CHAPTER - 3

MODIFIED HYDRODYNAMIC ROTATORY BENARD CONVECTION WITH SUSPENDED PARTICLES

3.1 THE PHYSICAL PROBLEM

The physical configuration to be investigated in this chapter is the following: a viscous finitely heat conducting Boussinesq fluid is statically confined between two horizontal boundaries z = 0 and z = d of infinite horizontal extension and finite vertical depth which are maintained at uniform temperatures T_0 and T_1 in the presence of a uniform rotation with suspended particles. The object of the present chapter is to mathematically investigate the onset of linear instability in the above configuration on the lines of chapter-2.

3.2 CONSTRUCTION OF THE MODIFIED SIMPLIFIED EQUATIONS GOVERNING THE PROBLEM

Proceeding exactly as in § 2.3, we derive the following modified simplified equations governing the problem of rotatory hydrodynamic Benard convection with suspended particles.

$$\frac{\partial U_j}{\partial x_j} = \tag{3.2.1}$$

$$\frac{\partial U_i}{\partial t} + U_j \frac{\partial U_i}{\partial x_j} = -\frac{\partial}{\partial x_i}\left[-\frac{p}{\rho_0} - \frac{1}{2}|\vec{\Omega} \times \vec{r}|^2\right] + \left(1 + \frac{\delta\rho}{\rho_0}\right)X_i + S'(u_i - U_i)$$
$$+ \nu_0 \nabla^2 U_i + 2\,\epsilon_{ijk} U_j \Omega_k \tag{3.2.2}$$

$$mN\left(\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j}\right) = K_1 N (U_i - u_i) \tag{3.2.3}$$

$$\frac{\partial N}{\partial t} + \frac{\partial}{\partial x_j}(N u_i) = 0 \tag{3.2.4}$$

$$(1-\alpha_2 T_0)\left(\frac{\partial T}{\partial t}+U_j\frac{\partial T}{\partial x_j}\right)+b\left(\frac{\partial T}{\partial t}+u_j\frac{\partial T}{\partial x_j}\right)=K_0\nabla^2 T \tag{3.2.5}$$

and

$$\rho = \rho_0\left[1+\alpha(T_0-T)\right] \tag{3.2.6}$$

where $\vec{\Omega}$ is the angular velocity, $\vec{r}$ is the position vector and all other symbols occuring in the above equations have the same meanings as in § 2.3.

3.3 THE INITIAL STATIONARY STATE SOLUTIONS

We now proceed to obtain the initial stationary state solution.

The initial stationary state whose stability we wish to examine is characterized by the following equations:

$$\left.\begin{aligned}(u,v,w) &= (0,0,0)=(v_1,v_2,v_3)\\ T &= T(z)\\ \rho &= \rho(z)\end{aligned}\right\} \tag{3.3.1}$$

The governing equations (3.2.1)-(3.2.6) yield the following initial stationary state solution

$$\left.\begin{aligned}(u,v,w) &= (0,0,0)=(v_1,v_2,v_3)\\ T &= T_0-\beta_1 z\\ \rho &= \rho_0\left[1+\alpha(T_0-T)\right]\\ &= \rho_0\left[1+\alpha\beta_1 z\right]\\ \text{and}\quad P &= p-\tfrac{1}{2}\rho_0|\vec{\Omega}\times\vec{r}|^2 = p_0-g\rho_0\left[z+\alpha\beta_1\frac{z^2}{2}\right]\end{aligned}\right\} \tag{3.3.2}$$

where all the symbols in (3.3.2) have the same meanings as in §2.4.

3.4 THE PERTURBATION EQUATIONS

Let the initial stationary state described by (3.3.2) be slightly perturbed so that the perturbed state is given by

$$\left.\begin{aligned} (\bar{u}, \bar{v}, \bar{w}) &= (0+u', \ 0+v', \ 0+w') \\ (\bar{v}_1, \bar{v}_2, \bar{v}_3) &= (0+v_1', \ 0+v_2', \ 0+v_3') \\ \bar{T} &= T_0 - \beta_1 z + \theta' \\ \bar{\rho} &= [1+\alpha(T_0 - T - \theta')] \\ \text{and} \quad p &= p_0 - g\rho_0[z + \alpha\beta_1 \tfrac{z^2}{2}] + \delta p' \end{aligned}\right\} \tag{3.4.1}$$

Then the linearized perturbation equations are respectively given by

$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} = 0 \tag{3.4.2}$$

$$\frac{\partial u'}{\partial t} = -\frac{1}{\rho_0}\frac{\partial}{\partial x}(\delta p') + \frac{\mu_0}{\rho_0}\nabla^2 u' + s'(v_1' - u') + 2\Omega v' \tag{3.4.3}$$

$$\frac{\partial v'}{\partial t} = -\frac{1}{\rho_0}\frac{\partial}{\partial y}(\delta p') + \frac{\mu_0}{\rho_0}\nabla^2 v' + s'(v_2' - v') - 2\Omega u' \tag{3.4.4}$$

$$\frac{\partial w'}{\partial t} = -\frac{1}{\rho_0}\frac{\partial}{\partial z}(\delta p') + \frac{\mu_0}{\rho_0}\nabla^2 w' + s'(v_3' - w') + g\alpha\theta' \tag{3.4.5}$$

$$(1-\alpha_2 T_0)\frac{\partial \theta'}{\partial t} + (1-\alpha_2 T_0)b\frac{\partial \theta'}{\partial t} - (1-\alpha_2 T_0)\beta_1 w' - (1-\alpha_2 T_0)\beta_1 b v_3' = K_0 \nabla^2 \theta' \tag{3.4.6}$$

$$\frac{m}{K_1}\frac{\partial v_1'}{\partial t} = u' - v_1' \tag{3.4.7}$$

$$\frac{m}{K_1}\frac{\partial v_2'}{\partial t} = v' - v_2' \tag{3.4.8}$$

$$\frac{m}{K_1}\frac{\partial v_3'}{\partial t} = w' - v_3' \qquad (3.4.9)$$

Further, combining (3.4.3) and (3.4.4), we derive

$$\frac{\partial \zeta'}{\partial t} = \frac{\mu_0}{\rho_0}\nabla^2\zeta' + S'\left(\frac{\partial v_1'}{\partial y} - \frac{\partial v_2'}{\partial x} - \zeta'\right) + 2\Omega\frac{\partial w'}{\partial z} \qquad (3.4.10)$$

where $\zeta' = \frac{\partial u'}{\partial y} - \frac{\partial v'}{\partial x}$ is the z component of vorticity.

3.5 THE NORMAL MODE ANALYSIS

Making use of the normal mode analysis as described in 2.6, we obtain the following linearized perturbation equations from equations (3.4.2)-(3.4.10):

$$ik_x u'' + ik_y v'' + \frac{d}{dz}w'' = 0 \qquad (3.5.1)$$

$$nu'' = -\frac{ik_x}{\rho_0}\delta p'' + \frac{\mu_0}{\rho_0}\left(\frac{d^2}{dz^2} - k^2\right)u'' - \frac{S'nu''m/K_1}{\frac{m}{K_1}n+1} + 2\Omega v'' \qquad (3.5.2)$$

$$nv'' = -\frac{ik_y}{\rho_0}\delta p'' + \frac{\mu_0}{\rho_0}\left(\frac{d^2}{dz^2} - k^2\right)v'' - \frac{S'nv''m/K_1}{\frac{m}{K_1}n+1} + 2\Omega u'' \qquad (3.5.3)$$

$$nw'' = -\frac{1}{\rho_0}\frac{\partial}{\partial z}\delta p'' + \frac{\mu_0}{\rho_0}\left(\frac{d^2}{dz^2} - k^2\right)w'' - \frac{S'nw''m/K_1}{\frac{m}{K_1}n+1} + g\alpha\theta'' \qquad (3.5.4)$$

$$(1-\alpha_2 T_0)\left[(b+1)n\theta'' - \frac{\beta_1\left(\frac{m}{K_1}n+b+1\right)w''}{\left(\frac{m}{K_1}n+1\right)}\right] = K_0\left(\frac{d^2}{dz^2} - k^2\right)\theta'' \qquad (3.5.5)$$

$$n\zeta'' = \frac{\mu_0}{\rho_0}\left(\frac{d^2}{dz^2} - k^2\right)\zeta'' - \frac{S'n\frac{m}{K_1}}{\frac{m}{K_1}n+1}\zeta'' + 2\Omega\frac{dw''}{dz} \qquad (3.5.6)$$

