CHAPTER - 4

MODIFIED ANALYSIS OF BÉNARD CONVECTION, GENERALIZED BÉNARD CONVECTION AND THERMOHALINE CONVECTION

4.1. The problem and the initial state

A viscous finitely heat conducting Boussinesq fluid of infinite horizontal extension is statically confined between two horizontal boundaries $z = 0$ and $z = d$ which are respectively maintained at uniform temperatures T_0 and T_1 ($< T_0$) and uniform concentrations S_0 and S_1 ($< S_0$). We wish to mathematically investigate the onset of linear hydrodynamic instability in the system.

The initial state in given by

$$\left.\begin{aligned} (u, v, w) &\equiv (0, 0, 0), \\ T &\equiv T(z), \\ S &\equiv S(z), \\ \text{and} \quad \rho &\equiv \rho(z). \end{aligned}\right\} \qquad (4.1.1)$$

4.2. The basic equations and the initial state solution

The governing equations as mentioned earlier are given by (3.2.36)- (3.2.40). These equations on the basis of (4.1.1) yield the initial state solution

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$$
\left.\begin{aligned}
(u,v,w) &= (0,\ 0,\ 0), \\
T &= T_0 - \beta_1 z , \\
S &= S_0 - \beta_2 z , \\
\text{and} \quad \rho &= \rho_c [1 + \alpha(T_0 - T) - \hat{\alpha}(S_0 - S)] , \\
&= \rho_c (1 + \alpha\beta_1 z - \hat{\alpha}\beta_2 z) ,
\end{aligned}\right) \quad (4.2.1)
$$

$$
\text{with} \quad P = P_0 - g\rho_c [z + (\alpha\beta_1 - \hat{\alpha}\beta_2) \frac{z^2}{2}]. \quad (4.2.2)
$$

4.3. The perturbation equations

Let the initial state described by (4.2.1) and (4.2.2) be slightly perturbed so that the perturbed state is given by

$$
\left.\begin{aligned}
(\bar{u},\ \bar{v},\ \bar{w}) &= (0+u',\ 0+v',\ 0+w'), \\
\bar{T} &= T_0 - \beta_1 z + \theta' , \\
\bar{S} &= S_0 - \beta_2 z + \varphi' , \\
\bar{\rho} &= \rho_c [1+\alpha(T_0 - T - \theta') - \hat{\alpha}(S_0 - S - \varphi')], \\
\text{and} \quad \bar{P} &= P_0 - g\rho_c [z + (\alpha\beta_1 - \hat{\alpha}\beta_2) \frac{z^2}{2}] + \delta p' .
\end{aligned}\right) \quad (4.3.1)
$$

Then, the linearized equations of perturbation of continuity, momentum, heat conduction and mass diffusion are respectively given by

$$
\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} = 0 , \quad (4.3.2)
$$

$$
\rho_c \frac{\partial u'}{\partial t} = - \frac{\partial \delta p'}{\partial x} + \mu_0 \nabla^2 u' , \quad (4.3.3)
$$

$$
\rho_c \frac{\partial v'}{\partial t} = - \frac{\partial \delta p'}{\partial y} + \mu_0 \nabla^2 v' , \quad (4.3.4)
$$

$$\rho_o \frac{\partial w'}{\partial t} = - \frac{\partial \delta p'}{\partial z} + u_o \nabla^2 w' + g \rho_o \alpha \Theta' - g \rho_o \hat{\alpha} \varphi', \qquad (4.3.5)$$

$$(1-T_o\alpha_2) \frac{\partial \Theta'}{\partial t} + T_o \hat{\alpha}_2 \frac{\partial \varphi'}{\partial t} -(1-T_o\alpha_2)\beta_1 w' - T_o\hat{\alpha}_2\beta_2 w' = K_o \nabla^2 \Theta', \qquad (4.3.6)$$

and
$$\frac{\partial \varphi'}{\partial t} - \beta_2 w' = \eta_o \nabla^2 \varphi'. \qquad (4.3.7)$$

4.4. The normal mode analysis

We shall now analyse an arbitrary perturbation into a complete set of normal modes and then examine the stability of each of these modes individually. For the system of equations (4.3.2) - (4.3.7) the analysis can be made in terms of two dimensional periodic waves of assigned wave numbers. Thus we ascribe to all quantities describing the perturbation a dependence on x, y and t of the form

$$\exp[i(k_x x + k_y y) + nt], \qquad (4.4.1)$$

where
$$k = +\sqrt{k_x^2 + k_y^2}\ , \qquad (4.4.2)$$

is the wave number of the perturbation, k_x and k_y being real constants and n is a constant which can be complex in general. The solution of the stability problem requires the specifications of the states for each k, which are characterized by the real part of n being zero.

The above considerations allow us to suppose that the perturbations u', v', w', Θ', φ' and $\delta p'$ have the following forms :

$$\left.\begin{aligned} u'(x,y,z,t) &= u''(z)\ \exp[i(k_x x + k_y y) + nt], \\ v'(x,y,z,t) &= v''(z)\ \exp[i(k_x x + k_y y) + nt], \\ w'(x,y,z,t) &= w''(z)\ \exp[i(k_x x + k_y y) + nt], \\ \Theta'(x,y,z,t) &= \Theta''(z)\ \exp[i(k_x x + k_y y) + nt], \\ \varphi'(x,y,z,t) &= \varphi''(z)\ \exp[i(k_x x + k_y y) + nt], \\ \text{and} & \\ \delta p'(x,y,z,t) &= \delta p''(z)\ \exp[i(k_x x + k_y y) + nt], \end{aligned}\right) \quad (4.4.3)$$

For functions with this dependence on x,y and t, we have

$$\left.\begin{aligned} \frac{\partial}{\partial t} &= n, \\ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} &= -k^2, \text{ and } \quad \nabla^2 = \frac{d^2}{dz^2} - k^2 \end{aligned}\right). \quad (4.4.4)$$

Equations (4.3.2) - (4.3.7) then become

$$i\,k_x u'' + i\,k_y v'' + \frac{dw''}{dz} = 0\ , \quad (4.4.5)$$

$$\rho_o n u'' = -i\,k_x \delta p'' + u_o\left(\frac{d^2}{dz^2} - k^2\right)u''\ , \quad (4.4.6)$$

$$\rho_o n v'' = -i\,k_y \delta p'' + u_o\left(\frac{d^2}{dz^2} - k^2\right)v'', \quad (4.4.7)$$

$$\rho_o n w'' = -\frac{d\delta p''}{dz} + \mu_o\left(\frac{d^2}{dz^2} - k^2\right)w'' + g\alpha\rho_o\Theta'' - g\hat{\alpha}\rho_o\varphi'', \quad (4.4.8)$$

$$(1-T_o\alpha_2)n\Theta'' + T_o\hat{\alpha}_2 n\varphi'' - (1-T_o\alpha_2)\beta_1 w'' - T_o\hat{\alpha}_2\beta_2 w''$$

$$= K_o\left(\frac{d^2}{dz^2} - k^2\right)\Theta'', \quad (4.4.9)$$

and $$n\varphi'' - \beta_2 w'' = \eta_0\left(\frac{d^2}{dz^2} - k^2\right)\varphi''. \qquad (4.4.10)$$

4.5. The boundary conditions

Since the flow domain under consideration is confined between two horizontal boundaries $z = 0$ and $z = d$, the field quantities must satisfy certain boundary conditions on them. Further because the above surfaces are fixed and maintained at constant temperature and concentration, we must have w', θ' and φ' vanish on them. Thus

$$\left.\begin{aligned} w' &= 0 \quad \text{at} \quad z = 0 \quad \text{and} \quad z = d, \\ \theta' &= 0 \quad \text{at} \quad z = 0 \quad \text{and} \quad z = d, \\ \text{and} \quad \varphi' &= 0 \quad \text{at} \quad z = 0 \quad \text{and} \quad z = d. \end{aligned}\right\} \qquad (4.5.1)$$

