

CHAPTER - 5

MODIFIED HYDRODYNAMIC THERMOHALINE CONVECTION WITH
SUSPENDED PARTICLES

5.1 THE PHYSICAL PROBLEM

The physical problem to be investigated in this chapter is the following: a viscous finitely heat conducting Boussinesq fluid is statically confined between two horizontal boundaries $z = 0$ and $z = d$ of infinite horizontal extension and finite vertical depth which are respectively maintained at uniform temperatures T_0 and T_1 , uniform concentrations S_0 and S_1 ($S_0 > S_1$) in the presence of a suspended particles. The concentration gradient thus maintained will be qualified as favourable because of its tendency to make the density of the fluid decrease vertically upwards. The object of the present chapter is to mathematically investigate the onset of linear instability in the above configuration on the lines of Chapter-2.

5.2 CONSTRUCTION OF THE MODIFIED SIMPLIFIED EQUATIONS GOVERNING THE PROBLEM

Proceeding exactly as in § 2.3, we derive the following modified simplified equations governing the problem of hydrodynamic thermohaline convection with suspended particles:

$$\frac{\partial U_j}{\partial x_j} = 0 \quad (5.2.1)$$

$$\frac{\partial U_i}{\partial t} + U_j \frac{\partial U_i}{\partial x_j} = -\frac{\partial}{\partial x} \left(\frac{p}{\rho_0} \right) + \left(1 + \frac{4\rho}{\rho} + \frac{4\rho'}{\rho} \right) \chi_i + S'(u_i - U_i) + \nu_0 \nabla^2 U_i \quad (5.2.2)$$

$$mN \left(\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} \right) = K_1 N (U_i - u_i) \quad (5.2.3)$$

$$\frac{\partial N}{\partial t} + \frac{\partial}{\partial x_j} (N u_j) = 0 \quad (5.2.4)$$

$$(1 - \alpha_2 T_0) \left[\frac{\partial T}{\partial t} + U_j \frac{\partial T}{\partial x_j} \right] + b \left[\frac{\partial T}{\partial t} + U_j \frac{\partial T}{\partial x_j} \right] + \hat{\alpha}_2 T \left(\frac{\partial S}{\partial t} + U_j \frac{\partial S}{\partial x_j} \right) = K_0 \nabla^2 T \quad (5.2.5)$$

$$\frac{\partial S}{\partial t} + U_j \frac{\partial S}{\partial x_j} = \hat{\gamma}_0 \nabla^2 S \quad (5.2.6)$$

and

$$\rho = \rho_0 [1 + \alpha(T_0 - T) - \hat{\alpha}(S_0 - S)] \quad (5.2.7)$$

where $\Delta \rho'$ is the variation in the density due to concentration, $\hat{\gamma}_0$ is the mass diffusivity, $\hat{\alpha}$ is the coefficient of volume expansion due to concentration, S is the concentration and all other symbols have the same meanings as in § 2.3.

5.3 THE INITIAL STATIONARY STATE SOLUTIONS

We now proceed to obtain the initial stationary state solutions:

The initial stationary state whose stability we wish to examine is characterized by the following equations:

$$\left. \begin{aligned} (u, v, w) &= (0, 0, 0) = (v_1, v_2, v_3) \\ T &= T(z) \\ S &= S(z) \\ \text{and} \\ \rho &= \rho(z) \end{aligned} \right\} \quad (5.3.1)$$

The governing equations (5.2.1)-(5.2.7) yield the following

initial stationary state solution:

$$\begin{aligned}
 (\mu, \nu, w) &= (0, 0, 0) = (\nu_1, \nu_2, \nu_3) \\
 T &= (T_0 - \beta_1 z) \\
 S &= (S_0 - \beta_2 z) \\
 \rho &= \rho_0 [1 + \alpha (T_0 - T) - \hat{\alpha} (S_0 - S)] \\
 &= \rho_0 [1 + \alpha \beta_1 z - \hat{\alpha} \beta_2 z] \\
 \text{and} \\
 P &= P_0 - g \rho_0 \left[z + (\alpha \beta_1 - \hat{\alpha} \beta_2) \frac{z^2}{2} \right]
 \end{aligned} \tag{5.3.2}$$

where $\beta_2 (= \frac{S_0 - S_1}{\alpha})$ is the uniform favourable concentration gradient, S_0 and S_1 , are the concentrations of the lower and upper boundaries, $S_0 > S_1$ and all other symbols occurring in § 5.3 have the same meanings as in § 2.4.

5.4 THE PERTURBATION EQUATIONS

Let the initial stationary state described by (5.3.2) be slightly perturbed so that the perturbed state is given by

$$\begin{aligned}
 (\bar{\mu}, \bar{\nu}, \bar{w}) &= (0 + \mu', 0 + \nu', 0 + w') \\
 (\bar{\nu}_1, \bar{\nu}_2, \bar{\nu}_3) &= (0 + \nu'_1, 0 + \nu'_2, 0 + \nu'_3) \\
 \bar{T} &= T_0 - \beta_1 z + \theta' \\
 \bar{S} &= S_0 - \beta_2 z + \phi \\
 \bar{\rho} &= [\rho_0 (1 + \alpha (T_0 - T - \theta') - \hat{\alpha} (S_0 - S - \phi'))] \\
 \text{and} \\
 \bar{P} &= P_0 - g \rho_0 \left[z + (\alpha \beta_1 - \hat{\alpha} \beta_2) \frac{z^2}{2} \right] + \delta p'
 \end{aligned} \tag{5.4.1}$$

Then the linearized perturbation equations are respectively given by

$$\frac{\partial \mu'}{\partial x} + \frac{\partial \nu'}{\partial y} + \frac{\partial w'}{\partial z} = 0 \tag{5.4.2}$$

$$\frac{\partial \mu'}{\partial t} = -\frac{1}{\rho_0} \frac{\partial}{\partial x} (\delta p') + \frac{\mu_0}{\rho_0} \nabla^2 \mu' + S' (\nu'_1 - \mu') \tag{5.4.3}$$

$$\frac{\partial v_1'}{\partial t} = -\frac{1}{\rho_0} \frac{\partial}{\partial y} (\delta p') + \frac{M_0}{\rho_0} \nabla^2 v_1' + S'(v_2' - v_1') \quad (5.4.4)$$

$$\frac{\partial w'}{\partial t} = -\frac{1}{\rho_0} \frac{\partial}{\partial z} (\delta p') + \frac{M_0}{\rho_0} \nabla^2 w' + S'(v_3' - w') + g\alpha\theta' - g\hat{\alpha}\phi' \quad (5.4.5)$$

$$(1-\alpha_2 T_0) \frac{\partial \theta'}{\partial t} + (1-\alpha_2 T_0) b \frac{\partial \theta'}{\partial t} + T_0 \hat{\alpha}_2 \frac{\partial \phi'}{\partial t} - (1-\alpha_2 T_0) \beta_1 w' - (1-\alpha_2 T_0) \beta_1 b v_3' - T_0 \hat{\alpha}_2 \beta_2 w' = K_0 \nabla^2 \theta' \quad (5.4.6)$$

$$\frac{\partial \phi}{\partial t} - \beta_2 w' = \hat{\gamma} \nabla^2 \phi \quad (5.4.7)$$

$$\frac{m}{\kappa_1} \frac{\partial v_1'}{\partial t} = u' - v_1' \quad (5.4.8)$$

$$\frac{m}{\kappa_1} \frac{\partial v_2'}{\partial t} = v' - v_2' \quad (5.4.9)$$

$$\frac{m}{\kappa_1} \frac{\partial v_3'}{\partial t} = w' - v_3' \quad (5.4.10)$$

