Chapter-II
Evaluation of eddy current loss in solid cores subjected to single excitation

2.1 INTRODUCTION

The study of eddy current loss in ferromagnetic cores subjected to alternating flux has been the subject over many years. The analysis associated with the penetration of alternating flux and the evaluation of eddy current distribution in ferromagnetic materials depends on the determination of the flux density distribution inside the medium. Such an analysis becomes difficult when the exact nature of the hysteresis loop or the normal magnetisation characteristic (B-H curve) of the material is taken into account. To gain an insight into the complexities of the phenomenon involved, the results of the linear theory [2] are examined.

Once the magnetic flux density and eddy current distributions are determined by the linear theory, the core losses constituting the eddy current and hysteresis losses can be evaluated in a straightforward manner. But the assumption of constant permeability is not always valid, because, the magnetisation curve of a material, in general is nonlinear in nature. So, to obtain expressions for the core loss, power factor etc., in the presence of saturation, the magnetisation curve of a material is approximated by a relax type curve. This theory (Limiting Nonlinear Theory) [3] has yielded good results.

Also in [4], the nonlinear dependence of flux density on the magnetic field strength is replaced by the equation \( B = kH^n \), where \( k \) and \( n \) are coefficients derived from the magnetisation curve. This type of dependence has the advantage that when \( n = 1 \) it becomes a linear theory and when \( n = 0 \) it reduces to limiting theory. When \( n \) has a value between 0 and 1, it represents a characteristic between the two limiting cases. With this type of approximation, the distribution of field, eddy current
loss and power factor are depending on the coefficients n and k. The nonlinear dependence of the form, \( B = \mu(H - \epsilon H^3) \) is also considered in [5], where, \( \epsilon \) being small. Based on this approximation, expressions are deduced for various quantities. Further, it is modified to take account of presence of hysteresis. Later numerical steady state solutions are developed by various authors to take exact shape of B-H curve into account. One of such methods is Crank-Nicholson Method [9]. In this chapter, a new classical numerical method called Pseudo-Spectral Method is presented. Upon touching the linear theory, Limiting Nonlinear Theory and Crank-Nicholson method, a comparison is made among the methods excluding Linear Theory.

2.2 FORMULATION OF PROBLEM

Consider an infinite half-space of iron subjected to an alternating magnetising force at the surface. An infinite half-space is defined as a region of a material which, for example, extends from \(-\infty\) to \(+\infty\) in the y and z- directions, and from 0 to \(\infty\) in the x-direction as shown in fig.2.1. The surface of iron is chosen as the y-z plane of a cartesian coordinate system, the x-axis is normal to the surface and extends into the material. The magnetizing field (H) at the surface is in the y-direction and is independent of z. Clearly, at any point inside the material, only H, and \( J_r \) (current density), exist, and they are purely functions of 'x' and time 't', owing to the assumptions of large width and length of iron along z and y-axes respectively.
Therefore, Maxwell's equations for this coordinate system are

\[ \frac{\partial H_x}{\partial x} = J, \quad (2.1) \]

\[ \frac{\rho}{\partial x} - \frac{\partial E_B}{\partial t} = \frac{\partial B_x}{\partial t}, \quad (2.2) \]

The above two equations give rise to equation of field distribution as

\[ \frac{\partial^2 H_x}{\partial x^2} = \frac{1}{\rho} \frac{\partial B_x}{\partial t} \]
(or) simply
\[
\frac{\partial^2 H}{\partial x^2} = \frac{1}{\rho} \frac{\partial B}{\partial t}
\]
(2.3)

Where \( \rho \) is specific resistance of the material.

If it is assumed that the material is homogenous and isotropic, the fundamental relationship between \( H \) and \( B \) is governed by the magnetisation characteristics of the material and can be written as

\[
B = f(H)
\]
(2.4)

The boundary conditions are

\[
\begin{align*}
(i) \text{ At } x = 0, H &= H_s \cos(T) \\
(ii) \text{ At } x = x_n, \frac{dH}{dx} &= 0 \text{ for all } "T" \\
\text{(or) As } x &\to 0, H = 0 \text{ for all } "T"
\end{align*}
\]
(2.5)

Where, \( T = \omega t \) and subscript 's' stands for surface value.

2.3 LINEAR THEORY

If the magnetisation curve is assumed to be linear, then the equation (2.4) becomes

\[
B = \mu H
\]
(2.6)

Where \( \mu \) is the permeability of the material, taken as constant.