The other boundary conditions to be satisfied however, depend on the nature of the boundaries at $z=0$ and $z=d$. It is to be noted at this point that condition (4.5.1) is independent of the nature of these surfaces. Throughout the present dissertation we shall deal with boundaries which are (a) rigid surfaces on which slip does not occur and (b) dynamically free surfaces on which tangential stresses do not act. Let us consider the case of a rigid boundary first. Since slip does not occur on this surface, it is clear that in addition to w', the horizontal components of velocity u' and v' must also vanish on it. Thus

$$\left.\begin{aligned} u' &= 0, \\ \text{and} \quad v' &= 0. \end{aligned}\right\} \text{ on a rigid surface.} \qquad (4.5.2)$$

Further, since this condition must be satisfied for all x and y on the surface, it follows from (4.3.2) that

$$\frac{\partial w'}{\partial z} = 0 \quad \text{on a rigid surface.} \qquad (4.5.3)$$

Next, we take up the case of a dynamically free surface. The conditions to be satisfied on it are

$$p_{xz} = 0, \qquad (4.5.4)$$

$$\text{and} \quad p_{yz} = 0, \qquad (4.5.5)$$

$$\text{where} \quad p_{ij} = -\hat{p}\delta_{ij} + 2\mu\, e_{ij}. \qquad (4.5.6)$$

The conditions (4.5.4) and (4.5.5) are equivalent to the vanishing of the components p_{xz} and p_{yz} of the viscous stress tensor

$$p_{xz} = \mu\left(\frac{\partial u'}{\partial z} + \frac{\partial w'}{\partial x}\right), \qquad (4.5.7)$$

$$\text{and} \quad p_{yz} = \mu\left(\frac{\partial v'}{\partial z} + \frac{\partial w'}{\partial y}\right). \qquad (4.5.8)$$

since the isotropic term $-\hat{p}\delta_{ij}$ has no transverse component. Now, as w vanishes for all x and y on the boundaries it follows from (4.5.7) and (4.5.8) that

$$\left.\begin{aligned} \frac{\partial u'}{\partial z} &= 0, \\ \text{and} \quad \frac{\partial v'}{\partial z} &= 0. \end{aligned}\right\} \text{ on a dynamically free surface.} \qquad (4.5.9)$$

Hence from (4.3.2) differentiated with respect to z, we conclude that

$$\frac{\partial^2 w'}{\partial z^2} = 0, \text{ on a dynamically free surface} \qquad (4.5.10)$$

Utilizing (4.4.3), we obtain from the above

$$\left.\begin{aligned} w'' &= 0, \\ \Theta'' &= 0, \\ \varphi'' &= 0, \\ \text{and} \quad \frac{dw''}{dz} &= 0. \end{aligned}\right) \text{ on a rigid boundary}, \qquad (4.5.11)$$

$$\left.\begin{aligned} \text{while} \quad w'' &= 0, \\ \Theta'' &= 0, \\ \varphi'' &= 0, \\ \text{and} \quad \frac{d^2w''}{dz^2} &= 0. \end{aligned}\right) \text{ on a dynamically free boundary.} \qquad (4.5.12)$$

4.6. <u>The characteristic value problem</u>

Multiplying (4.4.6) by k_x and (4.4.7) by k_y, adding the resulting equations and making use of (4.4.5), we obtain

$$\rho_o n \frac{dw''}{dz} = -k^2 \, \delta p'' + \mu_o\left(\frac{d^2}{dz^2} - k^2\right) \frac{dw''}{dz}. \qquad (4.6.1)$$

Eliminating $\delta p''$ between (4.4.8) and (4.6.1), it follows that

$$\left(\frac{d^2}{dz^2} - k^2\right)\left(\frac{d^2}{dz^2} - k^2 - \frac{n}{\nu_o}\right) w'' = \frac{g\alpha k^2 \Theta''}{\nu_o} - \frac{g\hat{\alpha} k^2 \varphi''}{\nu_o}. \qquad (4.6.2)$$

Equations (4.4.9) and (4.410) can be rewritten as

$$\left[\frac{d^2}{dz^2} - k^2 - \frac{n(1-T_o\alpha_2)}{K_o}\right]\theta'' - \frac{T_o\hat{\alpha}_2 n\varphi''}{K_o} = -\frac{(1-T_o\alpha_2)\beta_1 w''}{K_o} - \frac{T_o\hat{\alpha}_2\beta_2 w''}{K_o}, \qquad (4.6.3)$$

and

$$\left(\frac{d^2}{dz^2} - k^2 - \frac{n}{\eta_o}\right)\varphi'' = -\frac{\beta_2\, w''}{\eta_o}. \qquad (4.6.4)$$

We shall now introduce the non-dimensional quantities defined by

$$\left.\begin{aligned} z_* &= z/d\,, & \tau_* &= \eta_o/K_o\,,\\ a_* &= kd\,, & \sigma_* &= \nu_o/K_o\,,\\ D_* &= d.\frac{d}{dz}, & p_* &= nd^2/K_o\,. \end{aligned}\right\}. \qquad (4.6.5)$$

Using the above non-dimensional quantities and omitting the asterisks and the double dashes for simplicity, we can reduce (4.6.2), (4.6.3) and (4.6.4) to the following partially non-dimensional forms.

$$(D^2-a^2)(D^2-a^2-\frac{p}{\sigma})w = \frac{g\alpha a^2 d^2}{\nu_o}\theta - \frac{g\hat{\alpha} a^2 d^2}{\nu_o}\varphi, \qquad (4.6.6)$$

$$[D^2-a^2-p(1-T_o\alpha_2)]\theta - T_o\hat{\alpha}_2 p\varphi = -\frac{(1-T_o\alpha_2)\beta_1 d^2 w}{K_o} - \frac{T_o\hat{\alpha}_2\beta_2 d^2 w}{K_o}, \qquad (4.6.7)$$

and

$$(D^2-a^2-\frac{p}{\tau})\varphi = -\frac{\beta_2 d^2 w}{\eta_o} \qquad (4.6.8)$$

If we introduce more non-dimensional quantities defined by

$$\left.\begin{aligned} R_{1*} &= \frac{g\alpha\beta_1 d^4}{K_o \nu_o}, & w_* &= \frac{\beta_1 d^2 w}{K_o}, \\ R_{2*} &= \frac{g\hat{\alpha}\beta_2 d^4}{K_o \nu_o}, & \theta_* &= \theta, \\ R_3 &= \beta_2/\beta_1, & \varphi_* &= \varphi . \end{aligned}\right\} \tag{4.6.9}$$

and omit the asterisks for simplicity, we can further reduce (4.6.6), (4.6.7) and (4.6.8) to the following completely non-dimensional forms

$$(D^2-a^2)(D^2-a^2-\frac{p}{\sigma})w = R_1 a^2\theta - \frac{R_2 a^2\varphi}{R_3}, \tag{4.6.10}$$

$$[D^2-a^2-p(1-T_o\alpha_2)]\theta - T_o\hat{\alpha}_2 p\varphi = -(1-T_o\alpha_2)w - T_o\hat{\alpha}_2 R_3 w, \tag{4.6.11}$$

and $$(D^2-a^2-\frac{p}{\tau})\varphi = -\frac{R_3}{\tau} w. \tag{4.6.12}$$

In the subsequent analysis we shall proceed either with (4.6.6) - (4.6.8) or (4.6.10) - (4.6.12) which are equivalent forms. The boundary conditions (4.5.11) or (4.5.12) for the cases when (i) both boundaries are rigid (ii) both boundaries are dynamically free (iii) any one boundary is rigid and the other dynamically free, in the above framework respectively reduce to

$$\left.\begin{aligned} w &= 0, \\ \theta &= 0, \\ \varphi &= 0, \\ \text{and} \quad Dw &= 0. \end{aligned}\right\} \text{ at } z = 0 \text{ and } z = 1, \tag{4.6.13}$$