5.5 THE NORMAL MODE ANALYSIS

Making use of the normal mode analysis as described in §2.6, we obtain the following linearized perturbation equations from equations (5.4.2)-(5.4.10):

$$i k_x u'' + i k_y v'' + \frac{d w''}{dz} = 0 \quad (5.5.1)$$

$$n u'' = -\frac{i k_x}{\rho_0} \delta p'' + \frac{M_0}{\rho_0} \left(\frac{d^2}{dz^2} - k^2 \right) u'' - \frac{S' n u'' m / \kappa_1}{\frac{m}{\kappa_1} n + 1} \quad (5.5.2)$$

$$n v'' = -\frac{i k_y}{\rho_0} \delta p'' + \frac{M_0}{\rho_0} \left(\frac{d^2}{dz^2} - k^2 \right) v'' - \frac{S' n v'' m / \kappa_1}{\frac{m}{\kappa_1} n + 1} \quad (5.5.3)$$

$$\eta w'' = -\frac{1}{\rho_0} \frac{d}{dz} (\delta p'') + \frac{M_0}{\rho_0} \left(\frac{d^2}{dz^2} - k^2 \right) w'' + g \alpha \theta'' - g \hat{\alpha} \phi'' - \frac{S' \eta w'' m / \kappa_1}{\frac{m}{\kappa_1} \eta + 1} \quad (5.5.4)$$

$$(1 - \alpha_2 T_0) \left[(b+1) \eta \theta'' - \beta_1 \left(\frac{m}{\kappa_1} \eta + b+1 \right) w'' \right] + T_0 \hat{\alpha}_2 \eta \phi'' - T_0 \hat{\alpha}_2 \beta_2 w'' = k_0 \left(\frac{d^2}{dz^2} - k^2 \right) \theta \quad (5.5.5)$$

$$\eta \phi'' - \beta_2 w'' = \hat{\gamma}_0 \left(\frac{d^2}{dz^2} - k^2 \right) \phi'' \quad (5.5.6)$$

5.6 THE BOUNDARY CONDITIONS

The boundary conditions on w' , θ' and ϕ' are given by

$$w' = 0 = \frac{\partial w'}{\partial z} \quad (\text{on a rigid boundary}) \quad (5.6.1)$$

$$w' = 0 = \frac{\partial^2 w'}{\partial z^2} \quad (\text{on a dynamically free boundary}) \quad (5.6.2)$$

$$\theta' = 0 = \phi' \quad (\text{on a temperature and concentrationwise conducting boundary}) \quad (5.6.3)$$

$$\frac{\partial \theta'}{\partial z} = 0 = \frac{\partial \phi'}{\partial z} \quad (\text{on a temperature and concentrationwise insulating boundary}) \quad (5.6.4)$$

The above boundary conditions when analyzed in terms of normal modes, respectively become

$$w'' = 0 = \frac{dw''}{dz} \quad (\text{on a rigid boundary}) \quad (5.6.5)$$

$$w'' = 0 = \frac{d^2 w''}{dz^2} \quad (\text{on a dynamically free boundary}) \quad (5.6.6)$$

$$\theta'' = 0 = \phi'' \quad (\text{on a temperature and concentrationwise conducting boundary}) \quad (5.6.7)$$

$$\frac{d\theta''}{dz} = 0 = \frac{d\phi''}{dz} \quad (\text{on a temperature and concentrationwise insulating boundary}) \quad (5.6.8)$$

5.7 THE CHARACTERISTIC VALUE PROBLEM

Multiplying equation (5.5.2) by f_{2x} and equation (5.5.3) by f_{2y} , adding the resulting equations and making use of equation (5.5.1), we obtain

$$\eta \frac{dw''}{dz} = -\frac{f_2^2}{f_0} (s p'') + \frac{M_0}{f_0} \left(\frac{d^2}{dz^2} - f_2^2 \right) \frac{dw''}{dz} + \frac{S' \frac{m}{\kappa_1} n \frac{dw''}{dz}}{\frac{m}{\kappa_1} n + 1} \quad (5.7.1)$$

Eliminating $s p''$ between (5.7.1) and (5.5.4), it follows that

$$\left(\frac{d^2}{dz^2} - f_2^2 \right) \left[\frac{d^2}{dz^2} - f_2^2 - \frac{\eta}{\gamma_0} \left\{ \frac{\frac{m}{\kappa_1} (s' + n) + \dots}{\left(\frac{m}{\kappa_1} n + 1 \right)} \right\} \right] w'' = \frac{g \alpha f_2^2 \theta''}{\gamma_0} - \frac{g \hat{\alpha} f_2^2 \phi''}{\gamma_0} \quad (5.7.2)$$

Further equations (5.5.5)-(5.5.6) can be written as

$$(1 - \alpha_2 T_0) \left[(b+1) n \theta'' - \beta_1 \frac{\left(\frac{m}{\kappa_1} n + b + 1 \right) w''}{\left(\frac{m}{\kappa_1} n + 1 \right)} \right] - T_0 \hat{\alpha}_2 \beta_2 w'' = K_0 \left(\frac{d^2}{dz^2} - f_2^2 \right) \theta'' - T_0 \hat{\alpha}_2 n \phi'' \quad (5.7.3)$$

$$\left(\frac{d^2}{dz^2} - f_2^2 - \frac{\eta}{\gamma_0} \right) \phi'' = -\frac{\beta_2}{\gamma_0} w'' \quad (5.7.4)$$

We shall now introduce the non-dimensional quantities defined by

$$\left. \begin{aligned} z_* &= \frac{z}{d}, \quad a_* = f_2 d, \quad D_* = d \cdot \frac{d}{dz}, \quad \hat{\gamma}_* = \frac{\hat{\gamma}_0}{K_0}, \quad \sigma_* = \gamma_0 / K_0 \\ p_* &= \frac{\eta d^2}{K_0}, \quad S_* = \frac{S' d^2}{\gamma_0}, \quad \tau_* = \frac{m K_0}{\kappa_1 d^2}, \quad f_* = \tau_* S_* \sigma_*, \quad F_* = f_* + 1, \quad B_* = b + 1 \end{aligned} \right\} \quad (5.7.5)$$

Using the above non-dimensional quantities and omitting the asterisks and the double dashes for simplicity, we can reduce equations (5.7.2) - (5.7.4) respectively to the following

partially non-dimensional forms:

$$(D^2 - a^2) \left[D^2 - a^2 - \frac{\rho(\tau\rho + F)}{\sigma(\tau\rho + 1)} \right] W = \frac{g\alpha a^2 d^2 \theta}{\gamma_0} - \frac{g\hat{\alpha} a^2 d^2 \phi}{\gamma_0} \quad (5.7.6)$$

$$(D^2 - a^2 - \rho B) \theta - T_0 \hat{\alpha}_2 \rho \phi = - \frac{(1 - \alpha_2 T_0) \beta_1 d^2 (\tau\rho + B)}{K_0 (\tau\rho + 1)} W - \frac{T_0 \hat{\alpha}_2 \beta_2 d^2 W}{K_0} \quad (5.7.7)$$

$$(D^2 - a^2 - \frac{\rho}{\hat{\gamma}}) \phi = - \frac{\beta_2 d^2 W}{\hat{\gamma}_0} \quad (5.7.8)$$

If we introduce more non-dimensional quantities defined by

$$\left. \begin{aligned} R_{1*} &= \frac{g\alpha\beta_1 d^4}{K_0 \gamma_0}, & W_* &= \frac{\beta_1 d^2 W}{K_0}, & \theta^* &= R_{1*} a^2 \theta \\ R_{2*} &= \frac{g\hat{\alpha}\beta_2 d^4}{K_0 \gamma_0}, & \phi^* &= \phi, & R_3^* &= \frac{\beta_2}{\beta_1} \end{aligned} \right\} \quad (5.7.9)$$

and omit the asterisk for simplicity, the system of equations (5.7.6)-(5.7.8) and the boundary conditions (5.6.5)-(5.6.8) (after combining them appropriately and omitting the double dashes for simplicity) assume the following non-dimensional forms:

$$(D^2 - a^2) \left[D^2 - a^2 - \frac{\rho(\tau\rho + F)}{\sigma(\tau\rho + 1)} \right] W = \theta - \frac{R_2}{R_3} a^2 \phi \quad (5.7.10)$$

$$(D^2 - a^2 - \rho B) \theta - R_1 a^2 T_0 \hat{\alpha}_2 \rho \phi = - R_1 a^2 \left[\frac{(1 - \alpha_2 T_0)(\tau\rho + B)}{(\tau\rho + 1)} + R_2 T_0 \hat{\alpha}_2 \right] W \quad (5.7.11)$$

$$(D^2 - a^2 - \frac{\rho}{\hat{\gamma}}) \phi = - \frac{R_3}{\hat{\gamma}} W \quad (5.7.12)$$

and either

$$W=0=DW=D\theta=D\phi \quad \text{at } z=0 \quad \text{and } z=1 \quad (5.7.13)$$

$$W=0=D^2W=D\theta=D\phi \quad \text{at } z=0 \quad \text{and } z=1 \quad (5.7.14)$$

$$W=0=DW=\theta=\phi \quad \text{at } z=0 \quad \text{and } z=1 \quad (5.7.15)$$

$$W=0=D^2W=\theta=\phi \quad \text{at } z=0 \quad \text{and } z=1 \quad (5.7.16)$$

where $R_2 > 0$ is the concentration Rayleigh number and all other symbols have the same meanings as in § 2.8. Further the origin of z is translated to be mid way between the two boundaries for the sake of convenience.