Now the equation (2.3) can be written as

\[
\frac{\partial^2 H}{\partial x^2} = \frac{\mu}{\rho} \frac{\partial H}{\partial t}
\]
(2.7)

Assume a solution for \( H \) as

\[
H(x,t) = H(x)e^{i\omega t}
\]
(or) \[ H = H_0 e^{j\omega t} \] (2.8)

and substitute in equation (2.7), leads to

\[ \frac{\partial^2 H}{\partial x^2} = 2j\lambda^2 H \] (2.9)

where \( \lambda = \sqrt{\frac{\omega \mu}{2\rho}} \)

The solution of second order linear differential equation (2.9), simultaneously satisfying the boundary conditions of equation (2.5) is

\[ H = H_\xi e^{j\sqrt{2j} \lambda x} \] (2.10)

Then

\[ J = \frac{\partial H}{\partial x} = -j|\lambda_1| H_\xi e^{j\sqrt{2j} \lambda x} \] (2.11)

And flux

\[ \phi = \frac{\rho|J|}{\omega} = \frac{\rho|\lambda_1| H_\xi}{\omega} e^{j\sqrt{2j} \lambda x} \] (2.12)

Where

\[ \lambda_1 = \lambda \sqrt{2j} \]

The real and imaginary parts of the solution given by equation (2.10), corresponding to the surface magnetizing force being \( H_S \cos(\omega t) \) and \( H_S \sin(\omega t) \) respectively. If we assume that the surface magnetizing force is \( H_S \cos(\omega t) \), then the solution equation (2.10) becomes

\[ H = H_\xi e^{-\lambda x} \cos(\omega t - \lambda x) \] (2.13)
To see the variation of $H$ with $x$, from the above equation (2.13), consider two cases

(i) $\omega t = 0 \quad H = H_0 e^{-\lambda x} \cos(\lambda x)$

(ii) $\omega t = \frac{\pi}{2} \quad H = H_0 e^{\lambda x} \sin(\lambda x)$

The above two equations reveal that the $H$-wave is propagating in linear conducting medium. The phase and attenuation constants are equal and given by $\lambda$. The distance traveled by the plane electromagnetic wave in the medium is one period, is termed as its wavelength. In other words, it is the distance between the points of corresponding phases of two consecutive waves. In the present case, the wavelength is given by $2\pi/\lambda$, and the phase front is propagated at the phase velocity in the direction of $x$-axis.

The phase of a wave is decided by $(\omega t - \lambda x)$. Since the derivative of a constant is zero, the phase velocity is given by $(\omega/\lambda)$. In a linear magnetic material, with good conductivity and high permeability, both attenuation and phase propagation constants are high, whereas the wavelength and phase velocity are very low. Thus the flux penetration into the medium is viewed more as a process of magnetic diffusion rather than as electromagnetic wave propagation. Also, a close examination of expressions in this section leads to certain important physical interpretations. These are

1. The amplitude of magnetizing force and current density attenuates exponentially in the $x$-direction, besides phase shift. In fact, a constant

\[
\left(\frac{1}{\lambda}\right) \text{ can defined where, } \frac{1}{\lambda} = \frac{2\rho}{\mu} \text{ having the dimension of length and}
\]

signifying the depth of penetration of the flux inside the medium.
(ii) The magnetizing force at any layer below and parallel to the surface is equal to the total current (per unit length in the y-direction) below that layer,

(iii) The current density at any layer below and parallel to the surface leads the magnetizing force, in time phase by 45° (electrical), at the same layer.

(iv) The total flux (per unit length in the z-direction) below any layer lags the magnetizing force at that layer by 45°. Hence, the power factor is $1/\sqrt{2}$

2.3.1 Evaluation of eddy current loss

Two methods of approach are possible for the evaluation of eddy-current loss in the core

**Method 1:** The average loss over one cycle of time period 'T' is

$$P_c = \int \int \int \rho |\mathbf{J}|^2 \, dv \, dt$$  \hspace{1cm} (2.14)

The integration is over the volume 'V' of the material. For harmonic variation of $J_i$, the above expression (2.14) simplifies to

$$P_c = \frac{1}{2} \int \int \int \rho \left( J_i^2 + J_j^2 + J_k^2 \right) \, dv$$  \hspace{1cm} (2.15)

**Method 2:** The energy flow in the interval of time 'T' into the material of volume 'V', bounded by surface 'S', using the Poynting vector is

$$W = \int \int \int \left( \mathbf{E} \cdot \mathbf{H} \right) \, dv \, dt$$  \hspace{1cm} (2.16)

The average power loss is

$$P_c = \frac{1}{T} \int \int \int \left( \mathbf{E} \cdot \mathbf{H} \right) \, dv \, dt$$  \hspace{1cm} (2.17)
If \( E = E_S \sin(\omega t) \)

and \( H = H_S \sin(\omega t - \psi) \)

Then equation (2.17) becomes

\[
P_e = \left( \frac{1}{2} \right) \int_{-l}^{l} E_S H_S \sin(\theta) \cos(\psi) \, ds
\]

(2.18)

Where \( \theta \) is the angle between the vectors \( E \) and \( H \) and \( \psi \) is the time phase difference between \( E \) and \( H \).