3.6 THE BOUNDARY CONDITIONS

The boundary conditions on w' , θ' and ζ' are given by

$$w' = 0 = \frac{\partial w'}{\partial z} = \zeta' \quad \text{(on a rigid boundary)} \tag{3.6.1}$$

$$w' = 0 = \frac{\partial^2 w'}{\partial z^2} = \frac{\partial \zeta'}{\partial z} \quad \text{(on a free boundary)} \tag{3.6.2}$$

$$\theta' = 0 \quad \text{(on a thermally perfectly conducting boundary)} \tag{3.6.3}$$

$$\frac{\partial \theta'}{\partial z} = 0 \quad \text{(on a thermally insulating boundary)} \tag{3.6.4}$$

The above boundary conditions when analyzed in terms of normal modes, respectively become

$$w'' = 0 = \frac{dw''}{dz} = \zeta'' \quad \text{(on a rigid boundary)} \tag{3.6.5}$$

$$w'' = 0 = \frac{d^2 w''}{dz^2} = \frac{d\zeta''}{dz} \quad \text{(on a dynamically free boundary)} \tag{3.6.6}$$

$$\theta'' = 0 \quad \text{(on a thermally conducting boundary)} \tag{3.6.7}$$

$$\frac{d\theta''}{dz} = 0 \quad \text{(on a thermally insulating boundary)} \tag{3.6.8}$$

3.7 THE CHARACTERISTIC VALUE PROBLEM

Multiplying equation (3.5.2) by f_{2x} and equation (3.5.3) by f_{2y} , adding the resulting equations and making use of the equation (3.5.1), we obtain

$$n\frac{dw''}{dz} = -\frac{k^2}{\rho_0}\delta p'' + \frac{\mu_0}{\rho_0}\left(\frac{d^2}{dz^2}-k^2\right)\frac{dw''}{dz} + \frac{S'\frac{m}{K_1}nDw''}{\frac{m}{K_1}n+1} - 2\Omega\zeta'' \tag{3.7.1}$$

Eliminating $\delta p''$ between equations (3.7.1) and (3.5.4), it follows that

$$\left(\frac{d^2}{dz^2}-k^2\right)\left[\frac{d^2}{dz^2}-k^2-\frac{n}{\nu_0}\frac{\{\frac{m}{K_1}(S'+n)+1\}}{(\frac{m}{K_1}n+1)}\right]w'' = \frac{g\alpha k^2\theta''}{\nu_0} + \frac{2\Omega}{\nu_0}\frac{d\zeta''}{dz} \tag{3.7.2}$$

Further equations (3.5.5) and (3.5.6) can be written as

$$(1-\alpha_2T_0)\left[(b+1)n\theta'' - \frac{\beta_1(\frac{m}{K_1}n+b+1)w''}{(\frac{m}{K_1}n+1)}\right] = K_0\left(\frac{d^2}{dz^2}-k^2\right)\theta'' \tag{3.7.3}$$

$$\left[\frac{d^2}{dz^2}-k^2-\frac{n}{\nu_0}\frac{\{\frac{m}{K_1}(S'+n)+1\}}{(\frac{m}{K_1}n+1)}\right]\zeta'' = -\frac{2\Omega}{\nu_0}\frac{dw''}{dz} \tag{3.7.4}$$

We shall now introduce the non-dimensional quantities defined by

$$\left.\begin{aligned} & z_* = z/d,\ a_* = kd,\ D_* = d\cdot\frac{d}{dz},\ \sigma_* = \nu_0/K_0,\ p_* = \frac{nd^2}{K_0} \\ & S_* = \frac{S'd^2}{\nu_0},\ \tau_* = \frac{mK_0}{K_1d^2},\ f_* = \tau_*S_*\sigma_*,\ F_* = f_*+1,\ B_* = b+1 \end{aligned}\right\} \tag{3.7.5}$$

Using the above non-dimensional quantities and omitting the asterisks and the double dashes for simplicity, we can reduce equations (3.7.2)-(3.7.4) respectively to the following partially non-dimensional forms:

$$(D^2-a^2)\left[D^2-a^2-\frac{p(\tau p+F)}{\sigma(\tau p+1)}\right]w = \frac{g\alpha a^2d^2\theta}{\nu_0} + \frac{2\Omega d^3D\zeta}{\nu_0} \tag{3.7.6}$$

$$(D^2-a^2-pB)\theta = -\frac{(1-\alpha_2T_0)\beta_1d^2(\tau p+B)}{K_0(\tau p+1)} \tag{3.7.7}$$

$$\left[D^2-a^2-\frac{p}{\sigma}\frac{(\tau p+F)}{(\tau p+1)}\right]\zeta=-\frac{2\Omega d}{\nu_0}DW \tag{3.7.8}$$

If we introduce more non-dimensional quantities defined by

$$\left.\begin{aligned} R_{1*}&=\frac{g\alpha\beta_1 d^4}{K_0\nu_0}, \quad W_*=\frac{\beta_1 d^2 W}{K_0}, \quad \theta^*=R_{1*}a^2\theta \\ \hat{T}_*&=\frac{4\Omega^2 d^4}{\nu_0}, \quad \zeta_*=\frac{\beta_1 d\nu_0\zeta}{2\Omega K_0} \end{aligned}\right\} \tag{3.7.9}$$

and omit the asterisks for simplicity, the system of equations (3.7.6)-(3.7.8) and the boundary conditions (3.6.5)-(3.6.8) (after combining them appropriately and omitting the double dashes for simplicity) assume the following non-dimensional forms:

$$(D^2-a^2)\left[D^2-a^2-\frac{p}{\sigma}\frac{(\tau p+F)}{(\tau p+1)}\right]W=\theta+\hat{T}D\zeta \tag{3.7.10}$$

$$(D^2-a^2-pB)\theta=-(1-\alpha_2T_0)R_1a^2\frac{(\tau p+B)}{(\tau p+1)}W \tag{3.7.11}$$

$$\left[D^2-a^2-\frac{p}{\sigma}\frac{(\tau p+F)}{(\tau p+1)}\right]\zeta=-DW \tag{3.7.12}$$

and either

$W=0=\theta=D^2W=D\zeta$ at $z=0$ and $z=1$ (when both the boundaries are dynamically free) (3.7.13)

or

$W=0=\theta=DW=\zeta$ at $z=0$ and $z=1$ (when both the boundaries are rigid) (3.7.14)

or

$W=0=\theta=D^2W=D\zeta$ at $z=0$ (when the lower boundary is free and the upper is rigid)
$W=0=\theta=DW=\zeta$ at $z=1$ (3.7.15)

or

$W = 0 = \Theta = DW = \zeta$ at $z = 0$ (when the lower boundary is rigid and the upper

$W = 0 = \Theta = D^2W = D\zeta$ at $z = 1$ is free) (3.7.16)

where $\hat{T}$ is the Taylor number and all other symbols have the same meanings as in § 2.8. Further, the origin of z is translated to be mid way between the two boundaries for the sake of convenience.

It is important to note here that the system of equations and boundary conditions as given by equation (3.7.10)-(3.7.16) correspond to

(i) Simple Benard convection if $\alpha_2 = 0 = \hat{T} = \tau$ and $B = 1 = F$

(ii) Simple Benard convection with suspended particles if $\alpha_2 = 0 = \hat{T}$

(iii) Simple Benard convection with rotation if $\alpha_2 = 0 = \tau$ and $B = 1 = F$

(iv) Rotatory simple Benard convection with suspended particles if $\alpha_2 = 0$

(v) Modified simple Benard convection if $\tau = 0 = \hat{T}$ and $B = 1 = F$

(vi) Modified simple Benard convection with suspended particles if $\hat{T} = 0$

(vii) Rotatory modified simple Benard convection if $\tau = 0$ and $B = 1 = F$

System of equations (3.7.10)-(3.7.12) together with either of the boundary conditions (3.7.13)-(3.7.16) constitute an eigen-value problem for $p\,(=p_r+ip_i)$ for given values of the other parameters and a given state of the system is stable, neutral or positive. Further, if $p_r=0$ implies $p_i=0$ for all wave numbers a^2, then the principle of exchange of stabilities (PES) is valid otherwise we have overstability atleast when instability sets in as certain modes.