It is important to note here that the system of equations and boundary conditions as given by equations (5.7.10)-(5.7.16) correspond to

- (i) Simple Benard convection if $\alpha_2 = 0 = R_2 = \hat{\alpha}_2 = \tau$ and $B=1=F$
- (ii) Simple Benard convection with suspended particles if $\alpha_2 = R_2 = \hat{\alpha}_2$.
- (iii) Thermohaline convection of Veronis type if $\alpha_2 = 0 = \tau$ and $B=1=F$.
- (iv) Thermohaline convection with suspended particles if $\alpha_2 = 0$.
- (v) Modified simple Benard convection if $R_2 = 0 = \hat{\alpha}_2 = \tau$ and $B=1=F$.
- (vi) Modified simple Benard convection with suspended particles if $R_2 = 0 = \hat{\alpha}_2$.

(vii) Modified thermohaline convection of Veronis type if
 $\tau=0$ and $B=1=F$.

System of equations (5.7.10)-(5.7.12) together with either of the boundary conditions (5.7.13)-(5.7.16) constitute an eigen-value problem for $p (= p_r + i p_i)$ for given values of the other parameters and a given state of the system is stable, neutral or unstable according as p_r , the real part of p , is negative, zero or positive. Further, if $p_r=0$ implies $p_i=0$ for all wave numbers a^2 , then the principle of exchange of stabilities (PES) is valid otherwise we have overstability atleast when instability sets in as certain modes.

5.8 MATHEMATICAL ANALYSIS

CASE 1: $1 - \alpha_2 T_0 > 0$ with $R_1 > 0$

When $1 - \alpha_2 T_0 > 0$ with $R_1 > 0$, the governing equations and boundary conditions imply that we have a thermohaline convection of Veronis' (1965) type with suspended particles wherein a liquid layer is heated underside, in the parameter regime $\alpha_2 T_0 < 1$ and further that the Pellew and Southwell (1940) technique together with some non-trivial integral estimates obtained from the governing equations and boundary conditions given by (3.7.10)-(3.7.16) enable us to characterize the marginal state with the following result, namely

THEOREM 1: If (p, w, θ, ϕ) is a solution of equations (5.7.10)-(5.7.12) together with either of the boundary conditions (5.7.13)-(5.7.16) and $R_1 \leq \frac{27\pi^4}{4[(1-\alpha_2 T_0)B + \hat{\alpha}_2 R_3 T_0]}$ then

$$p_x = 0 \Rightarrow p_x \neq 0$$

Proof: If possible let $p_x = 0$ be allowed so that $p = 0$ and equations (5.7.10)-(5.7.12) assume the forms:

$$(D^2 - a^2)w = \theta - \frac{R_2 a^2 \phi}{R_3} \quad (5.8.1)$$

$$(D^2 - a^2)\theta = -R_1 a^2 [(1 - \alpha_2 T_0)B + T_0 \hat{\alpha}_2 R_3] w \quad (5.8.2)$$

$$(D^2 - a^2)\phi = -\frac{R_3}{\hat{\gamma}} w \quad (5.8.3)$$

Multiplying (5.8.1) by w^* (the complex conjugate of w) integrating the resulting equation over the vertical range of z and substituting for $\int_0^1 w^* \theta dz$ and $\int_0^1 w^* \phi dz$ in this equation from equations (5.8.2) and (5.8.3), and then integrating each term of the equation so obtained by parts for a suitable number of times with the help of the boundary conditions (5.7.13)-(5.7.16), we get

$$\int_0^1 |(D^2 - a^2)w|^2 dz = \frac{1}{[R_1 a^2 (1 - \alpha_2 T_0)B + \hat{\alpha}_2 R_3 T_0]} \int_0^1 [10|\theta|^2 + a^2 |\theta|^2] dz - \frac{\hat{\gamma} R_2 a^2}{R_3} \int_0^1 [10|\phi|^2 + a^2 |\phi|^2] dz \quad (5.8.4)$$

Multiplying equation (5.8.2) by its complex conjugate and integrating over the range of z , we get

$$\int_0^1 |D^2\theta|^2 dz + a^4 \int_0^1 |\theta|^2 dz + 2a^2 \int_0^1 |D\theta|^2 dz$$

$$= R_1^2 a^4 [(1-\alpha_2 T_0)B + \hat{\alpha}_2 R_3 T_0]^2 \int_0^1 |w|^2 dz \quad (5.8.5)$$

which upon using the Rayleigh-Ritz inequality given by equation, (3.8.8) gives

$$(\pi^2 + a^2) \int_0^1 |\theta|^2 dz \leq R_1^2 a^4 [(1-\alpha_2 T_0)B + \hat{\alpha}_2 R_3 T_0]^2 \int_0^1 |w|^2 dz \quad (5.8.6)$$

Now

$$\int_0^1 [|D\theta|^2 + a^2 |\theta|^2] dz = - \int_0^1 \theta^* (D^2 - a^2) \theta dz$$

$$= \left| - \int_0^1 \theta^* (D^2 - a^2) \theta dz \right|$$

$$\leq \int_0^1 |\theta| |D^2 - a^2| |\theta| dz$$

$$\leq \left\{ \int_0^1 |\theta|^2 dz \right\}^{1/2} \left\{ \int_0^1 |(D^2 - a^2) \theta|^2 dz \right\}^{1/2}$$

(By Schwartz's inequality)

Therefore, we have

$$\int_0^1 [|D\theta|^2 + a^2 |\theta|^2] dz \leq \frac{R_1^2 a^4 B^2 [(1-\alpha_2 T_0)B + \hat{\alpha}_2 R_3 T_0]^2}{(\pi^2 + a^2)} \int_0^1 |w|^2 dz \quad (5.8.7)$$

Equation (5.8.4) by using Rayleigh-Ritz inequality given by

equation (3.8.11) gives

$$\begin{aligned} (\pi^2 + a^2)^2 \int_0^1 |w|^2 dz + \frac{\hat{\gamma} R_2 a^2}{R_3^2} \int_0^1 [|\theta|^2 + a^2 |\phi|^2] dz \\ \leq \frac{1}{R_1 a^2 [(1 - \alpha_2 T_0) B + \hat{\alpha}_2 R_3 T_0]} \int_0^1 [|\theta|^2 + a^2 |\phi|^2] dz \quad (5.8.8) \end{aligned}$$

Inequalities (5.8.7) and (5.8.8), imply that

$$\left[\frac{(\pi^2 + a^2)^3}{a^2} - R_1 \{ (1 - \alpha_2 T_0) B + \hat{\alpha}_2 R_3 T_0 \} \right] \int_0^1 |w|^2 dz + \frac{\hat{\gamma} R_2 a^2}{R_3^2} \int_0^1 [|\theta|^2 + a^2 |\phi|^2] dz < 0 \quad (5.8.9)$$

Inequality (5.8.9) by virtue of the fact that

$$\min_{a^2} \left\{ \frac{(\pi^2 + a^2)^3}{a^2} \right\} = \frac{27\pi^4}{4} \quad \text{imply that}$$

$$\left[\frac{27\pi^4}{4} - R_1 \{ (1 - \alpha_2 T_0) B + \hat{\alpha}_2 R_3 T_0 \} \right] \int_0^1 |w|^2 dz + \frac{\hat{\gamma} R_2 a^2}{R_3^2} \int_0^1 [|\theta|^2 + a^2 |\phi|^2] dz < 0 \quad (5.8.10)$$

which obviously cannot hold under the condition of the theorem.

Hence $p_i \neq 0$

This proves the theorem.

Theorem 1 shows that for the problem under consideration, a sufficient condition for the validity of overstability is that $R_1 \leq \frac{27\pi^4}{4 [B(1 - \alpha_2 T_0) + \hat{\alpha}_2 R_3 T_0]}$ or equivalently a necessary condition for instability to set in as stationary convection is that $R_1 > \frac{27\pi^4}{4 [B(1 - \alpha_2 T_0) + \hat{\alpha}_2 R_3 T_0]}$. This result is valid

for quite general boundary conditions. In particular, it follows that for the hydromagnetic Benard convection a necessary condition for instability to set in as stationary convection is that $R_1 > \frac{27\pi^4}{4}$.