At this juncture, it is worthwhile to note that the power transferred by the magnetizing winding must appear as the total losses in the core. This power is equal to the product of the voltage required to balance the induced e.m.f in the winding, the current through the winding and the power factor of the winding. In general, the problem is one of determining the induced e.m.f and the current in the winding. The e.m.f induced in the winding per unit length in the \( z \)-direction is the same as the electrical intensity \( E_{zs} \), at the surface of the iron.

Thus

\[
E_{zs} = \rho J_{zs} = \rho |\lambda_z| H_S \, e^{i\omega t}
\]

(2.19)

Where \( J_{zs} \) is given by the equation (2.11), when \( x = 0 \).

Therefore, \( E_{zs} \) is the voltage applied per unit length of the winding. The current in the winding per unit length in the \( y \)-direction is \( H_{sy} = H_S \, e^{i\omega t} \). Thus the average power loss per unit surface area from the equation (2.18) is

\[
P_e = \left( \frac{1}{2} \right) E_{zs} H_{sy} \cos(E_{zs}, H_{sy})
\]

\[
= \left( \frac{1}{2} \right) (\rho |\lambda_z| H_S) H_S \cos(\pi/4)
\]

\[
= \left( \frac{\rho}{2\sqrt{2}} \right) |\lambda_z| H_S^2 \, \text{watts}
\]

(2.20)
Alternatively, the loss per unit surface area is given by equation (2.15), is

\[
P_x = \left( \frac{1}{2} \right) \rho \int \left| \mathbf{J}_s \right|^2 \, dx
\] (2.21)

Substituting equation (2.11) for \( \mathbf{J} \) and integrating results

\[
P_x = \left( \frac{\rho}{2\sqrt{2}} \right) \lambda_1 H_s^2 \text{ watts}
\] (2.22)

2.4 LIMITING NONLINEAR THEORY (LNT)

2.4.1 Graphical Construction

The magnetisation curve of the material is approximated by limiting curve, as shown in fig. 2.2 where it is assumed that the linearity ends at small values of \( H \). So the initial step for graphical construction is to assume a small value for \( H \) at which the linearity terminates. Using this \( H \), the corresponding flux \( \phi \) is found from the linear theory i.e. from equation (2.12). Where as 'x' may be taken as very small value say \( 1 \times 10^{-4} \) m. Then the values of \( H \) and \( \phi \) at an adjacent layer, at distance \( (x+\Delta x) \) are found by taking the vector sum of \( H \) and \( \Delta H \), \( \phi \) and \( \Delta \phi \) respectively. Where,

\[
\Delta H = \frac{(j\phi \omega \Delta x)}{\rho} \\
\text{and} \quad \Delta \phi = B_m \Delta x
\] (2.23)

\[
\Delta H_n \perp \text{to} \, \phi_n \quad (n = 1, 2, \ldots)
\]

\[
\Delta \phi_n \parallel \text{to} \, H_n \quad (n = 1, 2, \ldots)
\]

In this way \( H \) and \( \phi \) at successive layers are computed. To reduce the errors, the incremental depth \( \Delta x \) should be as small as possible. It is suggested that for a known value of surface excitation \( H_s \), the corresponding value of flux density from B-H curve may be chosen as \( B_m \).
Employing the above procedure, the loci of $H$ and $\phi$ are obtained in [3]. These are shown in fig 2.3.

Fig 2.2: Relay type B-H curve
Fig. 2.3: The geometry of $H$ and $\phi$ loci

From the geometry of $H$ and $\phi$ loci (fig. 2.3), it is followed that

$$\frac{dH}{\sin \psi} = \frac{\omega \phi}{\rho} dx$$

(2.24)

Based on physical consideration, the power loss per unit surface area is

$$P = \left(\frac{1}{2}\right) E H \sin \psi$$

(2.25)

Making use of equation (2.24) for $\sin \psi$ and $E = \omega \phi$, the equation (2.25) is modified as

$$P = \left(\frac{1}{2}\right) \rho H \frac{dH}{dx}$$

(2.26)
Also, the total power loss per unit surface area is

\[ P_i = \left( \frac{1}{2} \right) \int_0^1 (\omega \phi)^2 \, dx \]  

(2.27)

Equate equation (2.27) with (2.26) and solve for \( \phi \)

\[ \phi = \frac{\rho}{\omega} \left( H \frac{d^2 H}{dx^2} + \left( \frac{dH}{dx} \right)^2 \right)^{1/2} \]  

(2.28)

Now, substituting for \( \phi \) in equation (2.24), so as to get an equation for \( \sin \psi \)

\[ \sin \psi = \frac{\frac{dH}{dx}}{\left( H \frac{d^2 H}{dx^2} + \left( \frac{dH}{dx} \right)^2 \right)^{1/2}} \]  

(2.29)

2.4.2 Particular solution of the problem [3]

Let \( H = H \cos(\omega t + \theta) \)

and \( B = B \cos(\omega t + \theta) \)

substituting these two equations in (2.3) and equating similar terms, yields

\[ \frac{d^2 H}{dx^2} = H \left( \frac{d\theta}{dx} \right)^2 \]  

(2.30a)

\[ \frac{1}{H} \frac{d}{dx} \left( H \frac{d\theta}{dx} \right) = K B_m \]  

(2.30b)

Where \( K = \omega \rho \).