3.8 MATHEMATICAL ANALYSIS

We prove the following theorems

CASE 1 $1-\alpha_2 T_0>0$ with $R_1>0$

When $1-\alpha_2 T_0>0$ with $R_1>0$ the governing equations and boundary conditions imply that we have a rotatory Benard problem with suspended particles wherein a liquid layer is heated underside, in the parameter regime $T_0\alpha_2<1$ and further that the Pellew and Southwell (1940) technique together with some non-trivial integral estimates obtained from the governing equations and boundary conditions given by (3.7.10)-(3.7.16) enable us to characterize the marginal state with the following result, namely

THEOREM 1: If (p, W, Θ, ζ) is a solution of equations (3.7.10)-(3.7.12) together with either of the boundary conditions (3.7.13)-(3.7.16) and $R_1 \leq \dfrac{27\pi^4}{4B(1-\alpha_2 T_0)}$, then

$$p_r=0 \Rightarrow p_i \neq 0$$

Proof: If possible let $p_i = 0$ be allowed so that $p = 0$ and equations (3.7.10)-(3.7.12) assume the forms

$$(D^2-a^2)^2 W = \theta + T D\zeta \tag{3.8.1}$$

$$(D^2-a^2)\theta = -(1-\alpha_2 T_0) R_1 a^2 B W \tag{3.8.2}$$

$$(D^2-a^2)\zeta = -DW \tag{3.8.3}$$

Multiplying (3.8.1) by W^* (the complex conjugate of W) integrating the resulting equation over the vertical range of z and substituting for $\int_0^1 W^*\theta dz$ and $\int_0^1 W^* D\zeta dz$ in this equation from equations (3.8.2) and (3.8.3), and then integrating each term of the equation so obtained by parts for a suitable number of times with the help of the boundary conditions (3.7.13)-(3.7.16), we get

$$\int_0^1 |(D^2-a^2)W|^2 dz + T\int_0^1 [|D\zeta|^2 + a^2|\zeta|^2] dz = \frac{1}{R_1 a^2 (1-\alpha_2 T_0) B}\int_0^1 [|D\theta|^2 + a^2|\theta|^2] dz \tag{3.8.4}$$

Multiplying equation (3.8.2) by its complex conjugate and integrating over the range of z, we get

$$\int_0^1 |(D^2-a^2)\theta|^2 dz = (1-\alpha_2 T_0)^2 R_1 a^2 B \int_0^1 |W|^2 dz \tag{3.8.5}$$

Equation (3.8.5) implies that

$$\int_0^1 [|D^2\theta|^2 + a^4|\theta|^2 + 2a^2|D\theta|^2] dz = (1-\alpha_2 T_0)^2 B^2 R_1^2 a^4 \int_0^1 |W|^2 dz \tag{3.8.6}$$

Now

since $\theta(0) = 0 = \theta(1)$, therefore by Rayleigh-Ritz inequality (Schultz, 1973), we have

$$\begin{aligned}
\pi^2 \int_0^1 |\theta|^2 dz &\leq \int_0^1 |D\theta|^2 dz \\
&= -\int_0^1 \theta^* D^2\theta\, dz = |-\int_0^1 \theta^* D^2\theta\, dz| \\
&\leq \int_0^1 |\theta|\,|D^2\theta|\, dz \\
&\leq \left(\int_0^1 |\theta|^2 dz\right)^{1/2} \left(\int_0^1 |D^2\theta|^2 dz\right)^{1/2}
\end{aligned} \tag{3.8.7}$$

(By Schwartz's inequality)

so that

$$\int_0^1 |D^2\theta|^2 dz \geq \pi^4 \int_0^1 |\theta|^2 dz \tag{3.8.8}$$

Equation (3.8.6) together with inequalities (3.8.7) and (3.8.8) imply that

$$(\pi^2+a^2)^2 \int_0^1 |\theta|^2 dz \leq R_1^2 a^2 B^2 (1-\alpha_2 T_0)^2 \int_0^1 |W|^2 dz \tag{3.8.9}$$

Now

$$\begin{aligned}
\int_0^1 [|D\theta|^2 + a^2|\theta|^2] dz &= -\int_0^1 \theta^* (D^2-a^2)\theta\, dz \\
&= |-\int_0^1 \theta^* (D^2-a^2)\theta\, dz| \\
&\leq \int_0^1 |\theta|\, |(D^2-a^2)\theta|\, dz \\
&\leq \left\{\int_0^1 |\theta|^2 dz\right\}^{1/2} \left\{\int_0^1 |(D^2-a^2)\theta|^2 dz\right\}^{1/2}
\end{aligned}$$

(By Schwartz's inequality)

Therefore, we have

$$\int_0^1 [|D\theta|^2 + a^2|\theta|^2] dz \leq \frac{R_1^2 a^4 B^2 (1-\alpha_2 T_0)^2}{(\pi^2+a^2)} \int_0^1 |W|^2 dz \tag{3.8.10}$$

Equation (3.8.4) upon using Rayleigh-Ritz inequality, namely

$$\int_0^1 |Dw|^2 dz \geq \pi^2 \int_0^1 |w|^2 dz \qquad (\text{Since } w(0) = 0 = w(1)) \tag{3.8.11}$$

and

$$\int_0^1 |D^2 w|^2 dz \geq \pi^4 \int_0^1 |w|^2 dz$$

which is derived in a manner similar to derivation of (3.8.8) gives

$$(\pi^2+a^2)^2 \int_0^1 |w|^2 dz \;+\; \hat{T} \int_0^1 [|D\zeta|^2 + a^2|\zeta|^2] dz$$

$$\leq \frac{1}{R_1 a^2 B(1-\alpha_2 T_0)} \int_0^1 [|D\theta|^2 + a^2|\theta|^2] dz \tag{3.8.12}$$

Inequalities (3.8.10) and (3.8.12) imply that

$$\hat{T} \int_0^1 [|D\zeta|^2 + a^2|\zeta|^2] dz + \left[\frac{(\pi^2+a^2)^3}{a^2} - R_1 B(1-\alpha_2 T_0)\right] \int_0^1 |w|^2 dz < 0 \tag{3.8.13}$$

Inequality (3.8.13) by virtue of the fact that

$$\min_{a^2} \left\{ \frac{(\pi^2+a^2)^3}{a^2} \right\} = \frac{27\pi^4}{4}$$

imply that

$$\hat{T} \int_0^1 [|D\zeta|^2 + a^2|\zeta|^2] dz + \left[\frac{27\pi^4}{4} - R_1 B(1-\alpha_2 T_0)\right] \int_0^1 |w|^2 dz < 0 \tag{3.8.14}$$

which obviously cannot hold under the condition of the theorem

Hence $p_i \neq 0$

This proves the theorem.

Theorem 1 shows that for the problem under consideration a sufficient condition for the validity of overstability is that $R_1 \leq \frac{27\pi^4}{4B(1-\alpha_2 T_0)}$ or equivalently a necessary condition for instability to set in as stationary convection is that $R_1 > \frac{27\pi^4}{4B(1-\alpha_2 T_0)}$. In particular, it follows that for the rotatory hydrodynamic Benard convection a necessary condition for instability to set in as stationary convection is that $R_1 > \frac{27\pi^4}{4}$ This result is valid for quite general boundary conditions.

<u>THEOREM 2:</u> If (p , W , θ , ζ) is a solution of equations (3.7.10)-(3.7.12) together with either of the boundary conditions (3.7.13)-(3.7.16) and $[R_1\tau(1-\alpha_2 T_0)\pi^2\sigma + \hat{T}F] \leq F\pi^4$, then

$$p_r \geq 0 \Rightarrow p_i = 0$$

Proof: Multiplying (3.7.10) by W^* (the complex conjugate of W), integrating the resulting equation over the vertical range of z and substituting for $\int_0^1 W^*\theta\,dz$ and $\int_0^1 W^* D\zeta\,dz$ in this equation from equations (3.7.11) and (3.7.12) and then integrating each term of the equation so obtained by parts for a suitable number of times with the help of the boundary conditions (3.7.13)-(3.7.16) and putting $F = f+1$, we get

$$(\tau p+1)\int_0^1 |(D^2-a^2)W|^2 dz + \frac{(\tau p^2+Fp)}{\sigma}\int_0^1 [|DW|^2+a^2|W|^2]dz + \hat{T}(\tau p+1)\int_0^1 [|D\zeta|^2+a^2|\zeta|^2]dz$$
$$+\hat{T}(\tau p+1)\frac{p^*}{\sigma}\int_0^1 |\zeta|^2 dz + \frac{\hat{T}(\tau|p|^2+p^*)(\tau p+1)f}{\sigma|\tau p+1|^2}\int_0^1 |\zeta|^2 dz$$
$$= \frac{|\tau p+1|^2}{R_1 a^2(1-\alpha_2 T_0)|\tau p+B|^2}\left[(\tau p+B)\int_0^1 [|D\theta|^2+a^2|\theta|^2]dz + Bp^*(\tau p+B)\int_0^1 |\theta|^2 dz\right] \qquad (3.8.15)$$

