PART - II

CHAPTER - 3

BASIS AND CONSTRUCTION OF THE MODIFIED SIMPLIFIED EQUATIONS GOVERNING THE WORK IN THE SUBSEQUENT CHAPTERS

3.1. Basis of the modified simplified governing equations

The point that we wish to emphasise and utilize in the present dissertation is this that linear theoretical explanation of the phenomenon of gravity dominated thermal instability in a liquid layer heated underside (Bénard convection) should depend not only upon the Rayleigh number which is proportional to the uniform temperature difference maintained across the layer but also upon another parameter so that a provision could be made in the theory to recognise the fact that a relatively hotter layer with its heat diffusivity apparently increased/decreased as a consequence of an actual decrease/increase (depending upon the fluid) in its specific heat at constant volume must exhibit Bénard convection at a higher/lower temperature difference across the layer and hence at a higher/lower Rayleigh number than a cooler layer under identical conditions otherwise and further that this qualitative effect is not quantitatively insignificant.

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To make the argument more explicit consider two static liquid layers L_1 and L_2 of the same liquid and same depth with respective maintained uniform temperatures of lower and upper boundaries as $T_o^{L_1}$ and $T_o^{L_2}$, and $T_1^{L_1}$ and $T_1^{L_2}$ where $T_o^{L_1} > T_1^{L_1}$ and $T_o^{L_2} > T_1^{L_2}$ so that uniform adverse temperature differences

$$\left.\begin{aligned} \Delta T^{L_1} &= T_o^{L_1} - T_1^{L_1} , \\ \Delta T^{L_2} &= T_o^{L_2} - T_1^{L_2} . \end{aligned}\right\} \qquad (2.1.1)$$

are respectively acting on L_1 and L_2. An application of the consequences of Rayleigh's theory implies that by raising ΔT^{L_1} and ΔT^{L_2} within limits and thereby keeping all other parameters fixed in a suitable framework we can bring about thermal instability to manifest itself in L_1 and L_2 and further that when it occurs we must have $\Delta T^{L_1} = \Delta T^{L_2}$ irrespective of whether $T_o^{L_1} > T_o^{L_2}$ or $T_o^{L_1} < T_o^{L_2}$. In otherwords Rayleigh's theory of Bénard convection does not distinguish whether the layer is hotter or cooler and predicts the same Rayleigh number at the onset of instability in both the cases which is contrary to physical intuition and not in very good agreement with the experimental observations as mentioned earlier. The origin of the above conclusion lies in the Boussinesq approximation, which Rayleigh utilizes, wherein the variability in the density and in other coefficients is due to variations in temperature of only moderate amounts. Since the coefficient of volume expansion of liquids and gases such as we are mostly concerned with is in the range 10^{-3} to 10^{-4} it follows that for variations in temperature not exceeding 10° (say), the variations in the density are at most one percent and as a consequence the variations in the other coefficients must be of the same order. In the usual application of the Boussinesq approximation, the variations of this small amount are, in general, ignored in all the terms of the governing equations with the exception of the term that contains the external force and the theory that results does not either give a complete qualitative picture of Bénard convection or can be said to be in very good quantitative agreement with the experimental observations.

Remark 1 : We shall see later in this chapter that Rayleigh's utilization of the Boussinesq approximation as mentioned earlier overlooks a term in the equation of heat conduction which is on account of the variations in the specific heat at constant volume due to variations in temperature and which is such that in the usual circumstances it cannot be ignored if the Boussinesq approximation were to be consistently and relatively more accurately applied throughout the analysis. The essential argument on which this term finds a place in our modified theory is this that it is the temperature differences which are of moderate amounts but not necessarily the temperature itself and an incorporation of this term into the calculations adequately completes the qualitative and the quantitative gaps in Rayleigh's theory as pointed out above.

The extended argument when analysed in the context of generalised Bénard convection and the phenomenon of thermohaline convection shows the incompleteness of the well known theoretical treatments of Banerjee and Veronis respectively which are based on the usual application of the Boussinesq approximation and as a consequence do not distinguish whether the layer is hotter or cooler (due to temperature as in Bénard convection), or thinner or denser (due to concentration) and predict the same Rayleigh number at the onset of instability in all the cases in some appropriate framework. Similar remarks can be made on the mathematical investigations which are based on the usual application of the Boussinesq approximation of the more general flow situations wherein one considers the generalized Bénard convection and the phenomenon of thermohaline convection under the individual and simultaneous effects of rotation and magnetic field.