Or

$$\left.\begin{aligned} w &= 0 , \\ \Theta &= 0 , \\ \varphi &= 0 , \\ D^2 w &= 0 . \end{aligned}\right) \quad \text{at } z = 0 \text{ and } z = 1 ; \qquad (4.6.14)$$

Or

$$\left.\begin{aligned} w &= 0 , \\ \Theta &= 0 , \\ \varphi &= 0 , \\ \text{and} \quad Dw &= 0 . \end{aligned}\right) \quad \text{at } z = 0 \text{ (without loss of generality),} \qquad (4.6.15)$$

$$\left.\begin{aligned} \text{and} \quad w &= 0 , \\ \Theta &= 0 , \\ \varphi &= 0 , \\ \text{and} \quad D^2 w &= 0 . \end{aligned}\right) \quad \text{at } z = 1 . \qquad (4.6.16)$$

The governing perturbation equations (4.6.6)-(4.6.8) or (4.6.10) - (4.6.12) must be considered with proper boundary conditions on the flow variables. These are given by (4.6.13) or (4.6.14) or (4.6.15) and (4.5.16) according to the case under consideration. This poses a characteristic value problem for p for prescribed values of other parameters and a given normal mode is stable, marginal or unstable provided that the real part p_r of p is negative, zero or positive respectively. Further, if $p_r = 0$ implies p_i (the imaginary part of p) = 0 for every wave number 'a' then the 'principal of exchange of stabilities' is valid, otherwise, we will have overstability at least when instability sets in as certain modes.

4.7. Mathematical analysis

We prove the following theorems :

Theorem 1 : For $R_3 = 0$ and $\varphi = 0$, a necessary condition for the existence of nontrivial solution for w and θ satisfying (4.6.10) - (4.6.12) and (4.6.13) or (4.6.14) or (4.6.15) and (4.6.16) is that $p_i = 0$.

Proof : For $R_3 = 0$ and $\varphi = 0$, (4.6.12) is trivially satisfied, and (4.6.10) and (4.6.11) respectively reduce to

$$(D^2 - a^2)(D^2 - a^2 - \frac{p}{\sigma})\, w = R_1 a^2 \theta , \qquad (4.7.1)$$

and $$[D^2 - a^2 - p(1 - T_o\alpha_2)]\theta = -(1 - T_o\alpha_2) w. \qquad (4.7.2)$$

Multiplying both sides of (4.7.1) by w^* throughout and integrating the resulting equation over the vertical range of z, we obtain

$$\int_0^1 w^*(D^2 - a^2)(D^2 - a^2 - \frac{p}{\sigma}) w \, dz = R_1 a^2 \int_0^1 w^* \theta \, dz. \qquad (4.7.3)$$

Substituting the value of $\int_0^1 w^* \theta \, dz$ in (4.7.3) from (4.7.2), we have

$$\int_0^1 w^*(D^2 - a^2)(D^2 - a^2 - \frac{p}{\sigma}) w \, dz = R_1 a^2 [- \frac{1}{(1 - T_o\alpha_2)} \int_0^1 \theta \{ D^2 - a^2 - p^*(1 - T_o\alpha_2) \} \theta^* dz]. \qquad (4.7.4)$$

Making use of (4.6.13) or (4.6.14) or (4.6.15) and (4.6.16), the above equation becomes

$$\int_0^1 [|D^2w|^2 + 2a^2|Dw|^2 + a^4|w|^2 + \frac{p}{\sigma}(|Dw|^2 + a^2|w|^2)]dz =$$

$$\frac{R_1 a^2}{1-T_0\alpha_2}\int_0^1 [|D\theta|^2 + a^2|\theta|^2 + p^*(1-T_0\alpha_2)|\theta|^2]\, dz \,. \qquad (4.7.5)$$

Equating the imaginary parts of both sides of (4.7.5), we obtain

$$\frac{p_i}{\sigma}\int_0^1 (|Dw|^2 + a^2|w|^2)\, dz = -R_1 a^2 p_i \int_0^1 |\theta|^2\, dz. \qquad (4.7.6)$$

From (4.7.6), we conclude that $p_i = 0$, (4.7.7)

and this proves the theorem.

The essential content of Theorem 1, from the point of view of hydrodynamics, is that the 'principle of exchange of stabilities' is valid for the simple Bénard problem when considered in the present generalized framework and thus establishes the result due to Pellew and Southwell (1940) on a more firm basis.

Theorem 2 : For $\tau = 0$, a necessary condition for the existence of non-trivial solution for w, θ and φ satisfying (4.6.10) - (4.6.12), and (4.6.13) or (4.6.14) or (4.6.15) and (4.6.16) is that $p \neq 0$.

Proof : For $\tau = 0$, let $p = 0$ be allowed if possible. From (4.6.12), we conclude that $w \equiv 0$. (4.7.8)

Equations (4.6.10) and (4.6.11) then become

$$0 = R_1 a^2 \theta - \frac{R_2 a^2}{R_3} \varphi , \qquad (4.7.9)$$

and $$(D^2 - a^2)\,\theta = 0. \qquad (4.7.10)$$

The only solution of (4.7.10) which satisfies (4.6.13) or (4.6.14) or (4.6.15) and (4.6.16) is given by

$$\theta \equiv 0. \qquad (4.7.11)$$

Making use of (4.7.11), we conclude from (4.7.9) that

$$\varphi \equiv 0 . \qquad (4.7.12)$$

Equations (4.7.8), (4.7.11) and (4.7.12) imply that p cannot be zero and this proves the theorem.

The essential content of Theorem 2, from the point of view of hydrodynamics, is that the 'principle of exchange of stabilities is not valid for the generalised Bénard problem when considered in this generalized framework and this establishes the result due to Banerjee (1972) on a more firm basis.

Theorem 3 : For $R_3 = p_r = 0$ and $\varphi = 0$, an exact solution of (4.6.10) and (4.6.11) with (4.6.14) is given by

$$\left.\begin{aligned} w &= A \sin \pi z , \\ \theta &= \frac{A(1-T_0\alpha_2)\sin \pi z}{\pi^2 + a^2} , \\ a_c &= \pi/\sqrt{2} , \end{aligned}\right\} \qquad (4.7.13)$$

$$\text{and} \qquad R_{1c} = \frac{27\,\pi^4}{4} \cdot \frac{1}{1-T_o\alpha_2} \,. \quad \Big)$$

<u>Theorem 4</u> : For $R_3 = p_r = 0$ and $\varphi \equiv 0$, an exact solution of (4.6.10) and (4.6.11) with (4.6.13) is possible and a_c and R_{1c} are given by

$$\left.\begin{aligned} a_c &= 3.117 \,, \\ \text{and} \qquad R_{1c} &= \frac{1707.762}{1 - T_o\alpha_2} \,. \end{aligned}\right) \qquad (4.7.14)$$

<u>Theorem 5</u> : For $R_3 = p_r = 0$ and $\varphi \equiv 0$, an exact solution of (4.6.10) and (4.6.11) with (4.6.15) and (4.6.16) is possible and a_c and R_{1c} are given by

$$\left.\begin{aligned} a_c &= 2.682 \,, \\ \text{and} \qquad R_{1c} &= \frac{1100.65}{1 - T_o\alpha_2} \,. \end{aligned}\right) \qquad (4.7.15)$$

The proofs of <u>Theorem 3</u>, <u>Theorem 4</u> and <u>Theorem 5</u> follow at once since (4.6.10) and (4.6.11) in the present context reduce to

$$(D^2-a^2)^2 w = R_1\, a^2\, \theta \,, \qquad (4.7.16)$$

$$\text{and} \qquad (D^2-a^2)\, \theta = -(1-T_o\alpha_2)\, w, \qquad (4.7.17)$$

which are identical to the corresponding equations for the simple Bénard problem except that the Rayleigh number is to be replaced by a suitable multiple of it, the constant multiplying factor being $(1-T_o\alpha_2)$.

Theorem 3, Theorem 4 and Theorem 5 provide us with significant results from both qualitative as well as quantitative points of view. Irrespective of the nature of the bounding surfaces, the above three theorems explicitly derive the consequence that the hotter the liquid layer the more the postponement of the onset of instability provided α_2 is positive. Similarly, for a liquid layer for which α_2 is negative the onset of instability would occur earlier for the hotter one.