Equating the imaginary parts of the equation (3.8.15), we have for $p_i \neq 0$ the following equation

$$\tau\int_0^1 |(D^2-a^2)W|^2 dz + \left(\frac{2\tau p_r + F}{\sigma}\right)\int_0^1 [|DW|^2+a^2|W|^2]dz + \tau\hat{T}\int_0^1 [|D\zeta|^2+a^2|\zeta|^2]dz$$

$$+\frac{\tau^2|p|^2 f\hat{T}}{\sigma|\tau p+1|^2}\int_0^1 |\zeta|^2 dz + \frac{|\tau p+1|^2 B^2}{R_1 a^2(1-\alpha_2 T_0)|\tau p+B|^2}\int_0^1 |\theta|^2 dz$$

$$= \frac{|\tau p+1|^2 \tau}{R_1 a^2(1-\alpha_2 T_0)|\tau p+B|^2}\int_0^1 [|D\theta|^2+a^2|\theta|^2]dz + \frac{\hat{T}}{\sigma}\left[1+\frac{f}{|\tau p+1|^2}\right]\int_0^1 |\zeta|^2 dz \quad (3.8.16)$$

Multiplying equations (3.7.11) and (3.7.12) by W^* and ζ^* respectively and integrating over the range of z with the help of either of the boundary conditions (3.7.13)-(3.7.16) and putting $F=f+1$, we get

$$\frac{|\tau p+1|^2}{R_1^2 a^4(1-\alpha_2 T_0)^2|\tau p+B|^2}\left\{\int_0^1 [|D^2\theta|^2+a^4|\theta|^2+2a^2|D\theta|^2]dz + |p|^2 B^2\int_0^1 |\theta|^2 dz\right.$$

$$\left. + 2p_r B\int_0^1 [|D\theta|^2+a^2|\theta|^2]dz\right\} = \int_0^1 |W|^2 dz \quad (3.8.17)$$

and

$$\int_0^1 [|D\zeta|^2+a^2|\zeta|^2]dz + \frac{p}{\sigma}\left[1+\frac{f}{|\tau p+1|^2}\right]\int_0^1 |\zeta|^2 dz + \frac{\tau f|p|^2}{\sigma|\tau p+1|^2}\int_0^1 |\zeta|^2 dz = \int_0^1 \zeta^* DW dz \quad (3.8.18)$$

Since $p_r \geqslant 0$, equation (3.8.17) implies that

$$\frac{|\tau p+1|^2}{R_1^2 a^2(1-\alpha_2 T_0)^2|\tau p+B|^2}\int_0^1 [|D\theta|^2+a^2|\theta|^2]dz = \int_0^1 |W|^2 dz \quad (3.8.19)$$

which upon using Rayleigh-Ritz inequality namely

$$\int_0^1 |DW|^2 dz \geq \pi^2 \int_0^1 |W|^2 dz \quad (\text{Since } W(0) = 0 = W(1)) \tag{3.8.20}$$

gives

$$\frac{|\tau p+1|^2}{R_1^2 a^2 (1-\alpha_2 T_0)^2 |\tau p+B|^2} \int_0^1 [|D\theta|^2 + a^2|\theta|^2] dz \leq \frac{1}{\pi^2} \int_0^1 |DW|^2 dz \tag{3.8.21}$$

Equating the real parts of equation (3.8.18), we get

$$\int_0^1 [|D\zeta|^2 + a^2|\zeta|^2] dz + \frac{p_r}{\sigma}\left[1 + \frac{f}{|\tau p+1|^2}\right] \int_0^1 |\zeta|^2 dz + \frac{\tau f |p|^2}{\sigma |\tau p+1|^2} \int_0^1 |\zeta|^2 dz$$

$$= \text{Real part of } \left(\int_0^1 \zeta^* DW dz \right)$$

$$\leq \int_0^1 |\zeta| \, |DW| dz$$

$$\leq \left\{ \int_0^1 |\zeta|^2 dz \right\}^{1/2} \left\{ \int_0^1 |DW|^2 dz \right\}^{1/2} \tag{3.8.22}$$

(By Schwartz's inequality)

Since, $p_r \geq 0$ equation (3.8.22) gives

$$\int_0^1 [|D\zeta|^2] dz \leq \left\{ \int_0^1 |\zeta|^2 dz \right\}^{1/2} \left\{ \int_0^1 |DW|^2 dz \right\}^{1/2}$$

which upon using Rayleigh-Ritz inequality, namely

$$\int_0^1 |D\zeta|^2 dz \geq \pi^2 \int_0^1 |\zeta|^2 dz \quad (\text{Since } \zeta(0) = 0 = \zeta(1)) \tag{3.8.23}$$

gives

$$\pi^2 \int_0^1 |\zeta|^2 dz \leq \left\{ \int_0^1 |\zeta|^2 dz \right\}^{1/2} \left\{ \int_0^1 |DW|^2 dz \right\}^{1/2}$$

or

$$\int_0^1 |\zeta|^2 dz \leq \frac{1}{\pi^4} \int_0^1 |DW|^2 dz \tag{3.8.24}$$

Equation (3.8.16) together with inequalities (3.8.21), (3.8.24) and the fact that $F > 1 + \frac{f}{|\tau p + 1|^2}$ implies that

$$\tau \int_0^1 |(D^2 - a^2)W|^2 dz + \left(\frac{2p_r + F}{\sigma}\right) \int_0^1 a^2 |W|^2 dz + \tau \hat{T} \int_0^1 [|D\zeta|^2 + a^2|\zeta|^2] dz + \frac{\tau^2 |p|^2 f \hat{T}}{\sigma |\tau p + 1|^2} \int_0^1 |\zeta|^2 dz$$

$$+ \frac{|\tau p + 1|^2 B^2}{R_1 a^2 (1 - \alpha_2 T_0) |\tau p + B|^2} \int_0^1 |\theta|^2 dz + \left[\frac{F}{\sigma} - \frac{R_1 \tau (1 - \alpha_2 T_0)}{\pi^2} - \frac{\hat{T} F}{\pi^4 \sigma}\right] \int_0^1 |DW|^2 dz < 0 \quad (3.8.25)$$

Inequality (3.8.25` obviously cannot hold under the condition of theorem.

Hence $p_i = 0$

This proves the theorem

It is to be noted that in deriving inequality (3.8.25) the condition $p_r \geqslant 0$ has been used. Theorem 2 implies that for the problem under consideration an arbitrary neutral $(p_r = 0)$ or unstable $(p_r > 0)$ mode of the system is definitely non-oscillatory in character and therefore, in particular PES is valid if $R_1 \tau (1 - \alpha_2 T_0) \pi^2 \sigma + TF \leq F \pi^4$.

<u>**THEOREM 3:**</u> If (p, W, θ, ζ) is a solution of equations (3.7.10)-(3.7.12) together with either of the boundary conditions (3.7.13) and $p = 0$, then the critical Rayleigh number R_{1c} and the critical wave number a_c are given by

$$R_{1c} = \frac{\pi^4}{x(1 - \alpha_2 T_0) B} \left[(1 + x)^3 + \frac{\hat{T}}{\pi^4}\right] \quad (3.8.26)$$

$$a_c^2 = \pi^2 x \tag{3.8.27}$$

where $2x^3 + 3x^2 - (1 + \frac{T}{\pi^4}) = 0$

Proof: For $p = 0$, equations (3.7.10)-(3.7.12) imply that

$$(D^2 - a^2)^3 W + \hat{T} D^2 W = -R_1 a^2 (1 - \alpha_2 T_0) B W \tag{3.8.28}$$

The proper solution of equation (3.8.28) characterizing the lowest mode is

$$W = A \sin \pi z \tag{3.8.29}$$

where A is constant

Substituting the solution (3.8.29) in equation (3.8.28) we obtain

$$(\pi^2 + a^2)^3 + \hat{T} \pi^2 = R_1 a^2 (1 - \alpha_2 T_0) B \tag{3.8.30}$$

Letting $a^2 = \pi^2 x$

We can rewrite (3.8.30) in the form

$$R_1 = \frac{\pi^4}{x(1 - \alpha_2 T_0) B} \left[(1 + x)^3 + \frac{\hat{T}}{\pi^4} \right] \tag{3.8.31}$$

As a function of x , R_1 given by equation (3.8.31) attains its minimum value when

$$2x^3 + 3x^2 = 1 + \frac{\hat{T}}{\pi^4} \tag{3.8.32}$$

With x determined as the root of this cubic equation, equation (3.8.31) will give the required critical Rayleigh number R_{1c}.