THEOREM 2: If (β, w, θ, ϕ) is a solution of equations (5.7.10)-(5.7.12) together with either of the boundary conditions (5.7.13)-(5.7.16), $\hat{\alpha}_2 = 0$ and $\frac{R_2}{2\hat{\alpha}_2\pi^4} + \frac{R_1(1-\alpha_2 T_0)\tau}{\pi^2} \leq \frac{F}{\sigma}$ then

$$p_r \geq 0 \Rightarrow p_i = 0$$

Proof: For $\hat{\alpha}_2 = 0$, equation (5.7.11) becomes

$$[D^2 - \alpha^2 - \beta B]\theta = -R_1 \alpha^2 (1 - \alpha_2 T_0) \frac{(\tau\beta + B)}{(\tau\beta + 1)} w \quad (5.8.11)$$

Multiplying (5.7.10) by w^* (the complex conjugate of w), integrating the resulting equation over the vertical range of z and substituting for $\int_0^1 w^* \theta dz$ and $\int_0^1 w^* \phi dz$ in this equation from equations (5.8.11) and (5.7.12), and then integrating each term of the equation so obtained by parts for a suitable number of times with the help of the boundary conditions (5.7.13)-(5.7.16), we get

$$\begin{aligned}
& (\tau p + 1) \int_0^1 (D^2 - a^2) w^2 dz + \frac{(\tau p^2 + Fp)}{\sigma} \int_0^1 [Dw^2 + a^2 w^2] dz + \frac{(\tau p + 1) R_2 a^2 \hat{\gamma}}{R_3^2} \int_0^1 [D\phi^2 + a^2 \phi^2] dz \\
& + (\tau p + 1) p^* \frac{R_2 a^2}{R_3^2} \int_0^1 |\phi|^2 dz
\end{aligned}$$

$$= \frac{|\tau p + 1|^2}{R_1 a^2 (1 - \alpha_2 T_0) |\tau p + B|^2} \left\{ (\tau p + B) \int_0^1 [D\theta^2 + a^2 \theta^2] dz + B p^* (\tau p + B) \int_0^1 |\theta|^2 dz \right\} \quad (5.8.12)$$

Equating the imaginary parts of the equation (5.8.12), we have for $p_i \neq 0$ the following equation

$$\begin{aligned}
& \tau \int_0^1 (D^2 - a^2) w^2 dz + \frac{(2\tau p^2 + F)}{\sigma} \int_0^1 [Dw^2 + a^2 w^2] dz + \frac{B^2 |\tau p + 1|^2}{R_1 a^2 (1 - \alpha_2 T_0) |\tau p + B|^2} \int_0^1 |\theta|^2 dz \\
& + \frac{\tau R_2 a^2 \hat{\gamma}}{R_3^2} \int_0^1 [D\phi^2 + a^2 \phi^2] dz
\end{aligned}$$

$$= \frac{\tau |\tau p + 1|^2}{R_1 a^2 (1 - \alpha_2 T_0) |\tau p + B|^2} \left\{ \int_0^1 [D\theta^2 + a^2 \theta^2] dz + \frac{R_2 a^2}{R_3^2} \int_0^1 |\phi|^2 dz \right\} \quad (5.8.13)$$

Multiplying equations (5.8.11) and (5.8.12) by their complex conjugates and integrated over the vertical range of z by parts a suitable number of times and utilizing the boundary conditions on θ and ϕ as given in equations (5.7.13)-(5.7.16), we get

$$\begin{aligned}
& \frac{|\tau p + 1|^2}{R_1 a^2 |\tau p + B|^2 (1 - \alpha_2 T_0)^2} \left\{ \int_0^1 [D^2 \theta^2 + a^4 \theta^2 + 2a^2 D\theta^2] dz + |p|^2 B^2 \int_0^1 |\theta|^2 dz + 2p^2 B \int_0^1 [D\theta^2 + a^2 \theta^2] dz \right. \\
& \left. = \int_0^1 |w|^2 dz \right\} \quad (5.8.14)
\end{aligned}$$

$$\int_0^1 [|\partial^2 \phi|^2 + a^4 |\phi|^2 + 2a^2 |\partial \phi|^2] dz + \frac{2\beta_2}{\gamma^2} \int_0^1 [|\partial \phi|^2 + a^2 |\phi|^2] dz + \frac{|\beta_1|^2}{\gamma^2} \int_0^1 |\phi|^2 dz$$

$$= \frac{R_3^2}{\gamma^2} \int_0^1 |w|^2 dz \quad (5.8.15)$$

Since $\beta_2 \geq 0$, equation (5.8.14) implies that

$$\frac{|\tau\beta + 1|^2}{R_1^2 a^2 |\tau\beta + \beta_1|^2 (1 - \alpha_2 \tau_0)^2} \int_0^1 [|\partial \theta|^2 + a^2 |\theta|^2] dz < \int_0^1 |w|^2 dz \quad (5.8.16)$$

which upon using the Rayleigh-Ritz inequality given by equation (3.8.20) gives

$$\frac{|\tau\beta + 1|^2}{R_1^2 a^2 |\tau\beta + \beta_1|^2 (1 - \alpha_2 \tau_0)^2} \int_0^1 [|\partial \theta|^2 + a^2 |\theta|^2] dz < \frac{1}{\pi^2} \int_0^1 |\partial w|^2 dz \quad (5.8.17)$$

Since $\beta_2 \geq 0$, equation (5.8.15) implies that

$$2a^2 \int_0^1 |\partial \phi|^2 dz < \frac{R_3^2}{\gamma^2} \int_0^1 |w|^2 dz \quad (5.8.18)$$

which upon using the Rayleigh-Ritz inequality, namely

$$\int_0^1 |\phi|^2 dz \leq \frac{1}{\pi^2} \int_0^1 |\partial \phi|^2 dz \quad (\text{Since } \phi(0) = 0 = \phi(1)) \quad (5.8.19)$$

and the inequality given by equation (3.8.20) gives

$$a^2 \int_0^1 |\phi|^2 dz < \frac{R_3^2}{2\gamma^2 \pi^4} \int_0^1 |\partial w|^2 dz \quad (5.8.20)$$

Equation (5.8.13) together with inequalities (5.8.17) and (5.8.20) gives

$$\begin{aligned} & \tau \int_0^1 |(0^2 - \alpha^2) w|^2 dz + \frac{(2k_2 \tau + F)}{\sigma} \int_0^1 \alpha^2 |w|^2 dz + \frac{B^2 |\tau p + 1|^2}{(1 - \alpha_2 T_0) R_1 \alpha^2 |\tau p + B|^2} \int_0^1 |\theta|^2 dz \\ & + \frac{R_2 \alpha^2 \tau \hat{Y}}{R_3} \int_0^1 [|\partial \phi|^2 + \alpha^2 |\phi|^2] dz + \left[\frac{F}{\sigma} - \frac{R_2}{2 \hat{Y}^2 \pi^4} - \frac{R_1 (1 - \alpha_2 T_0) X}{\pi^2} \right] \int_0^1 |Dw|^2 dz < 0 \quad (5.8.21) \end{aligned}$$

Inequality (5.8.21) obviously cannot hold under the condition of the theorem.

Hence $p_i = 0$

This proves the theorem

It is to be noted that in deriving inequality (5.8.21), the condition $k_2 \geq 0$ has been used. Theorem 2 implies that for the problem under consideration an arbitrary neutral ($k_2 = 0$) or unstable ($k_2 > 0$) mode of the system is definitely non-oscillatory in character and therefore, in particular PES is valid if $\frac{R_2}{2 \hat{Y}^2 \pi^4} + \frac{R_1 (1 - \alpha_2 T_0) \tau}{\pi^2} \leq \frac{F}{\sigma}$ and this result holds for quite general boundary conditions. However, the result is restricted to the case when $\alpha_2 = 0$.