To elaborate these ideas further we calculate the values of R_{1c} according to Theorem 4 taking water as the theoretical fluid for which $\alpha_2 = + .0001$ and varying the temperatures of the lower and upper plates but maintaining the same temperature gradient. These calculations are given in the table below :

TABLE I

Calculations of critical Rayleigh number according to Theorem 4 for varying lower and upper plate temperatures

	Experiment 1	Experiment 2	Experiment 3	Experiment 4
Temperature of (upper plate	$5^{\circ}C$	$25^{\circ}C$	$45^{\circ}C$	$65^{\circ}C$
Temperature of (lower plate	$7^{\circ}C$	$27^{\circ}C$	$47^{\circ}C$	$67^{\circ}C$
R_c according to Theorem 4	1757	1760	1764	1768

The above calculations unmistakably show the point stated earlier and further strongly support the apparent result that for water the instability would manifest at higher values of critical Rayleigh number than the value 1708 predicted by Rayleigh's theory. Fortunately, water has been extensively used by various researchers as an experimental fluid for the above purpose and we reproduce below the experimental results of Schmidt and Milverton (1935) which bear directly on the detection of the onset of thermal instability and the determination of the critical Rayleigh number. These predictions according to Theorem 4, in the above situation are also produced in a separate table to provide a check against the above experimental observations.

TABLE II

The results of Schmidt and Milverton's experiments

		Experiment		
	1	2	3	4
Temperature of upper plate	23.2°C	22.75°C	19.80°C	17.63°C
Temperature of lower plate	24.9°C	23.75°C	20.85°C	18.47°C
R_c (experimental)	1970	1580	1850	1670

TABLE III

The results according to Theorem 4 for Schmidt and Milverton's experiments.

	Experiment			
	1	2	3	4
R_c (According to Theorem 4)	1760	1760	1759	1758

A comparison of the results of Schmidt and Milverton's observations with the theoretically predicted value 1708 according to Rayleigh's theory shows that the agreement is not good though surprisingly it has been taken as 'satisfactory' (Chandrasekhar, 1970). Indeed, it is true that there is definitely a scope to achieve greater precision in the above experiments of Schmidt and Milverton so that results would change a little but there is no denial of the unmistakable result shown by their observations that for water the value of the critical Rayleigh number should be definitely higher than 1708. It is striking to note in this regard that the mean value of the critical Rayleigh numbers from Schmidt and Milverton's results is 1767.5 which is almost precisely identical with the results from Theorem 4.

We now compute the upper and lower limits of the range of values of R_{1c} as predicted by Theorem 4 for fluids with which we are usually concerned and which are within the domain of applicability of Theorem 4. The two representative fluids which provide us with positive and negative admissible values of α_2 are water ($\alpha_2 \approx +.0001$) and heptane ($\alpha_2 \approx -.0001$) and a simple calculation according to Theorem 4 shows that for all fluids whose α_2 lies in the range -10^{-4} to $+10^{-4}$ under the usual laboratory conditions, instability would manifest for values of critical Rayleigh numbers R_{1c} that satisfy

$$1658 < R_{1c} < 1760 . \qquad (4.7.18)$$

In otherwords the critical Rayleigh number for the onset of instability for a wide range of fluids has the value

$$1700 - 42 < R_{1c}(\text{according to Theorem 4}) < 1700 + 60, \qquad (4.7.19)$$

and this is very satisfactorily supported by the precision experiments of Silveston (1958). In fact, according to Silveston who experimented with water, heptan, silicone oil AK 350, silicone oil AK 3 and ethylene glycol

$$R_{1c}(\text{Experimental}) = 1700 \pm 51 . \qquad (4.7.20)$$

Theorem 6 : An exact solution of (4.6.6), (4.6.7) and (4.6.8) with (4.6.14) is given by

$$w = A \sin n\pi z,$$

$$\varphi = \frac{A\beta_2 d^2}{\eta_0} . \frac{\sin n\pi z}{n^2\pi^2 + a^2 + \frac{p}{\tau}} ,$$

$$\theta = \frac{[(1-T_o\alpha_2)\beta_1 + T_o\hat{\alpha}_2\beta_2]\, d^2.A \sin n\pi z}{K_o\,[n^2\pi^2 + a^2 + p(1-T_o\alpha_2)]} -$$

$$\frac{T_o\hat{\alpha}_2 p\, \beta_2 d^2.\ A \sin n\pi z}{\eta_o(n^2\pi^2 + a^2 + \frac{p}{\tau})\,[n^2\pi^2 + a^2 + p(1-T_o\alpha_2)]},$$

$$[n^2\pi^2+a^2+p(1-T_o\alpha_2)](n^2\pi^2+a^2+\frac{p}{\tau})(n^2\pi^2+a^2)(n^2\pi^2+a^2+\frac{p}{\sigma}) =$$

$$-\frac{R_2a^2}{\tau}[n^2\pi^2+a^2+p(1-T_o\alpha_2)]+R_1a^2[B+(1-T_o\alpha_2)]$$

$$(n^2\pi^2+a^2+\frac{p}{\tau}) - R_1\ Bpa^2/\tau. \qquad (4.7.21)$$

Proof : In this case the boundary conditions (4.6.14) require that

$$w = \theta = \varphi = D^2w = 0 \text{ at } z = 0 \text{ and } z = 1\ , \qquad (4.7.22)$$

and therefore it follows from (4.6.6), (4.6.7) and (4.6.8) that

$$D^2\theta = D^2\varphi = D^4w = 0 \text{ at } z = 0 \text{ and } z = 1, \qquad (4.7.23)$$

Further, differentiating (4.6.6),(4,6,7) and (4.6.8) twice with respect to z and making use of (4.7.23), we conclude that

$$D^4\theta = D^4\varphi = D^6w = 0 \quad \text{at} \quad z = 0 \text{ and } z = 1. \qquad (4.7.24)$$

By further differentiation of (4.6.6), (4.6.7) and (4.6.8), we conclude, successively, that all the even derivatives of w vanish on the boundaries. Thus

$$D^{(2m)}w = 0 \text{ at } z = 0 \text{ and } z = 1 \text{ and } m=1,2,\ldots \qquad (4.7.25)$$

From this it follows that the required solution for w must be

$$w = A \sin n\pi z \qquad (n = 1,2,....). \qquad (4.7.26)$$

Eliminating φ between (4.6.7) and (4.6.8), we obtain

$$(D^2-a^2-\frac{p}{\tau})[D^2-a^2-p(1-T_o\alpha_2)]\theta + \frac{T_o\hat{\alpha}_2 p\beta_2 d^2 w}{\eta_o} =$$

$$-\frac{(1-T_o\alpha_2)\beta_1}{K_o} d^2(D^2-a^2-p/\tau)w - \frac{T_o\hat{\alpha}_2\beta_2 d^2(D^2-a^2-p/\tau)}{K_o} w. \qquad (4.7.27)$$

Operating on (4.6.6) by $[D^2-a^2-p(1-T_o\alpha_2)](D^2-a^2-p/\tau)$, we obtain

$$[D^2-a^2-p(1-T_o\alpha_2)](D^2-a^2-p/\tau)(D^2-a^2)(D^2-a^2-p/\sigma)w =$$

$$\frac{g\alpha a^2 d^2}{\nu_o}[D^2-a^2-p(1-T_o\alpha_2)](D^2-a^2-p/\tau)\theta -$$

$$\frac{g\hat{\alpha} a^2 d^2}{\nu_o}[D^2-a^2-p(1-T_o\alpha_2)](D^2-a^2-p/\tau)\varphi . \qquad (4.7.28)$$