Theorem 3 provides us the value of critical Rayleigh number when the instability sets in as stationary convection. Further, since

$$\frac{dR}{dB} = -\frac{\left[(\pi^2+a^2)^2+\hat{T}\pi^2\right]}{a^2(1-\alpha_2 T_0)B^2}$$

$$\frac{dR}{dT} = \frac{\pi^2}{a^2(1-\alpha_2 T_0)B}$$

therefore it follows that the suspended particles have a destabilizing effect whereas rotation has a stabilizing effect on the configuration in the parameter regime $\alpha_2 T_0 < 1$.

__THEOREM 4:__ If (p, W, θ, ζ), $p = p_r + i p_i$, $p_r \geq 0$, $p_i \neq 0$ is a solution of equations (3.7.10)-(3.7.12) together with either of the boundary conditions (3.7.13)-(3.7.16), then

$$|p|^2 < \max\left[\frac{T\sigma^2}{F}, \frac{R_1^2\tau^2\sigma^2(1-\alpha_2 T_0)^2}{F^2B^2}\right]$$

Proof: We know from Theorem 2 that for $p_i \neq 0$ and any of the boundary conditions (3.7.13)-(3.7.16), equation (3.8.16) holds good. Thus, for $p_i \neq 0$, we have

$$\tau\int_0^1 |(D^2-a^2)W|^2 dz + \frac{(2\tau p_r+F)}{\sigma}\int_0^1 [|DW|^2+a^2|W|^2]dz + \tau\hat{T}\int_0^1 [|D\zeta|^2+a^2|\zeta|^2]dz$$
$$+ \frac{\tau^2|p|^2 f\hat{T}}{\sigma|\tau p+1|^2}\int_0^1 |\zeta|^2 dz + \frac{|\tau p+1|^2 B^2}{R_1 a^2(1-\alpha_2 T_0)|\tau p+B|^2}\int_0^1 |\theta|^2 dz$$

$$= \frac{|\tau p+1|^2\tau}{R_1 a^2(1-\alpha_2 T_0)|\tau p+B|^2}\int_0^1 [|D\theta|^2+a^2|\theta|^2]dz + \frac{\hat{T}}{\sigma}\left(1+\frac{f}{|\tau p+1|^2}\right)\int_0^1 |\zeta|^2 dz \qquad (3.8.33)$$

Multiplying equations (3.7.11) and (3.7.12) by their complex conjugates and integrated over the vertical range of z by parts a suitable number of times and utilizing the boundary conditions on θ and ζ as given in equations (3.7.13)-(3.7.16) and putting $F=f+1$, we get

$$\int_0^1 [|D^2\theta|^2+a^4|\theta|^2+2a^2|D\theta|^2]dz+|p|^2B^2\int_0^1|\theta|^2dz+2p_rB\int_0^1[|D\theta|^2+a^2|\theta|^2]dz$$

$$=\frac{R_1^2a^4(1-\alpha_2T_0)^2|\tau p+B|^2}{|\tau p+1|^2}\int_0^1|W|^2dz \qquad (3.8.34)$$

and

$$\int_0^1|(D^2-a^2)\zeta|^2dz+\frac{2}{\sigma}\left[p_r+\frac{f\tau|p|^2}{|\tau p+1|^2}+\frac{f\,p_r}{|\tau p+1|^2}\right]\int_0^1[|D\zeta|^2+a^2|\zeta|^2]dz+\frac{|p|^2|\tau p+F|^2}{\sigma^2|\tau p+1|^2}\int_0^1|\zeta|^2dz$$

$$=\int_0^1|DW|^2dz \qquad (3.8.35)$$

Since $p_r \geqslant 0$, equation (3.8.34) implies that

$$\int_0^1|(D^2-a^2)\theta|^2dz < \frac{R_1^2a^4|\tau p+B|^2}{|\tau p+1|^2}(1-\alpha_2T_0)^2\int_0^1|W|^2dz \qquad (3.8.36)$$

$$\int_0^1|\theta|^2dz < \frac{R_1^2a^4|\tau p+B|^2}{|\tau p+1|^2|p|^2B^2}\int_0^1|W|^2dz \qquad (3.8.37)$$

Now

$$\begin{aligned}\int_0^1[|D\theta|^2+a^2|\theta|^2]dz &= -\int_0^1\theta^*(D^2-a^2)\theta\,dz\\ &= \left|-\int_0^1\theta^*(D^2-a^2)\theta\,dz\right|\\ &\leq \int_0^1|\theta|\,|(D^2-a^2)\theta|\,dz\\ &\leq \left\{\int_0^1|\theta|^2dz\right\}^{1/2}\left\{\int_0^1|(D^2-a^2)\theta|^2dz\right\}^{1/2}\end{aligned}$$

(By Schwartz's inequality)

which upon utilizing inequalities (3.8.36) and (3.8.37) gives

$$\int_0^1 [|D\theta|^2 + a^2|\theta|^2] dz < \frac{(1-\alpha_2 T_0)^2 |\tau p + B|^2 R_1^2 a^4}{|p|\, B\, |\tau p + 1|^2} \int_0^1 |W|^2 dz \tag{3.8.38}$$

Since $p_r \geq 0$, equation (3.8.35) implies that

$$\frac{|p|^2 |\tau p + F|^2}{\sigma^2 |\tau p + 1|^2} \int_0^1 |\zeta|^2 dz < \int_0^1 |DW|^2 dz \tag{3.8.39}$$

Inequality (3.8.39) further assume the form

$$\frac{|p|^2}{\sigma^2}\left[1 + \frac{f}{|\tau p + 1|^2}\right] \int_0^1 |\zeta|^2 dz < \int_0^1 |DW|^2 dz \tag{3.8.40}$$

Equation (3.8.33) together with inequalities (3.8.38), (3.8.39) and the fact that $F > 1 + \frac{f}{|\tau p + 1|^2}$ gives

$$\tau \int_0^1 |(D^2 - a^2)W|^2 dz + \left[\frac{F}{\sigma} - \frac{\hat{T}\sigma}{|p|^2}\right] \int_0^1 |DW|^2 dz + a^2\left[\frac{F}{\sigma} - \frac{R_1(1-\alpha_2 T_0)\tau}{|p|B}\right] \int_0^1 |W|^2 dz$$

$$+ \tau \hat{T} \int_0^1 [|D\zeta|^2 + a^2|\zeta|^2] dz + \frac{\tau^2 |p|^2 f \hat{T}}{\sigma |\tau p + 1|^2} \int_0^1 |\zeta|^2 dz + \frac{|\tau p + 1|^2 B^2}{R_1 a^2 (1-\alpha_2 T_0) |\tau p + B|^2} \int_0^1 |\theta|^2 dz < 0 \tag{3.8.41}$$

It follows from the inequality (3.8.41) that

$$\text{either } |p|^2 < \frac{T f^2}{F} \text{ or } |p|^2 < \frac{R_1^2 \tau^2 \sigma^2 (1-\alpha_2 T_0)^2}{F^2 B^2} \tag{3.8.42}$$

$$\text{Hence } |p|^2 < \max\left[\frac{T\sigma^2}{F}, \frac{R_1^2 \tau^2 \sigma^2 (1-\alpha_2 T_0)^2}{F^2 B^2}\right] \tag{3.8.43}$$

This proves the theorem.

Theorem 4 shows that for modified rotatory Benard convection with suspended particles, the complex growth rate $p (= p_r + i p_i)$ of an arbitrary oscillatory $(p_i \neq 0)$ perturbation, neutral $(p_r = 0)$ or unstable $(p_r > 0)$ lies within a semi-circle with centre at the origin and $(\text{radius})^2 = \max\left[\frac{\hat{T}\sigma^2}{F}, \frac{R_1^2 \tau^2 \sigma^2 (1-\alpha_2 T_0)^2}{F^2 B^2}\right]$ in the right half of the $p_r p_i$ -plane. Further, this result is valid for quite general boundary conditions.