THEOREM 3: If (p, w, θ, ϕ) , $p = k_2 + i p_i$, $k_2 \geq 0$, $p_i \neq 0$ is a solution of the equations (5.7.10)-(5.7.12) together with either of the boundary conditions (5.7.13)-(5.7.16), $\alpha_2 = 0$ and $F \hat{Y} \pi^4 > R_2 \sigma$ then

$$|p_i| < \frac{R_1 \tau (1 - \alpha_2 T_0) \sigma \hat{Y} \pi^4}{B (F \hat{Y} \pi^4 - R_2 \sigma)}$$

Proof: We know from Theorem 2 that for $p_i \neq 0$ any of the boundary conditions (5.7.13)-(5.7.16), equation (5.8.13) holds good. Thus for $p_i \neq 0$, we have

$$\begin{aligned} & \tau \int_0^1 (D^2 - a^2) |w|^2 dz + \frac{(2\tau p_2 + F)}{\sigma} \int_0^1 [D|w|^2 + a^2 |w|^2] dz + \frac{B^2 |\tau p + 1|^2}{R_2 a^2 (1 - \alpha_2 T_0) |\tau p + B|^2} \int_0^1 |\theta|^2 dz \\ & + \frac{\tau R_2 a^2 \hat{\gamma}}{R_3^2} \int_0^1 [D|\phi|^2 + a^2 |\phi|^2] dz \\ & = \frac{\tau |\tau p + 1|^2}{R_1 a^2 (1 - \alpha_2 T_0) |\tau p + B|^2} \int_0^1 [D|\theta|^2 + a^2 |\theta|^2] dz + \frac{R_2 a^2}{R_3^2} \int_0^1 |\phi|^2 dz \end{aligned} \quad (5.8.22)$$

Multiplying equations (5.8.11) and (5.8.12) by their complex conjugates and integrated over the vertical range of by parts a suitable number of times and utilizing the boundary conditions on θ and ϕ as given in equations (5.7.13)-(5.7.16), we get

$$\begin{aligned} & \frac{|\tau p + 1|^2}{R_1^2 a^2 |\tau p + 1|^2 (1 - \alpha_2 T_0)^2} \left\{ \int_0^1 [D^2 |\theta|^2 + a^4 |\theta|^2 + 2a^2 |\theta|^2] dz + |B|^2 \int_0^1 |\theta|^2 dz + 2p_2 B \int_0^1 [D|\theta|^2 + a^2 |\theta|^2] dz \right\} \\ & = \int_0^1 |w|^2 dz \end{aligned} \quad (5.8.23)$$

$$\begin{aligned} & \int_0^1 [D^2 |\phi|^2 + a^4 |\phi|^2 + 2a^2 |\phi|^2] dz + \frac{2p_2}{\hat{\gamma}} \int_0^1 [D|\phi|^2 + a^2 |\phi|^2] dz + \frac{|p_1|^2}{\hat{\gamma}^2} \int_0^1 |\phi|^2 dz \\ & = \frac{R_3^2}{\hat{\gamma}^2} \int_0^1 |w|^2 dz \end{aligned} \quad (5.8.24)$$

Since $\beta \geq 0$, equation (5.8.23) implies that

$$\int_0^1 |(D^2 - a^2)\theta|^2 dz < \frac{R_2^2 a^4 (1 - \alpha_2 T_0)^2 |\tau\beta + B|^2}{|\tau\beta + 1|^2} \int_0^1 |w|^2 dz \quad (5.8.25)$$

$$\int_0^1 |\theta|^2 dz < \frac{R_1^2 a^4 (1 - \alpha_2 T_0)^2 |\tau\beta + B|^2}{|B|^2 B^2 |\tau\beta + 1|^2} \int_0^1 |w|^2 dz \quad (5.8.26)$$

Now

$$\begin{aligned} \int_0^1 [|\theta|^2 + a^2 |\theta|^2] dz &= - \int_0^1 \theta^* (D^2 - a^2) \theta dz = 1 - \int_0^1 \theta^* (D^2 - a^2) \theta dz \\ &\leq \int_0^1 |\theta| |(D^2 - a^2)\theta| dz \\ &\leq \left\{ \int_0^1 |\theta|^2 dz \right\}^{1/2} \left\{ \int_0^1 |(D^2 - a^2)\theta|^2 dz \right\}^{1/2} \end{aligned}$$

(By Schwartz's inequality)

Therefore, we have

$$\int_0^1 [|\theta|^2 + a^2 |\theta|^2] dz < \frac{(1 - \alpha_2 T_0)^2 R_1^2 a^4 |\tau\beta + B|^2}{|B| B |\tau\beta + 1|^2} \int_0^1 |w|^2 dz \quad (5.8.27)$$

Since $\beta \geq 0$, equation (5.8.24) implies that

$$\int_0^1 |D^2 \phi|^2 dz < \frac{R_3^2}{\gamma^2} \int_0^1 |w|^2 dz \quad (5.8.28)$$

which upon using the Rayleigh-Ritz inequality (5.8.19) gives

$$\int_0^1 |\phi|^2 dz < \frac{R_3^2}{\gamma^2 \pi^4} \int_0^1 |w|^2 dz \quad (5.8.29)$$

Equation (5.8.22) together with inequalities (5.8.27) and (5.8.29) gives

$$\begin{aligned} & \tau \int_0^1 (D^2 - \alpha^2) |w|^2 dz + \frac{(2\tau p_r + F)}{\sigma} \int_0^1 |Dw|^2 dz + \alpha^2 \left[\frac{E}{\sigma} - \frac{R_1 \tau (1 - \alpha_2 T_0)}{|B| B} - \frac{R_2}{\hat{\gamma} \pi^2} \right] \int_0^1 |w|^2 dz \\ & + \frac{B^2 \tau p_r + 1}{|R_1| |1 - \alpha_2 T_0| |\tau p_r + B| \alpha^2} \int_0^1 |\theta|^2 dz + \frac{R_2^2 \alpha^2 \hat{\gamma}}{R_3^2} \int_0^1 [|\psi \phi|^2 + \alpha^2 |\phi|^2] dz < 0 \quad (5.8.30) \end{aligned}$$

It follows from inequality (5.8.30), that

$$|p| < \frac{R_1 \tau (1 - \alpha_2 T_0) \sigma \hat{\gamma} \pi^4}{B (F \hat{\gamma} \pi^4 - R_2 \sigma)} \quad (5.8.31)$$

This proves the theorem.

Theorem 3 shows that for modified thermohaline convection with suspended particles, the complex growth rate $p (= p_r + i p_i)$ of an arbitrary oscillatory ($p_r \neq 0$) perturbation, neutral ($p_r = 0$) or unstable ($p_r > 0$), lies within a semi-circle with centre at the origin and radius $= \frac{R_1 \tau (1 - \alpha_2 T_0) \sigma \hat{\gamma} \pi^4}{B [F \hat{\gamma} \pi^4 - R_2 \sigma]}$ in the right half of the $p_r p_i$ -plane. Further, this result is valid for quite general boundary conditions. However, the result is restricted to the case when $\hat{\alpha}_2 = 0$.

THEOREM 4: If (p, w, θ, ϕ) , $p = p_r + i p_i$, $p_r \geq 0$, $p_i \neq 0$ is a solution of equations (5.7.10)-(5.7.12) together with either of the boundary conditions (5.7.13)-(5.7.16), $\hat{\alpha}_2 = 0$ and

$$R_1^2 \tau^2 \sigma^2 \pi^4 \hat{\gamma}^2 (1 - \alpha_2 T_0)^4 > (\hat{\gamma} F \pi^4 - R_2 \sigma)^2, \text{ then}$$

$$|p|^2 < \frac{R_1^2 \tau^2 \sigma^2 \pi^8 \hat{\gamma}^2 (1 - \alpha_2 T_0)^2 - \pi^4 (\hat{\gamma} F \pi^4 - R_2 \sigma)^2}{B^2 (F \hat{\gamma} \pi^4 - R_2 \sigma)^2}$$