Substituting for $[D^2-a^2-p(1-T_o\alpha_2)](D^2-a^2-p/\tau)\theta$ from (4.7.27) in (4.7.28), we obtain

$$[D^2-a^2-p(1-T_o\alpha_2)](D^2-a^2-p/\tau)(D^2-a^2)(D^2-a^2-p/\sigma)w =$$

$$\frac{R_2 a^2}{\tau}[D^2-a^2-p(1-T_o\alpha_2)]w + R_1 a^2[-B-(1-T_o\alpha_2)](D^2-a^2-p/\tau)w -$$

$$\frac{R_1 B\ pa^2 w}{\tau} . \qquad (4.7.29)$$

Making use of (4.7.26), it follows from (4.7.29) that

$$[n^2\pi^2+a^2+p(1-T_o\alpha_2)](n^2\pi^2+a^2+p/\tau)(n^2\pi^2+a^2)(n^2\pi^2+a^2+p/\sigma) =$$

$$-\frac{R_2a^2}{\tau}[n^2\pi^2+a^2+p(1-T_o\alpha_2)]+R_1a^2[B+(1-T_o\alpha_2)](n^2\pi^2+a^2+p/\tau)-$$

$$\frac{R_1Bpa^2}{\tau}. \tag{4.7.30}$$

Further, substituting for w from (4.7.26) in (4.7.27), we get

$$\theta = \frac{[(1-T_o\alpha_2)\beta_1+T_o\hat{\alpha}_2\beta_2]\ d^2.A\ \sin n\pi z}{\kappa_o[n^2\pi^2+a^2+p(1-T_o\alpha_2)]} -$$

$$\frac{T_o\hat{\alpha}_2p\beta_2d^2.\ A\ \sin n\pi z}{\eta_o(n^2\pi^2+a^2+p/\tau)[n^2\pi^2+a^2+p(1-T_o\alpha_2)]}, \tag{4.7.31}$$

and utilizing (4.7.26) and (4.7.31), (4.6.6) gives

$$\varphi = \frac{A\beta_2d^2}{\eta_o}\cdot\frac{\sin n\pi z}{n^2\pi^2+a^2+p/\tau}. \tag{4.7.32}$$

The theorem is thus proved.

The essential content of Theorem 6 from the point of view of hydrodynamic instability is this that it provides us with the exact solutions for the vertical component of perturbation velocity, perturbation temperature, perturbation concentration and the complex frequency for the case when both the bounding surface are dynamically free.

Theorem 7 : For $p_r = 0$ and $p_i \neq 0$, the exact solution of Theorem 6 implies

$$R_1a^2[1-\tau B+\sigma(1-T_o\alpha_2)] = R_2a^2(\tau+\sigma)(1-T_o\alpha_2) +$$

$$(n^2\pi^2+a^2)^3[(\tau+\sigma+\frac{1}{1-T_o\alpha_2})\{1+\frac{\tau}{\sigma}+\tau(1-T_o\alpha_2)\}-\tau], \tag{4.7.33}$$

and

$$p_i^2\frac{(n^2\pi^2+a^2)}{\sigma}[1-\tau B+\sigma(1-T_o\alpha_2)] = R_2a^2[1-\tau B-\tau(1-T_o\alpha_2)] -$$

$$\frac{\tau(n\pi^2+a^2)^3}{\sigma(1-T_o\alpha_2)}[\{\tau B+\tau(1-T_o\alpha_2)\}\{1+\sigma(1-T_o\alpha_2)\}-B\sigma]. \tag{4.7.34}$$

Proof : For $p_r = 0$, we derive from (4.7.30) that

$$[n^2\pi^2+a^2+ip_i(1-T_o\alpha_2)](n^2\pi^2+a^2+ip_i/\tau)(n^2\pi^2+a^2)(n^2\pi^2+a^2+ip_i/\sigma)=$$

$$-\frac{R_2a^2}{\tau}[n^2\pi^2+a^2+ip_i(1-T_o\alpha_2)]+R_1a^2[B+(1-T_o\alpha_2)]$$

$$(n^2\pi^2+a^2+ip_i/\tau)-\frac{iR_1Bp_ia^2}{\tau}. \tag{4.7.35}$$

Equation (4.7.35) can be rearranged as

$$[(n^2\pi^2+a^2)^2\{(n^2\pi^2+a^2)^2-p_i^2/\tau\sigma\}-p_i^2(n^2\pi^2+a^2)^2(1-T_o\alpha_2)(\frac{1}{\tau}+\frac{1}{\sigma})]+$$

$$i[\{(n^2\pi^2+a^2)^2-\frac{p_i^2}{\tau\sigma}\}\{p_i(n^2\pi^2+a^2)(1-T_o\alpha_2)\}+p_i(n^2\pi^2+a^2)^3$$

$$(\frac{1}{\tau}+\frac{1}{\sigma})] = [-\frac{R_2a^2}{\tau}(n^2\pi^2+a^2)+R_1a^2(n^2\pi^2+a^2)\{B+(1-T_o\alpha_2)\}]+$$

$$i[-\frac{R_2a^2p_i(1-T_o\alpha_2)}{\tau}+\frac{R_1a^2p_i(1-T_o\alpha_2)}{\tau}]. \tag{4.7.36}$$

Equating the real and imaginary parts of both sides of (4.7.36), we obtain

$$(n^2\pi^2+a^2)^2[(n^2\pi^2+a^2)^2-\frac{p_i^2}{\tau\sigma}]-p_i^2(n^2\pi^2+a^2)^2(1-T_o\alpha_2)(\frac{1}{\tau}+\frac{1}{\sigma}) =$$

$$-\frac{R_2a^2}{\tau}(n^2\pi^2+a^2)+R_1a^2(n^2\pi^2+a^2)[B+(1-T_o\alpha_2)], \qquad (4.7.37)$$

and

$$[(n^2\pi^2+a^2)^2-\frac{p_i^2}{\tau\sigma}][p_i(n^2\pi^2+a^2)(1-T_o\alpha_2)]+p_i(n^2\pi^2+a^2)^3(\frac{1}{\tau}+\frac{1}{\sigma}) =$$

$$-\frac{R_2a^2p_i(1-T_o\alpha_2)}{\tau}+\frac{R_1a^2p_i(1-T_o\alpha_2)}{\tau} \qquad (4.7.38)$$

Equations (4.7.37) and (4.7.38) can be rearranged as

$$p_i^2\frac{(n^2\pi^2+a^2)}{\sigma}[1+(\tau+\sigma)(1-T_o\alpha_2)] = \tau(n^2\pi^2+a^2)^3-\tau R_1a^2$$

$$[B+(1-T_o\alpha_2)]+R_2a^2, \qquad (4.7.39)$$

and

$$R_1a^2 = \tau(n^2\pi^2+a^2)^3-\frac{p_i^2(n^2\pi^2+a^2)}{\sigma}+\frac{(n^2\pi^2+a^2)^3(1+\frac{\tau}{\sigma})}{(1-T_o\alpha_2)}+R_2a^2. \qquad (4.7.40)$$

Solving (4.7.39) and (4.7.40) for R_1 and p_i^2, we obtain

$$R_1a^2[1-\tau B+\sigma(1-T_o\alpha_2)] = R_2a^2(\tau+\sigma)(1-T_o\alpha_2)+(n^2\pi^2+a^2)^3$$

$$[(\tau+\sigma+\frac{1}{1-T_o\alpha_2})\left\{1+\frac{\tau}{\sigma}+\tau(1-T_o\alpha_2)\right\}-\tau], \qquad (4.7.41)$$

and

$$\frac{p_i^2(n^2\pi^2+a^2)}{\sigma}[1-\tau B+\sigma(1-T_0\alpha_2)] = R_2a^2[1-\tau B-\tau(1-T_0\alpha_2)] -$$

$$\frac{\tau(n^2\pi^2+a^2)^3}{\sigma(1-T_0\alpha_2)}[\{\tau B+\tau(1-T_0\alpha_2)\}\{1+\sigma(1-T_0\alpha_2)\} - B\sigma], \quad (4.7.42)$$

and this proves the Theorem 7.