<u>THEOREM 5:</u> If (p, w, θ, ζ), $p = p_r + i p_i$, $p_r \geq 0$, $p_i \neq 0$ is a solution of equations (3.7.10)-(3.7.12) together with either of the boundary conditions (3.7.13)-(3.7.16), then

$$|p|^2 < \max\left[\frac{\hat{T}\sigma^2}{F}, \frac{R_1 \tau^2 \sigma^2 (1-\alpha_2 T_0)^2 - \pi^4 F^2}{F^2 B^2}\right]$$

Proof: We know from Theorem 2 that for $p_i \neq 0$ and any of the boundary conditions (3.7.13)-(3.7.16), equation (3.8.16) holds good. Thus for $p_i \neq 0$, we have

$$\tau \int_0^1 |(D^2 - a^2) w|^2 dz + \frac{(2 p_r \tau + F)}{\sigma} \int_0^1 [|Dw|^2 + a^2 |w|^2] dz + \tau \hat{T} \int_0^1 [|D\zeta|^2 + a^2 |\zeta|^2] dz$$

$$+ \frac{\tau^2 |p|^2 f \hat{T}}{\sigma |\tau p + 1|^2} \int_0^1 |\zeta|^2 dz + \frac{|\tau p + 1|^2 B^2}{R_1 a^2 (1-\alpha_2 T_0) |\tau p + B|^2} \int_0^1 |\theta|^2 dz$$

$$= \frac{|\tau p + 1|^2 \tau}{R_1 a^2 (1-\alpha_2 T_0) |\tau p + B|^2} \int_0^1 [|D\theta|^2 + a^2 |\theta|^2] dz + \frac{\hat{T}}{\sigma}\left[1 + \frac{f}{|\tau p + 1|^2}\right] \int_0^1 |\zeta|^2 dz \qquad (3.8.44)$$

Since $p_r \geq 0$, equation (3.8.34) upon using Rayleigh-Ritz inequality (3.8.8) implies that

$$\int_0^1 |(D^2-a^2)\theta|^2 dz < \frac{|\tau p+B|^2}{|\tau p+1|^2} (1-\alpha_2 T_0)^2 R_1^2 a^4 \int_0^1 |W|^2 dz \tag{3.8.45}$$

and

$$[\pi^4+|p|^2 B^2] \int_0^1 |\theta|^2 dz < \frac{|\tau p+B|^2}{|\tau p+1|^2} R_1^2 a^4 (1-\alpha_2 T_0)^2 \int_0^1 |W|^2 dz \tag{3.8.46}$$

Now

$$\begin{aligned}
\int_0^1 [|D\theta|^2+a^2|\theta|^2] dz &= -\int_0^1 \theta^* (D^2-a^2)\theta\, dz \\
&= \left| -\int_0^1 \theta^* (D^2-a^2)\theta\, dz \right| \\
&\leq \int_0^1 |\theta|\, |(D^2-a^2)\theta|\, dz \\
&\leq \left\{ \int_0^1 |\theta|^2 dz \right\}^{1/2} \left\{ \int_0^1 |(D^2-a^2)\theta|^2 dz \right\}^{1/2}
\end{aligned}$$

(By Schwartz's inequality)

which upon utilizing inequalities (3.8.45) and (3.8.46) gives

$$\int_0^1 [|D\theta|^2+a^2|\theta|^2] dz < \frac{(1-\alpha_2 T_0)^2 |\tau p+B|^2}{|\tau p+1|^2 \{\pi^4+|p|^2 B^2\}^{1/2}} R_1^2 a^4 \int_0^1 |W|^2 dz \tag{3.8.47}$$

Equation (3.8.44) together with inequalities (3.8.40) and (3.8.47) and the fact that $F > 1+\frac{f}{|\tau p+1|^2}$ gives

$$\begin{aligned}
&\tau \int_0^1 |(D^2-a^2)W|^2 dz + \left[\frac{F}{\sigma} - \frac{\hat{T}\sigma}{|p|^2}\right] \int_0^1 |DW|^2 dz + a^2 \left[\frac{F}{\sigma} - \frac{R_1 \tau (1-\alpha_2 T_0)}{\{\pi^4+|p|^2 B^2\}^{1/2}}\right] \int_0^1 |W|^2 dz \\
&+ \tau \hat{T} \int_0^1 [|D\zeta|^2+a^2|\zeta|^2] dz + \frac{\tau^2 |p|^2 f \hat{T}}{\sigma |\tau p+1|^2} \int_0^1 |\zeta|^2 dz + \frac{|\tau p+1|^2 B^2}{R_1 a^2 (1-\alpha_2 T_0) |\tau p+B|^2} \int_0^1 |\theta|^2 dz < 0
\end{aligned} \tag{3.8.48}$$

It follows from the inequality (3.8.48) that

$$\text{either } |p|^2 < \frac{T\sigma^2}{F} \text{ or } |p|^2 < \frac{R_1^2\tau^2\sigma^2(1-\alpha_2 T_0)^2 - \pi^2 F^2}{F^2 B^2} \tag{3.8.49}$$

$$\text{Hence } |p|^2 < \max\left[\frac{T\sigma^2}{F}, \frac{R_1\tau^2\sigma^2(1-\alpha_2 T_0)^2 - \pi^2 F^2}{F^2 B^2}\right] \tag{3.8.50}$$

This proves the theorem

Theorem 5 clearly improves upon the result derived in Theorem 4 by reducing the radius of the semi-circular region therein.

CASE 2 $1-\alpha_2 T_0 > 0$ with $R_1 < 0$

When $1-\alpha_2 T_0 > 0$ with $R_1 < 0$, the governing equations and boundary conditions imply that we have a rotatory Benard problem with suspended particles wherein a liquid layer is heated overside, in the parameter regime $\alpha_2 T_0 < 1$. We prove the following theorems in this case:

THEOREM 6: If (p, w, θ, ζ), $p = p_r + ip_i$ is a solution of equations (3.7.10)-(3.7.12) together with either of the boundary conditions (3.7.13)-(3.7.16) and $p_i = 0$, then

$$p_r < 0$$

Proof: Equating the real part of the equation (3.8.15), we have

$$(\tau p_r + 1)\int_0^1 |(D^2-a^2)w|^2 dz + \left[\frac{\tau(p_r^2 - p_i^2) + F p_r}{\sigma}\right]\int_0^1 [|Dw|^2 + a^2|w|^2]dz$$
$$+ \hat{T}(\tau p_r + 1)\int_0^1 [|D\zeta|^2 + a^2|\zeta|^2]dz + \hat{T}\frac{\tau|p|^2}{\sigma}\left[1 + \frac{2f}{|\tau p+1|^2}\right]\int_0^1 |\zeta|^2 dz$$

$$+\frac{\hat{T}}{\sigma}p_r\left[1+\frac{f}{|\tau p+1|^2}+\frac{\tau^2|p|^2 f}{|\tau p+1|^2}\right]\int_0^1|\zeta|^2dz$$

$$=-\frac{|\tau p+1|^2}{|R_1|a^2(1-\alpha_2 T_0)|\tau p+B|^2}\left[(\tau p_r+B)\int_0^1[|D\theta|^2+a^2|\theta|^2]dz+\{B\tau|p|^2+B^2p_r\}\int_0^1|\theta|^2dz\right] \quad (3.8.51)$$

Now, if $p_i=0$ equation (3.8.51), implies that

$$(\tau p_r+1)\int_0^1|(D^2-a^2)w|^2dz+\frac{p_r(\tau p_r+F)}{\sigma}\int_0^1[|Dw|^2+a^2|w|^2]dz$$
$$+\hat{T}(\tau p_r+1)\int_0^1[|D\zeta|^2+a^2|\zeta|^2]dz+\frac{\hat{T}\tau|p|^2}{\sigma}\left[1+\frac{2f}{|\tau p+1|^2}\right]\int_0^1|\zeta|^2dz$$
$$+\frac{\hat{T}}{\sigma}p_r\left[1+\frac{f}{|\tau p+1|^2}+\frac{\tau^2|p|^2 f}{|\tau p+1|^2}\right]\int_0^1|\zeta|^2dz$$
$$=-\frac{|\tau p+1|^2}{|R_1|a^2(1-\alpha_2 T_0)|\tau p+B|^2}\left[(\tau p_r+B)\int_0^1[|D\theta|^2+a^2|\theta|^2]dz+B\{\tau|p|^2+Bp_r\}\int_0^1|\theta|^2dz\right] \quad (3.8.52)$$

Equation (3.8.52) under the conditions of the theorem clearly implies that

$$p_r<0$$

This proves the theorem

<u>THEOREM 7:</u> If (p, w, θ, ζ), $p=p_r+ip_i$ is a solution of equations (3.7.10)-(3.7.12) together with either of the boundary conditions (3.7.13)-(3.7.16) and $\hat{T}\le\pi^4$, $B\le\sqrt{\tau}\,\pi$ and $p_i=0$, then

$$p_r<0$$

Proof: For $p_i \neq 0$, equation (3.8.16) holds i.e.