Proof: We know from Theorem 2 that for $p_\lambda \neq 0$ any of the boundary conditions (5.7.13)-(5.7.16) equation (5.8.14) holds good. Thus for $p_\lambda \neq 0$, we have

$$\begin{aligned} & \tau \int_0^1 (\rho^2 - a^2) |w|^2 dz + \frac{(2\tau p_\lambda + F)}{\sigma} \int_0^1 [|\rho w|^2 + a^2 |w|^2] dz + \frac{B^2 |\tau p + 1|^2}{R_1 a^2 (1 - \alpha_2 T_0) |\tau p + B|^2} \int_0^1 |\theta|^2 dz \\ & \quad + \frac{\tau R_2 a^2 \hat{\gamma}}{R_3^2} \int_0^1 [|\rho \phi|^2 + a^2 |\phi|^2] dz \\ & = \frac{\tau |\tau p + 1|^2}{R_1 a^2 (1 - \alpha_2 T_0) |\tau p + B|^2} \int_0^1 [|\rho \theta|^2 + a^2 |\theta|^2] dz + \frac{R_2 a^2}{R_3^2} \int_0^1 |\phi|^2 dz \end{aligned} \quad (5.8.32)$$

Since $p_\lambda \geq 0$, equation (5.8.23), upon using the Rayleigh-Ritz inequality (3.8.8) implies that

$$\int_0^1 (\rho^2 - a^2) |\theta|^2 dz < \frac{R_1^2 a^4 |\tau p + B|^2}{|\tau p + 1|^2} (1 - \alpha_2 T_0)^2 \int_0^1 |w|^2 dz \quad (5.8.33)$$

and

$$[\pi^4 + |p|^2 B^2] \int_0^1 |\theta|^2 dz < \frac{|\tau p + B|^2}{|\tau p + 1|^2} R_1^2 a^4 (1 - \alpha_2 T_0)^2 \int_0^1 |w|^2 dz \quad (5.8.34)$$

Now

$$\begin{aligned} \int_0^1 [|\rho \theta|^2 + a^2 |\theta|^2] dz & = - \int_0^1 \theta^* (\rho^2 - a^2) \theta dz = | - \int_0^1 \theta^* (\rho^2 - a^2) \theta dz | \\ & \leq \int_0^1 |\theta| |(\rho^2 - a^2) \theta| dz \\ & \leq \left\{ \int_0^1 |\theta|^2 dz \right\}^{1/2} \left\{ \int_0^1 (\rho^2 - a^2) |\theta| dz \right\}^{1/2} \\ & \quad \text{(By Schwartz's inequality)} \end{aligned}$$

which upon utilizing inequalities (5.8.33) and (5.8.34) gives

$$\int_0^1 [|\theta|^2 + \alpha^2 |\theta'|^2] dz < \frac{R_1^2 \alpha^4 (1 - \alpha_2 T_0)^2 |\tau p + B|^2}{|\tau p + 1|^2 \{ \pi^2 + |p|^2 B^2 \}^{1/2}} \int_0^1 |w|^2 dz \quad (5.8.35)$$

Equation (5.8.32) together with inequalities (5.8.29) and (5.8.35) gives

$$\begin{aligned} \tau \int_0^1 |(D^2 - \alpha^2) w|^2 dz + \frac{(2\tau p + F)}{\sigma} \int_0^1 |Dw|^2 dz + \frac{B^2 |\tau p + 1|^2}{R_1^2 \alpha^2 |\tau p + B|^2 (1 - \alpha_2 T_0)} \int_0^1 |\theta|^2 dz \\ + \alpha^2 \left[\frac{F}{\sigma} - \frac{R_1 \tau (1 - \alpha_2 T_0)}{\{ \pi^2 + |p|^2 B^2 \}^{1/2}} - \frac{R_2}{\hat{\gamma} \pi^4} \right] \int_0^1 |w|^2 dz \\ + \frac{R_2 \alpha^2 \tau \hat{\gamma}}{R_3^2} \int_0^1 [|\phi|^2 + \alpha^2 |\phi'|^2] dz < 0 \end{aligned} \quad (5.8.36)$$

It follows from inequality (5.8.35)

$$|p|^2 < \frac{R_1^2 \tau \sigma^2 \hat{\gamma}^2 \pi^8 (1 - \alpha_2 T_0)^2 - \pi^4 (\hat{\gamma} F \pi^4 - R_2 \sigma)^2}{B^2 (F \hat{\gamma} \pi^4 - R_2 \sigma)^2} \quad (5.8.37)$$

This proves the theorem.

Theorem 4 clearly improves upon the result derived in Theorem 3 by reducing the radius of the semi-circular region therein.

CASE 2 $1 - \alpha_2 T_0 > 0$ with $R_1 < 0$

When $1 - \alpha_2 T_0 > 0$ with $R_1 < 0$, the governing equations and boundary conditions imply that we have a thermohaline convection of Veronis type with suspended particles wherein a

liquid layer is heated overside, in the parameter regime $\alpha_2 T_0 < 1$. We prove the following theorems in this case.

THEOREM 5: If (p, w, θ, ϕ) is a solution of equations (5.7.10)-(5.7.12) together with either of the boundary conditions (5.7.13)-(5.7.16), $\hat{\alpha}_2 = 0$ and $p_i = 0$, then

$$p_R < 0$$

Proof: Equating the real part of the equation (5.8.12), we get

$$\begin{aligned} & (\tau p_R + 1) \int_0^1 (D^2 - a^2) |w|^2 dz + \frac{[\tau(p_R^2 - p_i^2) + F p_R]}{\sigma} \int_0^1 (|Dw|^2 + a^2 |w|^2) dz \\ & + \frac{R_2 a^2}{R_3^2} (\tau |p|^2 + p_R)^2 \int_0^1 |\phi|^2 dz + \frac{R_2 a^2 \hat{\gamma} (\tau p_R + 1)}{R_3^2} \int_0^1 (|D\phi|^2 + a^2 |\phi|^2) dz \\ & = \frac{-\tau p + 11^2}{(R_1 a^2 (1 - \alpha_2 T_0) |\tau p + B|)^2} \left\{ (\tau p_R + B) \int_0^1 (|D\theta|^2 + a^2 |\theta|^2) dz \right. \\ & \quad \left. + B [\tau |p|^2 + B p_R] \int_0^1 |\theta|^2 dz \right\} \quad (5.8.38) \end{aligned}$$

Now, if $p_i = 0$, equation (5.8.38), implies that

$$\begin{aligned} & (\tau p_R + 1) \int_0^1 (|D^2 w|^2 + a^2 |w|^2 + 2a^2 |Dw|^2) dz + \frac{p_R (\tau p_R + F)}{\sigma} \int_0^1 (|Dw|^2 + a^2 |w|^2) dz \\ & + \frac{R_2 a^2}{R_3^2} (\tau |p|^2 + p_R) \int_0^1 |\phi|^2 dz + \frac{R_2 a^2 \hat{\gamma} (\tau p_R + 1)}{R_3^2} \int_0^1 (|D\phi|^2 + a^2 |\phi|^2) dz \end{aligned}$$

$$= \frac{-|\tau p + 1|^2}{|R_1| a^2 (1 - \alpha_2 T_0) |\tau p + B|^2} \left\{ (\tau p_2 + B) \int_0^1 [|\theta|^2 + a^2 |\theta|^2] dz + B [|\tau p_1 + B p_2|] \int_0^1 |\theta|^2 dz \right\} \quad (5.8.39)$$

Equation (5.8.39) under the conditions of the theorem clearly implies that

$$p_2 < 0$$

This proves the theorem.

THEOREM 6: If (ψ, w, θ, ϕ) is a solution of the equations (5.7.10)-(5.7.12) together with either of the boundary conditions (5.7.13)-(5.7.16), $\hat{\alpha}_2 = 0$, $\pi \hat{\tau} \hat{\gamma} \geq 1$, $B \leq \sqrt{\epsilon} \pi$ and $p_2 \neq 0$, then

$$p_2 < 0$$

Proof: For $p_2 \neq 0$, equation (5.8.13) holds

$$\begin{aligned} & \tau \int_0^1 (D^2 - a^2) w^2 dz + \frac{(2\tau p_2 + F)}{\sigma} \int_0^1 [|\theta w|^2 + a^2 |w|^2] dz + \frac{|\tau p + 1|^2}{|R_1| a^2 (1 - \alpha_2 T_0) |\tau p + B|^2} \int_0^1 [|\theta|^2 + a^2 |\theta|^2] dz \\ & + \frac{\tau R_2 a^2 \hat{\gamma}}{R_3^2} \int_0^1 [|\phi|^2 + a^2 |\phi|^2] dz \\ & = \frac{|B|^2 |\tau p + 1|^2}{|R_1| a^2 (1 - \alpha_2 T_0) |\tau p + B|^2} \int_0^1 |\theta|^2 dz + \frac{R_2 a^2}{R_3^2} \int_0^1 |\phi|^2 dz \end{aligned} \quad (5.8.40)$$