The essential content of Theorem 7 from the point of view of hydrodynamic instability is this that overstable solutions do exist when both the bounding surfaces are dynamically free and provides us with the exact calculations for the critical Rayleigh number and the frequency of oscillations of the overstable motions, at the marginal state with respect to them. It is important to note in this connection that in Banerjee's generalized Bénard model it is only the overstable motions that manifest at the marginal state while in Veronis's thermohaline model, the possibility of both the stationary as well as the overstable motions exist at the marginal state. However, Veronis' work gives an ample support to the proposition that overstable motions at the marginal state are the more likely ones and this is well supported by experiments

It is on the basis of the above works of Banerjee, and Veronis which are carried out for the case when both the bounding surfaces are dynamically free that we have taken $p_i \neq 0$ and $p_r = 0$ in Theorem 7 although solutions with $p_i=0$ when $p_r=0$ also exist as they exist in Veronis' thermohaline model.

For the simple Bénard problem $R_2 = \tau = B = \alpha_2 = 0 = p_i$. Equation (4.7.38) is then identically satisfied and thus the considerations of (4.7.42), which is derived from (4.7.38) on the assumption $p_i \neq 0$ are not required. We derive from (4.7.41) that

$$R_1 = \frac{(n^2\pi^2 + a^2)^3}{a^2}, \tag{4.7.43}$$

and this gives Rayleigh's result.

For the generalized Bénard problem $\tau = B = \alpha_2 = 0$, equations (4.7.41) and (4.7.42) then respectively give

$$R_1 = \frac{(n^2\pi^2 + a^2)^3}{a^2} + \frac{R_2\sigma}{1+\sigma}, \tag{4.7.44}$$

and

$$p_i^2 = \frac{R_2 a^2 \sigma}{(n^2\pi^2 + a^2)(1+\sigma)}, \tag{4.7.45}$$

and this gives Banerjee's result.

For the thermohaline problem $\alpha_2 = 0 = B$. Equations (4.7.41) and (4.7.42) then respectively give

$$R_1 = \frac{(n^2\pi^2 + a^2)^3[(\tau+\sigma+1)(1+\frac{\tau}{\sigma}+\tau)-\tau]}{a^2(1+\sigma)} + \frac{R_2(\tau+\sigma)}{1+\sigma}, \tag{4.7.46}$$

and

$$p_i^2 = \frac{\sigma}{(1+\sigma)(n^2\pi^2+a^2)}[R_2 a^2(1-\tau)] - \tau^2(n^2\pi^2+a^2)^2, \tag{4.7.47}$$

and this gives Veronis' result.

For given admissible values of a^2, τ, B, σ, $T_o\alpha_2$ and R_2, the lowest value of R_1 occurs when n=1. In this case (4.7.41) and (4.7.42) gives respectively

$$R_1 a^2[1-\tau B+\sigma(1-T_o\alpha_2)] = R_2 a^2(\tau+\sigma)(1-T_o\alpha_2) +$$

$$(\pi^2+a^2)^3[(\tau+\sigma+\frac{1}{1-T_o\alpha_2})\left\{1+\frac{\tau}{\sigma}+\tau(1-T_o\alpha_2)\right\}-\tau], \qquad (4.7.48)$$

and

$$p_i^2\,\frac{(\pi^2+a^2)}{\sigma}[1-\tau B+\sigma(1-T_o\alpha_2)] = R_2 a^2[1-\tau B-\tau(1-T_o\alpha_2)] -$$

$$\frac{\tau(\pi^2+a^2)^3}{\sigma(1-T_o\alpha_2)}[\left\{\tau B+\tau(1-T_o\alpha_2)\right\}\left\{1+\sigma(1-T_o\alpha_2)\right\}-B\sigma]. \qquad (4.7.49)$$

In the following table we show the variations in R_1 and p_i^2 as given by equations (4.7.48),(4.7.49) respectively for various values of $T_o\alpha_2$ when $a^2 = 4.9348$, $\tau = .01$, $B = -T_o\alpha_2$, $\sigma = 7$ and $R_2 = 500$.

TABLE IV

The variations in R_1 and p_i^2 with respect to $T_o\alpha_2$ for $\tau \neq 0$

$T_o\alpha_2$	R_1	p_i^2
0	1103	144
.30	1362	198
.31	1375	200
.32	1389	203
.33	1401	206

The above table clearly shows that for $\alpha_2 > 0$ the critical Rayleigh number and the square of the frequency of oscillations increase with increasing values of $T_0\alpha_2$. In other words hydrodynamic instability manifests itself earlier for a cooler layer than for a relatively hotter layer of the same fluid when the Rayleigh number is gradually and gradually increased and further the square of the frequency of oscillations also is smaller for a cooler layer than a relatively hotter layer of the same fluid. The quantitative changes brought about by the extended theory are surely significant as shown by the above table and definitely give the scope of being detected in a laboratory. These qualitative and quantitative aspects of the simple Bénard problem, the generalized Bénard problem and the thermohaline problem are not contained in the theories of Rayleigh, Banerjee and Veronis respectively because of the usual application of the Boussinesq approximation in their calculations.

Theorem 8 : For $\tau = B = 0$ and $n = 1$

$$R_1 = \frac{(\pi^2+a^2)^3}{a^2(1-T_0\alpha_2)} + \frac{R_2\sigma(1-T_0\alpha_2)}{[1+\sigma(1-T_0\alpha_2)]}, \qquad (4.7.50)$$

and

$$p_i^2 = \frac{R_2 a^2\sigma}{(\pi^2+a^2)[1+\sigma(1-T_0\alpha_2)]}. \qquad (4.7.51)$$

Proof : For $\tau = B = 0$ and $n = 1$, (4.7.33) and (4.7.34) become

$$R_1 a^2[1+\sigma(1-T_o\alpha_2)] = R_2 a^2\sigma(1-T_o\alpha_2)+(\pi^2+a^2)^3\left(\sigma+\frac{1}{1-T_o\alpha_2}\right), \tag{4.7.52}$$

and $$p_i^2\,\frac{(\pi^2+a^2)}{\sigma}\,[1+\sigma(1-T_o\alpha_2)] = R_2 a^2. \tag{4.7.53}$$

It follows from (4.7.52) (4.7.53) respectively that

$$R_1 = \frac{(\pi^2+a^2)^3}{a^2(1-T_o\alpha_2)} + \frac{R_2\sigma(1-T_o\alpha_2)}{[1+\sigma(1-T_o\alpha_2)]}, \tag{4.7.54}$$

and $$p_i^2 = \frac{R_2 a^2\sigma}{(\pi^2+a^2)[1+\sigma(1-T_o\alpha_2)]}, \tag{4.7.55}$$

and this proves the theorem.

Theorem 8 Specifically improves upon Banerjee's result concerning the generalized Benard model and we show this in the following table taking $a^2 = 4.9348$, $\sigma = 7$ and $R_2 = 500$.