Remark 2 : We shall see later in this chapter that these more general flow situations which include generalized Bénard convection and the phenomenon of thermohaline convection when investigated in the light of the extended argument give rise to interesting qualitative and quantitative possibilities since in all these cases the variations in the specific heat at constant volume is on account of the twin effects of the variations due to temperature and concentration and a usual application of the Boussinesq approximation not only overlooks a term that corresponds to that in Bénard convection but also ignores a term which is simultaneously temperature and concentration dependent, and these steps are not justified on previously stated accounts. As before, the essential argument on which these terms find a place in our modified theory is this that it is the temperature differences which are of moderate amounts but not necessarily the temperature itself and an incorporation of these terms into the calculations adequately completes the qualitative and quantitative gaps in the theories mentioned above though we do not, to the best of our knowledge, as yet have the precise experimental results to give support to our assertions.

3.2. Construction of the modified simplified governing equations

In the following investigation we construct the modified simplified equations governing the flow situations stated above by making a more careful mathematical analysis and appropriately retaining in the governing equations the variations in the density and in the other coefficients on account of variations in temperature of moderate amounts.

(a) Modified equations for Bénard convection

The basic hydrodynamic equations that govern the problem are

(i) The equation of continuity

$$\frac{\partial \rho}{\partial t} + u_j \frac{\partial \rho}{\partial x_j} + \rho \frac{\partial u_j}{\partial x_j} = 0 . \qquad (3.2.1)$$

(ii) The equation of motion

$$\rho\left(\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j}\right) = \rho X_i - \frac{\partial P}{\partial x_i} + \frac{\partial}{\partial x_j}\left\{\mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) - \frac{2}{3}\mu \frac{\partial u_k}{\partial x_k}\right\} . \qquad (3.2.2)$$

(iii) The equation of heat conduction

$$\frac{\partial}{\partial t}(\rho c_v T) + \frac{\partial}{\partial x_j}(\rho c_v T u_j) = \frac{\partial}{\partial x_j}\left(K \frac{\partial T}{\partial x_j}\right) - \frac{p\,\partial u_j}{\partial x_j} + \varphi , \qquad (3.2.3)$$

where

$$\varphi = 2\mu e_{ij}^2 - \frac{2}{3} u (e_{jj})^2, \qquad (3.2.4)$$

and

$$e_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \qquad (3.2.5)$$

and making use of (3.2.1), we can simplify (3.2.3) to the form

$$\rho\left[\frac{\partial}{\partial t}(c_v T) + u_j \frac{\partial}{\partial x_j}(c_v T)\right] = \frac{\partial}{\partial x_j}\left(K \frac{\partial T}{\partial x_j}\right) - p \frac{\partial u_j}{\partial x_j} + \varphi. \qquad (3.2.6)$$

The above basic hydrodynamic equations must be supplemented by equations of state and for substances with which we shall be primarily concerned, we can write

$$\left.\begin{aligned} \rho &= \rho_o [1 + \alpha(T_o - T)], \\ \mu &= \mu_o [1 + \alpha_1(T_o - T)], \\ c_v &= c_{v_o} [1 + \alpha_2(T_o - T)], \\ K &= K_o [1 + \alpha_3(T_o - T)], \\ \text{and } \alpha &= \alpha_o [1 + \gamma (T_o - T)], \text{ etc.} \end{aligned}\right\}, \qquad (3.2.7)$$

where α, α_1, α_2, α_3, γ etc. are in the range 10^{-3} to 10^{-4}. Applying the usual Boussinesq approximation which in essence amounts to neglecting terms which are of order 10^{-3} at most as compared to 1 for variations in temperature of order 1^o (say), we obtain, as in Rayleigh's theory the simplified forms of (3.2.1) and (3.2.2), namely

$$\frac{\partial u_i}{\partial x_j} = 0, \tag{3.2.8}$$

and

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho_o}\frac{\partial P}{\partial x_i} + (1 + \frac{\delta\rho}{\rho_o})X_i + \frac{\mu_o}{\rho_o}\nabla^2 u_i, \tag{3.2.9}$$

$$\text{where} \quad \delta\rho = -\rho_o \alpha_o (T - T_o). \tag{3.2.10}$$

The simplification of the equation of heat conduction is the point where our system of equations departs from Rayleigh's and we make this point explicitly clear by showing a step by step analysis in the following detailed calculations.