$$\int_0^1 |(D^2-a^2)W|^2 dz + \left(\frac{2\tau p_r + F}{\sigma}\right)\int_0^1 [|DW|^2 + a^2|W|^2]dz + \tau\hat{T}\int_0^1 [|D\zeta|^2 + a^2|\zeta|^2]dz$$
$$+ \frac{\tau^2 |p|^2 f\hat{T}}{\sigma|\tau p+1|^2}\int_0^1 |\zeta|^2 dz + \frac{|\tau p+1|^2 \tau}{|R_1|a^2(1-\alpha_2 T_0)|\tau p+B|^2}\int_0^1 [|D\theta|^2 + a^2|\theta|^2]dz$$
$$= \frac{|\tau p+1|^2 B^2}{|R_1|a^2(1-\alpha_2 T_0)|\tau p+B|^2}\int_0^1 |\theta|^2 dz + \frac{\hat{T}}{\sigma}\left[1+\frac{f}{|\tau p+1|^2}\right]\int_0^1 |\zeta|^2 dz \tag{3.8.53}$$

Equation (3.8.53) upon using inequality (3.8.24) and the fact that $F > 1+\frac{f}{|\tau p+1|^2}$ and the Rayleigh-Ritz inequality, namely

$$\int_0^1 |D\theta|^2 dz \geq \pi^2 \int_0^1 |\theta|^2 dz \qquad (\text{Since } \theta(0) = 0 = \theta(1)) \tag{3.8.54}$$

or

$$-\int_0^1 |D\theta|^2 dz \leq -\pi^2 \int_0^1 |\theta|^2 dz$$

gives

$$\tau\int_0^1 |(D^2-a^2)W|^2 dz + \frac{F}{\sigma}\left[1-\frac{\hat{T}}{\pi^4}\right]\int_0^1 |DW|^2 dz + \left(\frac{2\tau p_r + F}{\sigma}\right)\int_0^1 a^2|W|^2 dz$$
$$+\hat{T}\tau\int_0^1 [|D\zeta|^2 + a^2|\zeta|^2]dz + \frac{\tau^2|p|^2 f\hat{T}}{\sigma|\tau p+1|^2}\int_0^1 |\zeta|^2 dz + \frac{|\tau p+1|^2\tau}{|R_1|(1-\alpha_2 T_0)}\int_0^1 |\theta|^2 dz$$
$$< \frac{|\tau p+1|^2}{|R_1|a^2(1-\alpha_2 T_0)|\tau p+B|^2}\left\{B^2 - \tau\pi^2\right\}\int_0^1 |\theta|^2 dz \tag{3.8.55}$$

Inequality (3.8.55) under the condition of the theorem clearly implies that

$$p_r < 0$$

This proves the theorem.

Theorem 6 and 7 establish the stability of the system if $\hat{T} \leq \pi^4$ and $B \leq \sqrt{2}\pi$ when the liquid layer under consideration is heated from above in the parameter regime $1-\alpha_2 T_0 > 0$.

CASE 3 $1-\alpha_2 T_0 < 0$ with $R_1 < 0$

When $1-\alpha_2 T_0 < 0$ with $R_1 < 0$ the governing equations and boundary conditions imply that we have a rotatory Benard problem with suspended particles wherein a liquid layer is heated overside, in the parameter regime $1-\alpha_2 T_0 < 0$. Since R_1 is negative the initial distribution of density is bottom heavy and therefore statically gravitationally stable and this stabilizing effect together with the joint stabilizing effect of viscosity and conduction is expected to impart, in the usual parameter regime characterized by $1-\alpha_2 T_0 > 0$, an overall stabilizing effect to the system. The nature of the problem, however, is completely different in the regime $1-\alpha_2 T_0 < 0$ in which case we have with $R_1 < 0$, $R_1(1-\alpha_2 T_0) > 0$ and this in turn introduce some new results namely a sufficient condition for the validity of overstability, condition for the validity of PES and semi-circle theorems for arresting the complex growth rate.

We prove the following theorems

THEOREM 8: If (p, w, θ, ζ) is a solution of equations (3.7.10)-(3.7.12) together with either of the boundary conditions (3.7.13)-(3.7.16) and $|R_1| \leq \dfrac{27\pi^4}{4B|1-\alpha_2 T_0|}$ then

$$p_r = 0 \Rightarrow p_i \neq 0$$

Proof: Supposing that $p_i = 0$ and proceeding exactly as in the proof of Theorem 1, we have in place of inequality (3.8.14) the following inequality:

$$T\int_0^1 [|D\zeta|^2 + a^2|\zeta|^2]dz + \left[\frac{27\pi^4}{4} - |R_1| B|1-\alpha_2 T_0|\right]\int_0^1 |W|^2 dz < 0 \qquad (3.8.56)$$

Inequality (3.8.56) obviously cannot hold under the condition of the theorem.

Hence $p_i \neq 0$

This proves the theorem

Theorem 8 shows that for the problem under consideration, a sufficient condition for the validity of overstability is that $|R_1| \leq \frac{27\pi^4}{4B|1-\alpha_2 T_0|}$ or equivalently a necessary condition for instability to set in as stationary convection is that $|R_1| > \frac{27\pi^4}{4B|1-\alpha_2 T_0|}$. This result is valid for quite general boundary conditions. In particular, it follows that for the rotatory hydrodynamic simple Benard convection a necessary condition for instability to set in as stationary convection is that $|R_1| > \frac{27\pi^4}{4}$. This result is valid for quite general boundary conditions.

<u>**THEOREM 9:**</u> If (p , W, θ , ζ) is a solution of equations (3.7.10)-(3.7.12) together with either of the boundary condition (3.7.13)-(3.7.16) and $\left[|R_1||1-\alpha_2 T_0|\tau\pi^2\sigma + \hat{T}F\right] \leq \pi^2 F$,
then

$$p_r \geq 0 \implies p_i = 0$$

Proof: Proceeding exactly as in Theorem 2, we have inplace of inequality (3.8.25), the following inequality

$$\tau\int_0^1 |(D^2-a^2)w|^2 dz + \frac{(2\tau p_r+F)}{\sigma}\int_0^1 a^2|w|^2 dz + \tau\hat{T}\int_0^1 [|D\zeta|^2+a^2|\zeta|^2]dz$$

$$+\frac{\tau^2|p|^2 F\hat{T}}{\sigma|\tau p+1|^2}\int_0^1|\zeta|^2 dz + \frac{|\tau p+1|^2 B^2}{|R_1||1-\alpha_2 T_0|a^2|\tau p+B|^2}\int_0^1|\theta|^2 dz$$

$$+\left[\frac{F}{\sigma}-\frac{|R_1|\tau|1-\alpha_2 T_0|}{\pi^2}-\frac{\hat{T}F}{\pi^4\sigma}\right]\int_0^1|Dw|^2 dz < 0 \qquad (3.8.57)$$

Inequality (3.8.57) obviously cannot hold under the conditions of the theorem.

Hence $p_i = 0$

This proves the theorem.

Theorem 9 implies that for the problem under consideration an arbitrary neutral $(p_r=0)$ or unstable $(p_r>0)$ mode of the system is definitely non-oscillatory in character and therefore, in particular PES is valid if $[|R_1|\tau|1-\alpha_2 T_0|\pi^2\sigma+F\hat{T}]\leq\pi^2 F$

<u>THEOREM 10:</u> If (p, w, θ, ζ), $p=0$ is a solution of equations (3.7.10)-(3.7.12) with boundary conditions (3.7.13) then the expression for $|R_1|$ is given by

$$|R_1| = \frac{\pi^4}{x|1-\alpha_2 T_0|B}\left[(1+x)^3+\frac{T}{\pi^4}\right] \qquad (3.8.58)$$

Proof: Proceeding exactly as in Theorem 3, we get the result.