Equation (5.8.40), upon using the Rayleigh-Ritz inequality, namely

$$\int_0^1 |D\phi|^2 dz \geq \pi^2 \int_0^1 |\phi|^2 dz$$

or

$$-\int_0^1 |D\phi|^2 dz \leq -\pi^2 \int_0^1 |\phi|^2 dz$$

(5.8.41)

and the inequality (3.8.55)

gives

$$\begin{aligned} & \tau \int_0^1 |(\sigma^2 - a^2)w|^2 dz + \frac{(2\tau p_2 + F)}{\sigma} \int_0^1 [|Dw|^2 + a^2 |w|^2] dz + \frac{R_2 a^2}{R_3} (\tau \hat{\gamma} - \frac{1}{\pi^2}) \int_0^1 |\theta|^2 dz \\ & + \frac{\tau R_2 a^4 \hat{\gamma}}{R_3} \int_0^1 |\phi|^2 + \frac{|\tau p_2 + 1|^2}{|R_1| (1 - \alpha_2 T_0) |\tau p_2 + B|^2} \int_0^1 |\theta|^2 dz \\ & < \frac{|\tau p_2 + 1|^2}{|R_1| a^2 (1 - \alpha_2 T_0) |\tau p_2 + B|^2} [B^2 - \tau \pi^2] \int_0^1 |\theta|^2 dz \end{aligned} \quad (5.8.42)$$

Inequality (5.8.42) under the condition of the theorem clearly implies that

$$p_2 < 0$$

This proves the theorem.

Theorems 5 and 6 establish the stability of the system if $\pi^2 \hat{\gamma} \gg 1$ and $B \leq \sqrt{\epsilon} \pi$ when the liquid layer under consideration is heated from above in the parameter regime

$1 - \alpha_2 T_0 > 0$. However, the result is restricted to the case when $\hat{\alpha}_2 = 0$.

CASE 3 $1 - \alpha_2 T_0 < 0$ with $R_1 < 0$

When $1 - \alpha_2 T_0 < 0$ with $R_1 < 0$ the governing equations and boundary conditions imply that we have a thermohaline convection with suspended particles wherein a liquid layer is heated overside, in the parameter regime $1 - \alpha_2 T_0 < 0$. Since R_1 is negative the initial distribution of density is bottom heavy and therefore statically gravitationally stable and this stabilizing effect together with the joint stabilizing effect of viscosity and conduction is expected to impart, in the usual parameter regime characterized by $1 - \alpha_2 T_0 > 0$, an overall stabilizing effect to the system. The nature of the problem, however, is completely different in the regime $1 - \alpha_2 T_0 < 0$ in which case we have with $R_1 < 0$, $R_1(1 - \alpha_2 T_0) > 0$ and this in turn introduce some new results, namely a sufficient condition for the validity of overstability, condition for the validity of PES and semi-circle theorems for arresting the complex growth rate.

We prove the following theorems.

THEOREM 7: If (ψ, w, θ, ϕ) is a solution of equations (5.7.10)-(5.7.12) together with either of the boundary conditions (5.7.13)-(5.7.16), and $|R_1| \leq \frac{27\pi^4}{4[1 - \alpha_2 T_0 |B - \hat{\alpha}_2 R_3 T_0]}$ then

$$p_\lambda = 0 \Rightarrow p_\lambda \neq 0$$

Proof: Supposing that $p_\lambda = 0$ and proceeding exactly as in the proof of Theorem 1, we have in place of inequality (5.8.10),

the following inequality.

$$\left[\frac{27\pi^4}{4} - |R_1| \left\{ |1 - \alpha_2 T_0| B - \hat{\alpha}_2 R_3 T_0 \right\} \right] \int_0^1 |\omega|^2 dz + \frac{\hat{\gamma} R_2 \alpha^2}{R_3^2} \int_0^1 [|\psi|^2 + \alpha^2 |\phi|^2] dz < 0 \quad (5.8.43)$$

Inequality (5.8.42) obviously cannot hold under the condition of the theorem

Hence $p_i \neq 0$

This proves the theorem.

Theorem 7 shows that for the problem under consideration, a sufficient condition for the validity of overstability is that $|R_1| \leq \frac{27\pi^4}{4[|1 - \alpha_2 T_0| B - \hat{\alpha}_2 R_3 T_0]}$ or equivalently a necessary condition for instability to set in as stationary convection is that $|R_1| > \frac{27\pi^4}{4[|1 - \alpha_2 T_0| B - \hat{\alpha}_2 R_3 T_0]}$. This result is valid for quite general boundary conditions. In particular, it follows that for the Benard convection a necessary condition for instability to set in as stationary convection is that $|R_1| > \frac{27\pi^4}{4}$ when the liquid layer under consideration is heated overside in the parameter regime $1 - \alpha_2 T_0 < 0$.

THEOREM 8: If $(p, \omega, \theta, \phi)$ is a solution of equations (5.7.10)-(5.7.12) together with either of the boundary conditions (5.7.13)-(5.7.16), $\hat{\alpha}_2 = 0$ and $\frac{R_2}{2\hat{\gamma}^2 \pi^4} + \frac{|R_1| |1 - \alpha_2 T_0| C}{\pi^2} \leq \frac{E}{r}$ then

$$p_i > 0 \Rightarrow p_i = 0.$$

Proof: Proceeding exactly as in Theorem 2, we have in place of inequality (5.8.21) the following inequality

$$\begin{aligned} & \tau \int_0^1 |(D^2 a^2) w|^2 dz + \frac{(2\tau p_2 + F)}{\sigma} \int_0^1 a^2 |w|^2 dz + \frac{B^2 |\tau p + 1|^2}{|1 - \alpha_2 T_0| |R_1| a^2 |\tau p + B|^2} \int_0^1 |\theta|^2 dz \\ & + \frac{R_2 a^2 \tau \hat{\gamma}}{R_3} \int_0^1 [(D\phi)^2 + a^2 \phi^2] dz + \left[\frac{F}{\sigma} - \frac{R_2}{2\hat{\gamma}^2 \pi^2} - \frac{|R_1| |1 - \alpha_2 T_0| \tau}{\pi^2} \right] \int_0^1 |Dw|^2 dz < 0 \quad (5.8.44) \end{aligned}$$

Inequality (5.8.44) obviously cannot hold under the condition of the theorem.

Hence $p_2 = 0$

This proves the theorem

It is to be noted that in deriving inequality (5.8.44) the condition $p_2 \geq 0$ has been used. Theorem 8 implies that for the problem under consideration an arbitrary neutral ($p_2 = 0$) or unstable ($p_2 > 0$) mode of the system is definitely non-oscillatory in character and therefore, in particular PES is valid if $\frac{R_2}{2\hat{\gamma}^2 \pi^4} + \frac{|R_1| |1 - \alpha_2 T_0| \tau}{\pi^2} \leq \frac{F}{\sigma}$. Further, this result is valid for quite general boundary conditions. However, the result is restricted to the case when $\hat{\alpha}_2 = 0$.

THEOREM 9: If (p, w, θ, ϕ) , $p = p_2 + i p_1$, $p_2 \geq 0$, $p_1 \neq 0$ is a solution of the equations (5.7.10)-(5.7.12) together with either of the boundary conditions (5.7.13)-(5.7.16),

$\hat{\alpha}_2 = 0$ and $F\hat{\gamma}\pi^2 > R_2\sigma$, then

$$|\beta| < \frac{|R_1|\tau||-\alpha_2 T_0|\sigma\hat{\gamma}\pi^4}{B[F\hat{\gamma}\pi^4 - R_2\sigma]}$$

Proof: Proceeding exactly as in Theorem 3, we have in place of inequality (5.8.30) the following inequality

$$\begin{aligned} & \tau \int_0^1 |(\partial_z^2 - a^2)w|^2 dz + \frac{(2\tau p_2 + F)}{\sigma} \int_0^1 |w|^2 dz + a^2 \left[\frac{F}{\nu} - \frac{|R_1|\tau||-\alpha_2 T_0|}{|\beta|B} - \frac{R_2}{\hat{\gamma}\pi^4} \right] \int_0^1 |w|^2 dz \\ & + \frac{B^2|\tau p + 1|^2}{|R_1|\alpha^2||-\alpha_2 T_0|\tau p + B|^2} \int_0^1 |\theta|^2 dz + \frac{R_2 a^2 \tau \hat{\gamma}}{R_3} \int_0^1 [|\phi|^2 + a^2|\phi|^2] dz < 0 \end{aligned} \quad (5.8.45)$$

It follows from inequality (5.8.45), that

$$|\beta|^2 < \frac{|R_1|\tau||-\alpha_2 T_0|\sigma\hat{\gamma}\pi^4}{B(F\hat{\gamma}\pi^4 - R_2\sigma)} \quad (5.8.46)$$

This proves the theorem.