Using (3.2.7), we obtain from (3.2.6)

$$\rho_o\left\{1 + \alpha(T_o - T)\right\}\left[\frac{\partial}{\partial t}\left\langle c_{v_o}\left\{1 + \alpha_2(T_o-T)\right\}T\right\rangle + u_j\frac{\partial}{\partial x_j}\left\langle c_{v_o}\left\{1 + \alpha_2(T_o-T)\right\}T\right\rangle\right] = \frac{\partial}{\partial x_j}\left\langle K_o\left\{1+\alpha_3(T_o-T)\right\}\frac{\partial T}{\partial x_j}\right\rangle - p\frac{\partial u_j}{\partial x_j} + 2\mu_o\left\{1+\alpha_1(T_o-T)\right\}e_{ij}^2 - \frac{2}{3}\mu_o\left\{1+\alpha_1(T_o-T)\right\}(e_{jj})^2. \tag{3.2.11}$$

We now confine our attention on the right hand side of (3.2.11) which can be simplified to the form

$$\frac{\partial}{\partial x_j}\left\langle K_o\left\{1+\alpha_3(T_o-T)\right\}\frac{\partial T}{\partial x_j}\right\rangle + 2\mu_o\left\{1 + \alpha_1(T_o-T)\right\}e_{ij}^2, \tag{3.2.12}$$

by making use of (3.2.8) and therefore the above simplification is an outcome of the usual Boussinesq approximation. The expression in (3.2.12) can be written in the more elaborate form

$$K_o \left\{ 1+\alpha_3(T_o-T)\right\} \frac{\partial^2 T}{\partial x_j^2} - K_o\alpha_3\left(\frac{\partial T}{\partial x_j}\right)^2 + 2\mu_o\left\{1+\alpha_1(T_o-T)\right\} e_{ij}^2 \qquad (3.2.13)$$

and since $d \sim 1$ cm and the temperature differences are of moderate amounts, we can approximate the above expression with the help of the usual Boussinesq approximation by

$$K_o \frac{\partial^2 T}{\partial x_j^2} + 2\mu_o e_{ij}^2 , \qquad (3.2.14)$$

of which the first term represents the contribution due to heat conduction while the second represents the contribution due to viscous heat dissipation. Further, according to (3.2.9) and (3.2.10) the prevailing velocities are of the order $\sqrt{\alpha_o \Delta T\, |X|\, d}$ and consequently the term in heat dissipation in (3.2.14) is of the order

$$\mu_o\alpha_o\, |X|\, d/K_o , \qquad (3.2.15)$$

relative to the term arising from heat conduction in (3.2.14) and this ratio for ordinary liquids (such as water and mercury) is 10^{-7} or 10^{-8} for $|X| \sim g$. Under these circumstances the right hand side of the equation of heat conduction simplifies to

$$K_o \frac{\partial^2 T}{\partial x_j^2} , \qquad (3.2.16)$$

and this is precisely the result in Rayleigh's theory also. The left hand side of (3.2.11) when written in the elaborate form gives

$$\rho_o c_{v_o}\left\{1+\alpha(T_o-T)\right\} \left[\left\{1+\alpha_2(T_o-T)\right\} \frac{\partial T}{\partial t} - \alpha_2 T \frac{\partial T}{\partial t} + u_j \left\{1+\alpha_2(T_o-T)\right\} \frac{\partial T}{\partial x_j} - \alpha_2 T\, u_j \frac{\partial T}{\partial x_j}\right], \qquad (3.2.17)$$

which can be written in the form

$$\rho_o c_{v_o}\left\{1+\alpha(T_o-T)\right\}\left\{1+\alpha_2(T_o-T) - \alpha_2 T\right\}\left[\frac{\partial T}{\partial t} + u_j \frac{\partial T}{\partial x_j}\right]. \qquad (3.2.18)$$

Now applying the usual Boussinesq approximation to the expression in (3.2.18) we can approximate it, and as a consequence the left hand side of the equation of the heat conduction can be approximated by

$$\rho_o c_{v_o} (1-\alpha_2 T) \left(\frac{\partial T}{\partial t} + u_j \frac{\partial T}{\partial x_j}\right) , \qquad (3.2.19)$$

and combining the results obtained in (3.2.11), (3.2.16) and (3.2.19), we obtain the modified simplified form of the equation of heat conduction as

$$(1 - \alpha_2 T)\left(\frac{\partial T}{\partial t} + u_j \frac{\partial T}{\partial x_j}\right) = \kappa_o \nabla^2 T . \qquad (3.2.20)$$

