_2$	Q = 0	Q = 10	Q = 100	Q = 1000	Q =10,000
0	1095.0067	1333.7941	3360.1718	21571.142	201151.84
.3	1354.5509	1619.1397	3631.3275	20861.617	191297.95
.31	1367.1452	1632.6479	3644.4302	20829.453	190847.69
.32	1380.1175	1646.5421	3653.9440	20796.661	190387.36

TABLE - 3

Values of p_i^2 for $B=T_o\alpha_2$, τ =.01, σ=7, σ_1=10, R_2=500, a^2=4.9348

$T_o\alpha_2$	Q = 0	Q = 10	Q = 100	Q=1000	Q=10,000
0	144.2693	154.9827	274.6648	1859.253	18178.775
.3	198.0776	226.7675	550.6319	4299.794	42120.859
.31	200.5456	230.1202	563.6261	4411.367	43214.676
.32	203.0835	233.5752	595.483	4525.924	44337.582

TABLE - 4

Values of R_1 for $B=T_o\alpha_2$, τ= .01, σ=7, σ_1=10, R_2=500, a^2=4.9348

$T_o\alpha_2$	Q = 0	Q = 10	Q = 100	Q = 1000	Q=10,000
0	1103.604	1342.12	3366.42	21570.74	201123.84
.3	1361.69	1624.97	3630.08	20847.95	191198.33
.31	1374.234	1638.38	3642.06	20815.42	190745.78
32	1387.168	1652.17	3549.1201	20781.68	190277.54

which $\tau \neq 0$. However, in this latter case, the values of R_1 are larger than the corresponding values of R_1 when $\tau = 0$, while the values of p_i^2 are larger in the case of thermohaline convection for which $\tau \neq 0$ than the corresponding values of p_i^2 when $\tau = 0$, except in the case when $T_o\alpha_2 = 0$ and Q takes increasing values that p_i^2 shows the reverse behaviour.

6.8. Contributions of Chapter-6

(i) Shows, for the first time to the best of our knowledge, that a more consistent and relatively more accurate application of Boussinesq approximation leads to a formulation of thermal and thermohaline instability problems under the magnetic effects which does significantly depend upon whether the fluid layer is hotter or cooler and this is on account of the variations in the specific heat at constant volume due to variations in temperature and/or concentration.

(ii) Establishes, on a more sound basis, the results of Banerjee et al (1976) regarding the character of the marginal state in the generalized Bénard problem under magnetic effects.

(iii) Exactly solves the problem of generalized Bénard convection and thermohaline convection under magnetic effects with dynamically free and perfectly conducting boundaries for oscillatory modes and distinguishes between a hotter and a cooler layer as regards the onset of instability.