TABLE V

The variations in R_1 and p_i^2 with respect to $T_o\alpha_2$ for $\tau = 0$

$T_o\alpha_2$	R_1	p_i^2
0	1095	146
.30	1355	198
.31	1367	200
.32	1380	203
.33	1393	205

Theorem 9 : For $\tau = B = 0$ and $n = 1$

(i) $\dfrac{\partial R_1}{\partial T_o} > 0$ for $\alpha_2 > 0$ provided $R_2 < \dfrac{27}{4}\pi^4\sigma$, (4.7.56)

and (ii) $\dfrac{\partial R_1}{\partial T_o} < 0$ for $\alpha_2 < 0$ provided $R_2 < \dfrac{27}{4}\pi^4\sigma$. (4.7.57)

Proof : For $\tau = B = 0$ and $n = 1$, R_1 is given by (4.7.54). From this expression for R_1, we obtain upon differentiation with respect to T_o

$$\frac{\partial R_1}{\partial T_o} = \frac{(\pi^2+a^2)^3\ \alpha_2}{a^2(1-T_o\alpha_2)^2} - \frac{R_2\sigma\ \alpha_2}{[1+\sigma(1-T_o\alpha_2)]^2}\ , \tag{4.7.58}$$

or
$$\frac{1}{\alpha_2}\frac{\partial R_1}{\partial T_o} = \frac{(\pi^2+a^2)^3}{a^2(1-T_o\alpha_2)^2} - \frac{R_2\sigma}{[1+\sigma(1-T_o\alpha_2)]^2}\ , \tag{4.7.59}$$

or
$$\frac{1}{\alpha_2}\frac{\partial R_1}{\partial T_o} = \frac{1}{(1-T_o\alpha_2)^2}\left[\frac{(\pi^2+a^2)^3}{a^2} - \frac{R_2\sigma(1-T_o\alpha_2)^2}{[1+\sigma(1-T_o\alpha_2)]^2}\right], \tag{4.7.60}$$

or
$$\frac{1}{\alpha_2}\frac{\partial R_1}{\partial T_o} = \frac{1}{(1-T_o\alpha_2)^2}\left[\frac{(\pi^2+a^2)^3}{a^2} - \frac{R_2\sigma^2(1-T_o\alpha_2)^2}{\sigma[1+\sigma(1-T_o\alpha_2)]^2}\right]. \tag{4.7.61}$$

Now, since $\dfrac{(\pi^2+a^2)^3}{a^2} \geq \dfrac{27}{4}\pi^4$ for all admissible values of a^2, (4.7.62)

and $\dfrac{\sigma^2(1-T_o\alpha_2)^2}{[1+\sigma(1-T_o\alpha_2)]^2} \leq 1$ for all admissible values of $T_o\alpha_2$, (4.7.63)

it follows from (4.7.61) that

$$\frac{1}{\alpha_2}\frac{\partial R_1}{\partial T_o} \geq \frac{1}{(1-T_o\alpha_2)^2}\left[\frac{27\pi^4}{4} - \frac{R_2}{\sigma}\right]. \tag{4.7.64}$$

Thus we derive from (4.7.64) that

$$\text{(i)} \quad \frac{\partial R_1}{\partial T_o} \geq 0 \text{ for } \alpha_2 > 0 \text{ provided } R_2 < \frac{27}{4}\pi^4\sigma, \tag{4.7.65}$$

and (ii) $\frac{\partial R_1}{\partial T_o} < 0$ for $\alpha_2 < 0$ provided $R_2 < \frac{27}{4}\pi^4\sigma$, (4.7.66)

and this proves Theorem 9.

Theorem 9 implies that hydrodynamic instability in the generalised Bénard model in our extended framework manifests itself earlier or is postponed for $\alpha_2 < 0$ or $\alpha_2 > 0$ respectively for increasing values of T_o provided the initial nonhomogeneity number R_2 is less than σ times $\frac{27}{4}\pi^4$.

Theorem 10 : For $\tau = B = 0$ and $n = 1$

$$\frac{\partial R_1}{\partial T_o} > 0 \quad \text{for } \alpha_2 > 0 \text{ provided } R_2 < \frac{27\pi^4}{4\sigma}, \tag{4.7.67}$$

Proof : For $\tau = B = 0$ and $n=1$, we have from (4.7.59) that

$$\frac{1}{\alpha_2}\frac{\partial R_1}{\partial T_o} = \frac{(\pi^2+a^2)^3}{a^2(1-T_o\alpha_2)^2} - \frac{R_2\sigma}{[1+\sigma(1-T_o\alpha_2)]^2}. \tag{4.7.68}$$

Now, since $(1-T_o\alpha_2)^2 \leq 1$ for all positive admissible values of $T_o\alpha_2$, (4.7.69) and $\frac{1}{[1+\sigma(1-T_o\alpha_2)]^2} \leq 1$ for all positive admissible values of $T_o\alpha_2$, (4.7.70)

It follows from (4.7.68) that

$$\frac{1}{\alpha_2}\frac{\partial R_1}{\partial T_o} \geq \frac{(\pi^2+a^2)^3}{a^2} - R_2\sigma . \qquad (4.7.71)$$

Further since $\frac{(\pi^2+a^2)^3}{a^2} \geq \frac{27\pi^4}{4}$ for all admissible value of a^2,

(4.7.72)

We derive from (4.7.71) that

$$\frac{1}{\alpha_2}\frac{\partial R_1}{\partial T_o} \geq \frac{27}{4}\pi^4 - R_2\sigma . \qquad (4.7.73)$$

The conclusion of Theorem 10 then follows from (4.7.73). This proves the theorem. The nature of Theorem 10 is similar to that of Theorem 9.

Theorem 11 : For $\tau = B = 0$ and $n = 1$

$$\frac{\partial R_1}{\partial T_o} > 0 \quad \text{for } \alpha_2 > 0 \text{ provided } a^2 < \frac{\pi^6}{R_2\sigma} . \qquad (4.7.74)$$

Proof : For $\tau = B = 0$ and $n=1$, we have from (4.7.59) that

$$\frac{1}{\alpha_2}\frac{\partial R_1}{\partial T_o} = \frac{(\pi^2+a^2)^3}{a^2(1-T_o\alpha_2)^2} - \frac{R_2\sigma}{[1+\sigma(1-T_o\alpha_2)]^2} , \qquad (4.7.75)$$

or $$\frac{1}{\alpha_2}\frac{\partial R_1}{\partial T_o} > \frac{\pi^6}{a^2(1-T_o\alpha_2)^2} - \frac{R_2\sigma}{[1+\sigma(1-T_o\alpha_2)]^2} . \qquad (4.7.76)$$

Now, since $\frac{1}{(1-T_o\alpha_2)^2} \geq 1$ for all positive admissible values of $T_o\alpha_2$, (4.7.77)

and $\frac{1}{[1+\sigma(1-T_o\alpha_2)]^2} \leq 1$ for all positive admissible values of $T_o\alpha_2$, (4.7.78)

it follows from (4.7.76) that

$$\frac{1}{\alpha_2}\ \frac{\partial R_1}{\partial T_o} > \frac{\pi^6}{a^2} - R_2\sigma\ . \tag{4.7.79}$$

Therefore if $a^2 < \pi^6/R_2\sigma$ and $\alpha_2 > 0$, we derive from (4.7.79) that $\partial R_1/\partial T_o > 0$ and this proves Theorem 11.

The nature of Theorem 11 is similar to that of Theorem 9 and Theorem 10 except that the results contained in Theorem 11 are dependent on the square of the wave number a^2 but independent of the nonhomogeneity number R_2 in the sense that it can take any admissible prescribed value.

Theorem 12: (A circle theorem for modified generalized Bénard convection)

For $\tau = 0 = \hat{\alpha}_2$ and $p_i \neq 0$, a necessary condition for the existence of nontrivial solutions for w, ϑ and φ satisfying (4.6.13) or (4.6.14) or (4.6.15) is that

$$|p|^2 < R_2\sigma. \tag{4.7.80}$$

Proof : For $\tau = 0$, we have from (4.6.12) that

$$p\,\varphi = R_3 w. \tag{4.7.81}$$

Making use of (4.7.81) and $p_i \neq 0$, (4.6.10) and (4.6.11) respectively become

$$(D^2-a^2)(D^2-a^2-\frac{p}{\sigma})w = R_1 a^2 \theta - \frac{p^* R_2 a^2 w}{|p|^2}, \qquad (4.7.82)$$

and $$[D^2-a^2-p(1-T_o\alpha_2)]\,\theta = -(1-T_o\alpha_2)w. \qquad (4.7.83)$$

Multiplying both sides of (4.7.82) by w^* and integrating the resulting equation over the vertical range of z, we obtain

$$\int_0^1 w^*(D^2-a^2)(D^2-a^2-\frac{p}{\sigma})w\,dz = R_1 a^2 \int_0^1 w^* \theta\,dz - \frac{p^* R_2 a^2}{|p|^2}\int_0^1 |w|^2 dz. \qquad (4.7.84)$$