The above equation differs from its classical counterpart by the multiplication of the factor $(1-\alpha_2 T)$ instead of 1 on the left hand side and since it is the temperature differences and not the temperature itself which are of moderate amounts, we must not ignore without justification $\alpha_2 T$ as compared to 1. To understand the implications of the neglect of $\alpha_2 T$ as compared to 1, as is done in Rayleigh's theory, in its correct perspective, we make an explicit calculation of the various quantities involved in the experiments of Schmidt and Milverton who used water and rigid boundaries. In their set up $d \sim 1$ cm,

$$\alpha = 2x10^{-3}, \quad \alpha_2 \sim 10^{-4}, \quad T_0 - T \sim 1^{\circ} \text{ absolute}$$

and $T_0 = 290^{\circ}$ absolute and therefore $\alpha(T_0 - T) \sim 2x10^{-3}$ while $\alpha_2 T \sim 2.9x10^{-2}$. The usual application of the Boussinesq approximation in this case implies that quantitites of the order 10^{-3} (due to $\alpha(T_0 - T)$ can be ignored as compared to 1 but that certainly does not permit, with justification, the neglect of a quantity of order 10^{-2} (due to $\alpha_2 T$) as compared to 1 and this is precisely where we wish to modify Rayleigh's theory. In fact, for the problems of the type of Bénard convection, the neglect of this term cannot be justified because the term $(1 - \alpha_2 T)$ gets multiplied with g and produces important contributions and what is perhaps most likely is that Rayleigh and subsequent researchers never derived the simplified version of the equation of heat conduction in the form

(3.2.20) so that the unsatisfactory omission could be rectified.

The modified simplified equations governing Bénard convection are thus given by

$$\frac{\partial u_i}{\partial x_j} = 0, \qquad (3.2.21)$$

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = - \frac{1}{\rho_0} \frac{\partial P}{\partial x_i} + (1 + \frac{\delta\rho}{\rho_0}) X_i + \nu_0 \nabla^2 u_i, \qquad (3.2.22)$$

$$(1-\alpha_2 T) \left(\frac{\partial T}{\partial t} + u_j \frac{\partial T}{\partial x_j}\right) = \kappa_0 \nabla^2 T, \qquad (3.2.23)$$

and

$$\rho = \rho_0 [1 + \alpha(T_0 - T)] , \qquad (3.2.24)$$

which differs from its classical counterpart by the multiplicative factor $(1 - \alpha_2 T)$ instead of 1 on the left hand side of the equation of heat conduction.

(b) Modified equations for generalized Bénard convection/ thermohaline convection

The basic hydrodynamic equations that govern the problem are

(i) The equation of continuity

$$\frac{\partial \rho}{\partial t} + u_j \frac{\partial \rho}{\partial x_j} + \rho \frac{\partial u_j}{\partial x_j} = 0. \qquad (3.2.25)$$

(ii) The equation of motion

$$\rho\left(\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j}\right) = \rho X_i - \frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j}\left[\mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) - \frac{2}{3}\mu \frac{\partial u_k}{\partial x_k}\right]. \qquad (3.2.26)$$

(iii) The equation of heat conduction

$$\rho\left[\frac{\partial}{\partial t}(c_v T) + u_j \frac{\partial}{\partial x_j}(c_v T)\right] = \frac{\partial}{\partial x_j}\left(K \frac{\partial T}{\partial x_j}\right) - p \frac{\partial u_j}{\partial x_j} + \varphi . \qquad (3.2.27)$$

(iv) The equation of mass diffusion

$$\frac{\partial S}{\partial t} + u_j \frac{\partial S}{\partial x_j} = \eta \nabla^2 S. \qquad (3.2.28)$$