THEOREM 11: If (p, w, θ, ζ), $p = p_r + i p_i$, $p_r \geq 0$, $p_i \neq 0$ is a solution of equations (3.7.10)-(3.7.12) together with either of the boundary conditions (3.7.13)-(3.7.16), then

$$|p|^2 < \max\left[\frac{\hat{T}\sigma^2}{F}, \frac{|R_1|^2\tau^2\sigma^2|1-\alpha_2 T_0|^2}{F^2B^2}\right]$$

Proof: Proceeding exactly as in Theorem 4, we have in place of inequality (3.8.41) the following inequality

$$\tau\int_0^1 |(D^2-a^2)w|^2 dz + \left[\frac{F}{\sigma}-\frac{\hat{T}\sigma}{|p|^2}\right]\int_0^1 |Dw|^2 dz + a^2\left[\frac{F}{\sigma}-\frac{|R_1||1-\alpha_2 T_0|\tau}{|p|B}\right]\int_0^1 |Dw|^2 dz$$

$$+\tau\hat{T}\int_0^1 [|D\zeta|^2+a^2|\zeta|^2]dz + \frac{\tau^2|p|^2 F\hat{T}}{\sigma|\tau p+1|^2}\int_0^1 |\zeta|^2 dz + \frac{|\tau p+1|^2 B^2}{|R_1|a^2|1-\alpha_2 T_0||\tau p+B|^2}\int_0^1 |\theta|^2 dz < 0 \qquad (3.8.59)$$

It follows from the inequality (3.8.59) that

$$\text{either } |p|^2 < \frac{\hat{T}\sigma^2}{F} \text{ or } |p|^2 < \frac{|R_1|^2\tau^2\sigma^2|1-\alpha_2 T_0|^2}{F^2B^2} \qquad (3.8.60)$$

$$\text{Hence } |p|^2 < \max\left[\frac{\hat{T}\sigma^2}{F}, \frac{|R_1|^2\tau^2\sigma^2|1-\alpha_2 T_0|^2}{F^2B^2}\right] \qquad (3.8.61)$$

This proves the theorem.

Theorem 11 shows that for modified rotatory Benard convection with suspended particles wherein a liquid layer is heated overside, in the parameter regime $1-\alpha_2 T_0 < 0$, the complex growth rate $p\,(=p_r + i p_i)$ of an arbitrary oscillatory $(p_i \neq 0)$ perturbation, neutral $(p_r = 0)$ or unstable $(p_r > 0)$ lies within a semi-circle with centre at the origin and

$(\text{radius})^2 = \max\left[\frac{\hat{T}\sigma^2}{F}, \frac{|R_1|^2\tau^2\sigma^2|1-\alpha_2 T_0|^2}{F^2B^2}\right]$ in the right half of the

$p_r p_i$ -plane. Further, this result is valid for quite general boundary conditions.

THEOREM 12: If (p , w , θ , ζ), $p = p_r + i p_i$, $p_i \neq 0$, $p_r \geq 0$ is a solution of equations (3.7.10)-(3.7.12) together with either of the boundary conditions (3.7.13)-(3.7.16), then

$$|p|^2 < \max\left[\frac{\hat{T}\sigma^2}{F}, \frac{|R_1|^2 \tau^2 \sigma^2 |1-\alpha_2 T_0|^2 - \pi^4 F^2}{F^2 B^2}\right]$$

Proof: Proceeding exactly as in Theorem 5, we have in place of inequality (3.8.48) the following inequality

$$\tau\int_0^1 |(D^2-a^2)w|^2 dz + \left[\frac{E}{\sigma} - \frac{\hat{T}\sigma}{|p|^2}\right]\int_0^1 |Dw|^2 dz + a^2\left[\frac{E}{\sigma} - \frac{|R_1|\tau|1-\alpha_2 T_0|}{\{\pi^4 + |p|^2 B^2\}^{1/2}}\right]\int_0^1 |w|^2 dz$$

$$+\tau\hat{T}\int_0^1 [|D\zeta|^2 + a^4|\zeta|^2]dz + \frac{\tau^2 |p|^2 f \hat{T}}{\sigma|\tau p+1|^2}\int_0^1 |\zeta|^2 dz + \frac{|\tau p+1|^2 B^2}{|R_1| a^2 |1-\alpha_2 T_0| |\tau p+B|^2}\int_0^1 |\theta|^2 dz < 0 \quad (3.8.62)$$

It follows from the inequality (3.8.62) that

$$\text{either } |p|^2 < \frac{\hat{T}\sigma^2}{F} \text{ or } |p|^2 < \frac{|R_1|^2 \tau^2 \sigma^2 |1-\alpha_2 T_0|^2 - \pi^4 F^2}{F^2 B^2} \quad (3.8.63)$$

$$\text{Hence } |p|^2 < \max\left[\frac{\hat{T}\sigma^2}{F}, \frac{|R_1|^2 \tau^2 \sigma^2 |1-\alpha_2 T_0|^2 - \pi^4 F^2}{F^2 B^2}\right] \quad (3.8.64)$$

This proves the theorem.

Theorem 12 clearly improves upon the result derived in Theorem 11 by reducing the radius of the semi-circular region therein.

CASE 4 $1-\alpha_2 T_0 < 0$ with $R_1 > 0$

When $1-\alpha_2 T_0 < 0$ with $R_1 > 0$ governing equations and boundary conditions imply we have a rotatory Benard convection with suspended particles. Wherein a liquid layer is heated underside, in the parameter regime $1-\alpha_2 T_0 < 0$. Since R_1 is positive the initial distribution of density is top heavy and therefore statically potentially gravitationally unstable and this destabilizing effect together with the joint stabilizing effects of viscosity and conduction is expected to impart, in the usual parameter regime characterized by $1-\alpha_2 T_0 > 0$, the character of a rotatory Benard problem with suspended particles in which the liquid layer is heated from below. The nature of the problem, however is completely different in the regime $1-\alpha_2 T_0 < 0$, in which case we have, with $R_1 > 0$, $R_1(1-\alpha_2 T_0) < 0$ and this in turn forces all the perturbation to decay and thus making the system stable. We prove the following theorems.

THEOREM 13: If (p , w , θ , ζ), $p = p_r + i p_i$ is a solution of equations (3.7.10)-(3.7.12) together with either of the boundary conditions (3.7.13)-(3.7.16) and $p_i = 0$, then

$$p_r < 0$$

Proof: Proceeding exactly as in Theorem 6, we have in place of equation (3.8.52) the following equation.

$$(\tau p_r+1)\int_0^1 |(D^2-a^2)W|^2 dz + \frac{p_r(\tau p_r+F)}{\sigma}\int_0^1 [|DW|^2+a^2|W|^2]dz + \hat{T}(\tau p_r+1)\int_0^1 [|D\zeta|^2+a^2|\zeta|^2]dz$$

$$+\frac{\hat{T}\tau|p|^2}{\sigma}\left[1+\frac{2f}{|\tau p+1|^2}\right]\int_0^1 |\zeta|^2 dz + \frac{\hat{T}}{\sigma}p_r\left[1+\frac{f}{|\tau p+1|^2}+\frac{\tau^2|p|^2 f}{|\tau p+1|^2}\right]\int_0^1 |\zeta|^2 dz$$

$$= -\frac{|\tau p+1|^2}{R_1 a^2|1-\alpha_2 T_0||\tau p+B|^2}\left[(\tau p_r+B)\int_0^1 [|D\theta|^2+a^2|\theta|^2]dz + B[\tau|p|^2+Bp_r]\int_0^1 |\theta|^2 dz\right] \quad (3.8.65)$$

Equation (3.8.65) under the conditions of the theorem clearly implies that

$$p_r < 0$$

This proves the theorem

<u>**THEOREM 14:**</u> If (p, W, θ, ζ), $p = p_r + ip_i$ is a solution of equations (3.7.10)-(3.7.12) together with either of the boundary conditions (3.7.13)-(3.7.16) and $p_i \neq 0$, $\hat{T} \leq \pi^4$ and $B \leq \sqrt{\tau}\,\pi$, then

$$p_r < 0$$

Proof: Proceeding exactly as in Theorem 7, we have in place of inequality (3.8.55) the following inequality

$$\tau\int_0^1 |(D^2-a^2)W|^2 dz + \frac{F}{\sigma}\left(1-\frac{\hat{T}}{\pi^4}\right)\int_0^1 |DW|^2 dz + \left(\frac{2\tau p_r+F}{\sigma}\right)\int_0^1 a^2|W|^2 dz$$

$$+\hat{T}\tau\int_0^1 [|D\zeta|^2+a^2|\zeta|^2]dz + \frac{\tau^2|p|^2 f\hat{T}}{\sigma|\tau p+1|^2}\int_0^1 |\zeta|^2 dz + \frac{\tau|\tau p+1|^2}{|R_1||1-\alpha_2 T_0|}\int_0^1 |\theta|^2 dz$$

$$< \frac{|\tau p+1|^2}{|R_1|a^2|1-\alpha_2 T_0||\tau p+B|^2}[B^2-\tau\pi^2]\int_0^1 |\theta|^2 dz \quad (3.8.66)$$

Inequality (3.8.66) under the conditions of the theorem clearly implies that

$$p_r < 0$$

This proves the theorem

Theorem 13 and 14 establish the stability of the system if $\hat{T} \leq \pi^4$ and $B \leq \sqrt{\tau}$ when the liquid layer under consideration is heated from below in the parameter regime $1-\alpha_2 T_0 < 0$.