Theorem 9 shows that for modified thermohaline convection with suspended particles wherein a liquid layer is heated upside, in the parameter regime $|-\alpha_2 T_0| < 0$, the complex growth rate $\beta (= p_R + i p_I)$ of an arbitrary oscillatory ($p_I \neq 0$) perturbation, neutral ($p_R = 0$) or unstable ($p_R > 0$) lies within a semi-circle with centre at the origin and

$$\text{radius} = \frac{|R_1|\tau||-\alpha_2 T_0|\sigma\hat{\gamma}\pi^4}{B(F\hat{\gamma}\pi^4 - R_2\sigma)} \quad \text{in the right half of the}$$

$p \in \mathbb{C}$ -plane. Further, this result is valid for quite general boundary conditions. However, the result is restricted to the case when $\hat{\alpha}_2 = 0$

THEOREM 10: If (p, w, θ, ϕ) , $p = p_r + i p_i$, $p_r \geq 0$, $p_i \neq 0$ is a solution of equations (5.7.10)-(5.7.12), together with either of the boundary conditions (5.7.13)-(5.7.16), $\hat{\alpha}_2 = 0$ and $|R_1|^2 \tau^2 \sigma^2 \pi^4 \hat{\gamma} ||-\alpha_2 T_0|^2 > (\hat{\gamma} F \pi^4 - R_2 \sigma)^2$, then

$$|p|^2 < \frac{|R_1|^2 \tau^2 \sigma^2 \pi^8 \hat{\gamma} ||-\alpha_2 T_0|^2 - \pi^4 (\hat{\gamma} F \pi^4 - R_2 \sigma)^2}{B^2 (F \hat{\gamma} \pi^4 - R_2 \sigma)^2}$$

Proof: Proceeding exactly as in Theorem 4.

We have in place of inequality (5.8.36) the following inequality

$$\begin{aligned} & \tau \int_0^1 |(D^2 - a^2)w|^2 dz + \frac{(2\tau p_r + F)}{\sigma} \int_0^1 |Dw|^2 dz + \frac{B^2 |\tau p + 1|^2}{|R_1|^2 \alpha^2 ||-\alpha_2 T_0| |\tau p + B|^2} \int_0^1 |\theta|^2 dz \\ & + a^2 \left[\frac{F}{\sigma} - \frac{|R_1| ||-\alpha_2 T_0| \tau}{\{ \pi^4 + |p|^2 B^2 \} \gamma_2} - \frac{R_2}{\hat{\gamma} \pi^4} \right] \int_0^1 |w|^2 dz \\ & + \frac{R_2 a^2 \tau \gamma}{R_3^2} \int_0^1 [|D\phi|^2 + a^2 |\phi|^2] dz < 0 \end{aligned} \quad (5.8.47)$$

It follows from inequality (5.8.46), that

$$|P|^2 < \frac{|R_1|^2 \tau^2 \sigma^2 \pi^8 \hat{\gamma} |1 - \alpha_2 T_0|^2 - \pi^4 (\hat{\gamma} F \pi^4 - R_2 \sigma)^2}{B^2 (F \hat{\gamma} \pi^4 - R_2 \sigma)^2} \quad (5.8.48)$$

This proves the theorem.

Theorem 10 clearly improves upon the result derived in Theorem 9 by reducing the radius of the semi-circular region therein.

CASE 4 $1 - \alpha_2 T_0 < 0$ with $R_1 > 0$

When $1 - \alpha_2 T_0 < 0$ with $R_1 > 0$ governing equations and boundary conditions imply we have a thermohaline convection of Veronis' type with suspended particles wherein a liquid layer is heated underside, in the parameter regime $1 - \alpha_2 T_0 < 0$. Since R_1 is positive the initial distribution of density is top heavy and therefore statically potentially gravitationally unstable and this destabilizing effect together with the joint stabilizing effects of viscosity and conduction is expected to impart, in the usual parameter regime characterized by $1 - \alpha_2 T_0 > 0$, the character of a modified thermohaline convection with suspended particles in which the liquid layer is heated from below. The nature of the problem, however is completely different in the regime $1 - \alpha_2 T_0 < 0$ in which case we have, with $R_1 > 0$, $R_1(1 - \alpha_2 T_0) < 0$ and this in turn forces all the perturbation to decay and thus making the system stable.

This prove the following theorems.

THEOREM 11: If (β, w, θ, ϕ) is a solution of equations (5.7.10)-(5.7.12) together with either of the boundary conditions (5.7.13)-(5.7.16) $\beta_1 = 0$, and $\hat{\alpha}_2 = 0$, then

$$\beta_2 < 0$$

Proof: Proceeding exactly as in Theorem 5.

We have in place of equation (5.8.39) the following equation

$$\begin{aligned} & (\tau \beta_2 + 1) \int_0^1 [|Dw|^2 + a^2 |w|^2 + 2a^2 |Dw|^2] dz + \beta_2 \frac{[\tau \beta_2 + F]}{\sigma} \int_0^1 [|Dw|^2 + a^2 |w|^2] dz \\ & + \frac{R_2 a^2}{R_3^2} [|\beta|^2 \tau + \beta_2] \int_0^1 |\phi|^2 dz + \frac{R_2 a^2 \hat{\gamma} (\tau \beta_2 + 1)}{R_3^2} \int_0^1 [|D\phi|^2 + a^2 |\phi|^2] dz \\ & = \frac{-|\tau \beta + 1|^2}{R_1 a^2 |1 - \alpha_2 \tau_0| |\tau \beta + B|^2} \left\{ (\tau \beta_2 + B) \int_0^1 [|D\theta|^2 + a^2 |\theta|^2] dz \right. \\ & \quad \left. + B [\tau |\beta|^2 + B \beta_2] \int_0^1 |\theta|^2 dz \right\} \end{aligned} \quad (5.8.49)$$

Equation (5.8.49) under the conditions of the theorem clearly implies that

$$\beta_2 < 0$$

This proves the theorem.

THEOREM 12: If (β, w, θ, ϕ) is a solution of the equations (5.7.10)-(5.7.12) with boundary conditions (5.7.13)-(5.7.16), $\hat{\alpha}_2 = 0$, $\tau \hat{\gamma} \gg 1$, $B \leq \sqrt{2} \pi$ and $\beta_1 \neq 0$ then

$$\beta_2 < 0$$

Proof: Proceeding exactly as in Theorem 6.

We have in place of inequality (5.8.42) the following inequality

$$\begin{aligned}
 & \tau \int_0^1 |(D^2 - a^2) w|^2 dz + \frac{(2\tau p_2 + F)}{\sigma} \int_0^1 [Dw|^2 + a^2 |w|^2] dz + \frac{R_2 a^2}{R_3^2} (\tau \hat{\gamma} - \frac{1}{\pi^2}) \int_0^1 |D\phi|^2 dz \\
 & + \frac{\tau R_2 a^4 \hat{\gamma}}{R_3^2} \int_0^1 |\phi|^2 dz + \frac{|\tau p + 1|^2}{R_1 |1 - \alpha_2 T_0| |\tau p + B|^2} \int_0^1 |\theta|^2 dz \\
 & < \frac{|\tau p + 1|^2}{R_1 a^2 |1 - \alpha_2 T_0| |\tau p + B|^2} [B^2 - \tau \pi^2] \int_0^1 |\theta|^2 dz \quad (5.8.50)
 \end{aligned}$$

Inequality (5.8.50) under the conditions of the theorem clearly implies that

$$p_2 < 0$$

This proves the theorem

Theorem 11 and 12 establish the stability of the system if

$\tau \hat{\gamma} \geq 1$ and $B \leq \sqrt{\tau} \pi$ when the liquid layer under consideration is heated from below in the parameter regime $1 - \alpha_2 T_0 < 0$. However, the result is restricted to the case when $\hat{\alpha}_2 = 0$.