The above basic hydrodynamic equations must be supplemented by equations of state and for substances with which we shall be primarily concerned we can write

$$\left.\begin{aligned}
\rho &= \rho_0[1+\alpha(T_0-T) - \hat{\alpha}(S_0-S)], \\
\mu &= \mu_0[1+\alpha_1(T_0-T) - \hat{\alpha}_1(S_0-S)], \\
c_v &= c_{v_0}[1+\alpha_2(T_0-T) - \hat{\alpha}_2(S_0-S)], \\
K &= K_0[1+\alpha_3(T_0-T) - \hat{\alpha}_3(S_0-S)], \\
\eta &= \eta_0[1+\alpha_4(T_0-T) - \hat{\alpha}_4(S_0-S)], \\
\text{and } \alpha &= \alpha_0[1+\gamma(T_0-T) - \hat{\gamma}(S_0-S)], \text{etc.}
\end{aligned}\right\}, \qquad (3.2.29)$$

where α, $\hat{\alpha}$, α_1, $\hat{\alpha}_1$, α_2, $\hat{\alpha}_2$, α_3, $\hat{\alpha}_3$, α_4, $\hat{\alpha}_4$, γ, $\hat{\gamma}$ etc. are in the range 10^{-3} to 10^{-4}.

The simplified forms of the equations of continuity and motion on the basis of the usual Boussinesq approximation are

$$\frac{\partial u_i}{\partial x_j} = 0, \tag{3.2.30}$$

and

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho_o}\frac{\partial P}{\partial x_i} + \left(1+\frac{\delta\rho}{\rho_o}+\frac{\hat{\delta\rho}}{\rho_o}\right)X_i + \nu_o \nabla^2 u_i, \tag{3.2.31}$$

where $\delta\rho = -\rho_o \alpha_o (T-T_o),$ (3.2.32)

and $\hat{\delta\rho} = \rho_o \hat{\alpha}_o (S-S_o).$ (3.2.33)

Simplifying the equation of heat conduction on the basis of the remarks made earlier we obtain from (3.2.27), (3.2.29) and (3.2.30)

$$(1-\alpha_2 T)\left(\frac{\partial T}{\partial t} + u_j \frac{\partial T}{\partial x_j}\right) + \hat{\alpha}_2 T\left(\frac{\partial S}{\partial t} + u_j \frac{\partial S}{\partial x_j}\right) = K_o \nabla^2 T. \tag{3.2.34}$$

The equation of mass diffusion becomes

$$\frac{\partial S}{\partial t} + u_j \frac{\partial S}{\partial x_j} = \eta_o \nabla^2 S. \tag{3.2.35}$$

The modified simplified equations governing generalized Bénard convection ($\eta_o = 0$) and the phenomenon of thermohaline convection are thus given by

$$\frac{\partial u_j}{\partial x_j} = 0, \tag{3.2.36}$$

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho_o}\frac{\partial p}{\partial x_i} + \left(1 + \frac{\delta\rho}{\rho_o} + \frac{\hat{\delta\rho}}{\rho_o}\right)X_i + \nu_o \nabla^2 u_i \, , \quad (3.2.37)$$

$$(1-\alpha_2 T)\left(\frac{\partial T}{\partial t} + u_j \frac{\partial T}{\partial x_j}\right) + \hat{\alpha}_2 T\left(\frac{\partial S}{\partial t} + u_j \frac{\partial S}{\partial x_j}\right) = K_o \nabla^2 T, \quad (3.2.38)$$

$$\frac{\partial S}{\partial t} + u_j \frac{\partial S}{\partial x_j} = \eta_o \nabla^2 S, \quad (3.2.39)$$

and

$$\rho = \rho_o\,[1+\alpha(T_o - T) - \hat{\alpha}(S_o - S)]. \quad (3.2.40)$$

(c) Modified equations for generalized Bénard convection/ thermohaline convection under rotation

The modified simplified equations governing generalized Bénard' convection ($\eta_o = 0$) and the phenomenon of thermohaline convection under rotation are given by

$$\frac{\partial u_i}{\partial x_j} = 0, \quad (3.2.41)$$

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{\partial}{\partial x_i}\left(\frac{p}{\rho_o} - \frac{1}{2}|\underline{\Omega} \times \underline{r}|^2\right) + \left(1 + \frac{\delta\rho}{\rho_o} + \frac{\hat{\delta\rho}}{\rho_o}\right) X_i +$$

$$\nu_o \nabla^2 u_i + 2\varepsilon_{ijk}\, u_j \Omega_k, \quad (3.2.42)$$

$$(1-\alpha_2 T)\left(\frac{\partial T}{\partial t} + u_j \frac{\partial T}{\partial x_j}\right) + \hat{\alpha}_2 T\left(\frac{\partial S}{\partial t} + u_j \frac{\partial S}{\partial x_j}\right) = K_o \nabla^2 T, \quad (3.2.43)$$

$$\frac{\partial S}{\partial t} + u_j \frac{\partial S}{\partial x_j} = \eta_o \nabla^2 S, \quad (3.2.44)$$

and $\rho = \rho_0 \, [1+\alpha(T_0-T) - \hat{\alpha}(S_0-S)]$. (3.2.45)