Substituting the value of $\int_0^1 w^*\,dz$ in (4.7.84) from (4.7.83), we have

$$\int_0^1 w^*(D^2-a^2)(D^2-a^2-\frac{p}{\sigma})w\,dz = R_1 a^2\Big[-\frac{1}{1-T_o\alpha_2}\int_0^1 \theta\{D^2-a^2-p^*(1-T_o\alpha_2)\}\theta^* dz - \frac{p^* R_2 a^2}{|p|^2}\int_0^1 |w|^2 dz\Big]. \qquad (4.7.85)$$

Making use of (4.6.13) or (4.6.14) or (4.6.15), the above equation becomes

$$\int_0^1 [|D^2 w|^2 + 2a^2|Dw|^2 + a^4|w|^2]dz + \frac{p}{\sigma}\int_0^1 [|Dw|^2+a^2|w|^2]dz =$$

$$\frac{R_1 a^2}{1-T_o\alpha_2}\int_0^1 [|D\theta|^2+a^2|\theta|^2] + R_1 a^2 p^* \int_0^1 |\theta|^2 dz - \frac{p^* R_2 a^2}{|p|^2}\int_0^1 |w|^2 dz. \qquad (4.7.86)$$

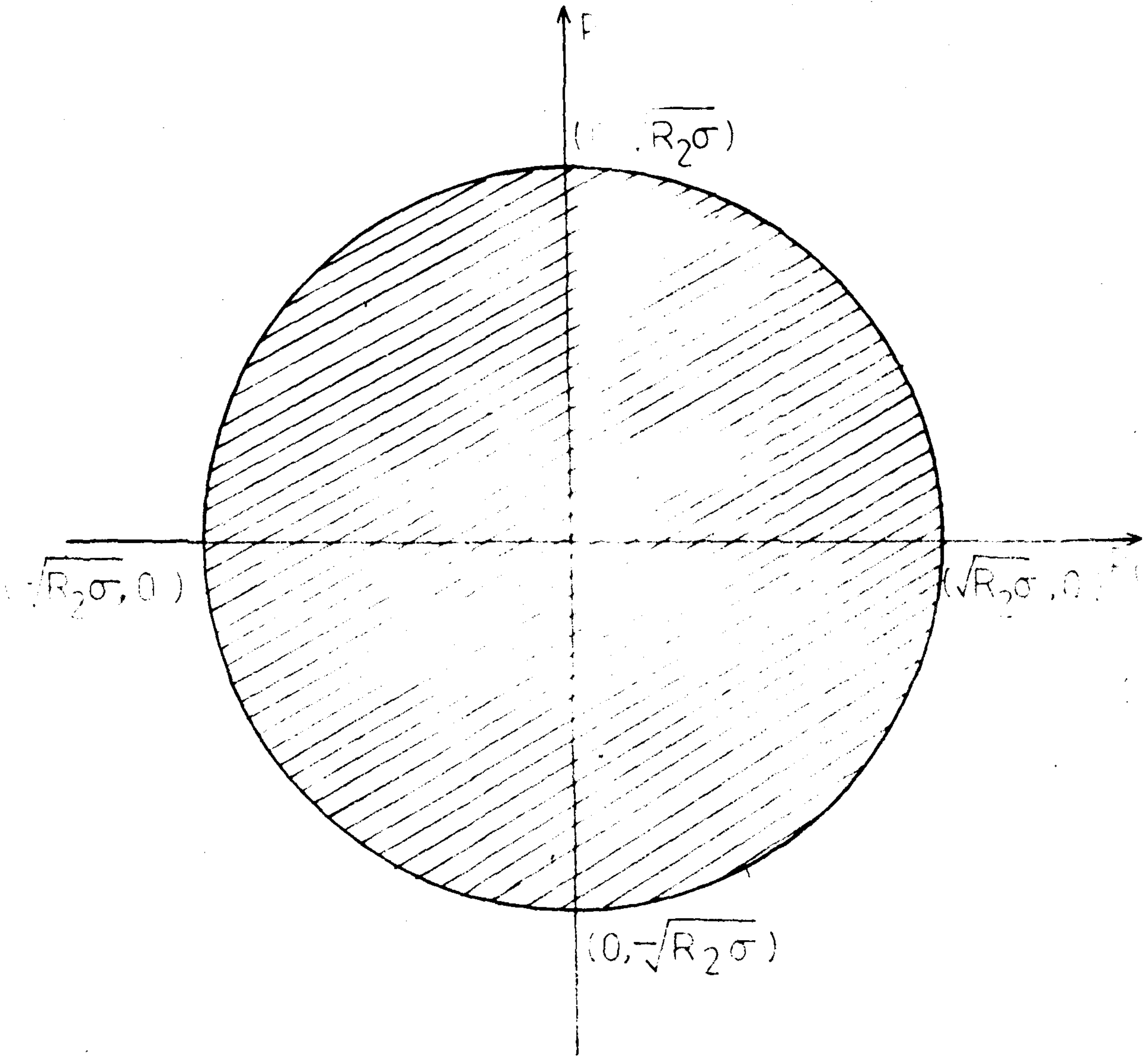

Fig. : Shaded area shows the region in which the complex growth rate of an arbitrary oscillatory mode, neutral or ...table, in generalized Bénard configuration must li...

Equating the imaginary parts of both sides of (4.7.86) and using the constraint $p_i \neq 0$,

$$\frac{1}{\sigma}\int_0^1 |Dw|^2 dz + a^2\int_0^1 \left(\frac{1}{\sigma} - \frac{R_2}{|p|^2}\right)|w|^2 dz + R_1 a^2 \int_0^1 |\Theta|^2 dz = 0. \tag{4.7.87}$$

Equation (4.7.87) implies that

$$|p|^2 < R_2\sigma \qquad \text{(see Fig.(e))}, \tag{4.7.88}$$

and this proves Theorem 12.

The essential content of Theorem 12 from the point of view of hydrodynamic instability is this that it provides us with an upper bound for the complex growth rate of an arbitrary oscillatory perturbation in the generalized Bénard model in the present framework. The theorem states that the complex growth rate of an arbitrary oscillatory perturbation in the generalized Bénard model (in the present framework) in the $p_r p_i$ plane must lie inside a circle whose centre is the origin and radius $\sqrt{R_2\sigma}$. It is noted here in this connection that for the case when both the bounding surfaces are free the exact solution for p_i^2 (p_r=0) which is given by (4.7.51) and depends on $T_o\alpha_2$ does satisfy (4.7.88) i.e.

$$p_i^2 = \frac{R_2 a^2\sigma}{(\pi^2+a^2)[1+\sigma(1-T_o\alpha_2)]} < R_2\sigma. \tag{4.7.89}$$

Further, (4.7.88) holds good also, as shown above, for the cases when both the boundaries are rigid or one boundary rigid and one boundary free. Specially in these cases (4.7.88) would be very useful since calculations become complicated.

4.8. Contributions of Chapter -4

(i) Shows that there is a nontrivial coupling, not pointed out as yet to the best of our knowledge, between temperature and concentration fields through the equation of heat conduction in thermohaline instability problems.

(ii) Shows, for the first time to the best of our knowledge, that a more consistent and relatively more accurate application of Boussinesq approximation leads to a formulation of thermal and thermohaline instability problems which does significantly depend upon whether the fluid layer is relatively hotter or cooler and this is on account of the variations in the specific heat at constant volume due to variations in temperature and/or concentration.

(iii) Establishes, on a more sound basis, the results of Pellew and Southwell (1940) and Banerjee (1972) regarding the character of the marginal state in the simple Bénard problem and generalised Bénard problem respectively.

(iv) Shows that a better agreement could be achieved between theory and experiments in the field of thermal instability of a liquid layer heated underside provided the classical theory is modified along presently suggested lines.

(v) Exactly solves the problems of generalized Bénard convection and thermohaline convection with free boundaries for oscillatory modes and distinguishes between a hotter and a cooler layer as regards the onset of instability.

(vi) Establishes bounds for the growth rate of an arbitrary oscillatory perturbation in the generalized Bénard problem.