(d) Modified equations for generalized Bénard convection/ thermohaline convection under magnetic field

The modified simplified equations governing generalized Bénard convection ($\eta_0 = 0$) and the phenomenon of thermohaline convection under magnetic field are given by

$$\frac{\partial u_i}{\partial x_j} = 0 \,, \tag{3.2.46}$$

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} - \frac{\mu_0}{4\pi \rho_0} H_j \frac{\partial H_i}{\partial x_j} = - \frac{\partial}{\partial x_i}\left(\frac{p}{\rho_0} + \frac{\mu_0 |H|^2}{8\pi \rho_0}\right) +$$

$$\left(1 + \frac{\delta\rho}{\rho_0} + \frac{\hat{\delta\rho}}{\rho_0}\right) X_i + \nu \nabla^2 u_i \,, \tag{3.2.47}$$

$$(1-\alpha_2 T)\left(\frac{\partial T}{\partial t} + u_j \frac{\partial T}{\partial x_j}\right) + \hat{\alpha}_2 T\left(\frac{\partial S}{\partial t} + u_j \frac{\partial S}{\partial x_j}\right) = \kappa_0 \nabla^2 T, \tag{3.2.48}$$

$$\frac{\partial H_i}{\partial t} + u_j \frac{\partial H_i}{\partial x_j} = H_j \frac{\partial u_i}{\partial x_j} + \gamma \nabla^2 H_i, \tag{3.2.49}$$

$$\frac{\partial H_i}{\partial x_i} = 0, \tag{3.2.50}$$

$$\frac{\partial S}{\partial t} + u_j \frac{\partial S}{\partial x_j} = \eta_0 \nabla^2 S, \tag{3.2.51}$$

and $\rho = \rho_0 [1+\alpha(T_0-T) - \hat{\alpha} \, (S_0-S)]$. (3.2.52)

(e) Modified equations for generalized Bénard convection/ thermohaline convection under rotation and magnetic field

The modified simplified equations governing generalized Benard convection ($\eta_o = 0$) and the phenomenon of thermohaline convection under rotation and magnetic field are given by

$$\frac{\partial u_j}{\partial x_j} = 0, \qquad (3.2.53)$$

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} - \frac{\mu_o H_j}{4\pi \rho_o} \frac{\partial H_i}{\partial x_j} = - \frac{\partial}{\partial x_i}\left(\frac{p}{\rho_o} - \frac{1}{2}|\Omega \times \underline{r}|^2 + \frac{\mu_o |\underline{H}|^2}{8\pi\rho_o}\right) +$$

$$\left(1 + \frac{\delta\rho}{\rho_o} + \frac{\delta\hat{\rho}}{\rho_o}\right) X_i + \nu_o \nabla^2 u_i + 2\varepsilon_{ijk} u_j \Omega_k , \qquad (3.2.54)$$

$$(1-\alpha_2 T)\left(\frac{\partial T}{\partial t} + u_j \frac{\partial T}{\partial x_j}\right) + \hat{\alpha}_2 T\left(\frac{\partial S}{\partial t} + u_j \frac{\partial S}{\partial x_j}\right) = \kappa_o \nabla^2 T , \qquad (3.2.55)$$

$$\frac{\partial H_i}{\partial t} + u_j \frac{\partial H_i}{\partial x_j} = u_j \frac{\partial u_i}{\partial x_j} + \eta \nabla^2 H_i , \qquad (3.2.56)$$

$$\frac{\partial H_i}{\partial x_i} = 0, \qquad (3.2.57)$$

$$\frac{\partial S}{\partial t} + u_j \frac{\partial S}{\partial x_j} = \eta_o \nabla^2 S, \qquad (3.2.58)$$

and

$$\rho = \rho_o [1+\alpha(T_o-T) - \hat{\alpha}(S_o-S)]. \qquad (3.2.59)$$