If \( H \) assumed to be \( H = K B_m H' \), then the equation (2.30) can be normalised as

\[ \frac{d^2 H'}{dx^2} - H' \left( \frac{d\theta}{dx} \right)^2 = 0 \]  

(2.31a)
The normalized step-by-step graphical construction outlined in section 2.4.1, is in effect the solution of nonlinear simultaneous differential equations (2.31a) and (2.31b) respectively.

There appears to be no general solution with arbitrary constants. However, one could recognize the above two equations as analogous to the dynamics of motion of a particle with radial and transverse accelerations. It is well known result that, if a particle moves in an equiangular spiral and if the radial acceleration is zero, then the transverse acceleration is proportional to square of distance. Stated explicitly, if $H'$ is proportional to $e^{\omega x}$, then the above equations (2.31a) and (2.31b) holds simultaneously.

As a consequence,

$$H' = \frac{1}{3\sqrt{2}} (x + d)^{\frac{3}{2}}$$

(2.32)

Where $d$ is an arbitrary constant.

If it is assumed that $d=0$, the equation (2.32) reduces to

$$H = \frac{1}{3\sqrt{2}} x$$

(2.33)

So,

$$H = K B_m H' = \frac{\omega B_m x^2}{3\rho\sqrt{2}} = a x^2$$

(2.34)

Where $a = \omega B_m / (3\rho\sqrt{2})$. 
Once the law of variation of $H$ with $x$ is known, the relevant quantities could be deduced using the equations (2.26), (2.28), (2.29) and (2.34). The derived equations are as follows:

**Loss per unit surface area**

$$P_i = \frac{\rho \omega B_m H_s^{3/2}}{3\sqrt{2}} \text{ Watts/m}^2$$  \hspace{1cm} (2.35)

**Flux per unit length of perimeter**

$$\phi_s = \frac{\sqrt{2}\rho B_m H_s}{\omega} \text{ Webers/m}$$  \hspace{1cm} (2.36)

**Power factor**

$$\sin\psi_s = \frac{2}{\sqrt{3}}$$  \hspace{1cm} (2.37)

**Total depth of penetration**

$$X = \frac{3\sqrt{2}\rho H_s}{\omega B_m} \text{ meters}$$  \hspace{1cm} (2.38)

Where the subscript 's' is an indication of surface value.

**2.4.3 Limitations**

(i) The actual distortion of $H$ at each layer cannot be visualized, since it is assumed that the magnetising force at each layer is having only fundamental component.

(ii) The results of the graphical construction cannot be used directly for plates of finite thickness.
2.5 CRANK-NICHOLSON METHOD (CNM)

An implicit scheme for a parabolic partial differential equation (2.41) leads to a system of algebraic equations after replacing $\partial H/\partial t$ by forward finite difference approximation and $\partial^2 H/\partial x^2$ by the arithmetic mean of its central difference approximations on the $i^{th}$ and $(i+1)^{th}$ time-rows. The figure 2.4 shows the discretization. The computation of unknown pivotal values requires the solution of a system of linear equations. Though the implicit scheme is iterative in nature, but the method will converge for all finite values of $\Delta T/\Delta x^2$. Whereas the explicit scheme is valid only for $(\Delta T/\Delta x^2) \leq 0.5$. Therefore, this restriction, necessitates a very small time step, making the method computationally uneconomical.

![Fig 2.4: Grid showing boundary and initial conditions for parabolic PDE](image-url)
The magnetic curve has been represented by the Frohlich equation, given by

\[ B = \frac{\alpha|H|}{\gamma + |H|} \]  

(2.39)

The equation (2.3) can be written as

\[ \frac{\partial^2 H}{\partial x^2} = \frac{1}{\rho} \left( \frac{\partial B}{\partial H} \right) \frac{\partial H}{\partial t} \]  

(2.40)

Where \( \partial B/\partial H \) is the slope of the curve at a point calculated by taking the derivative of equation (2.39). Hence, the equation (2.40) reduces to magnetic diffusion equation,

\[ \frac{\partial^2 H}{\partial x^2} = \frac{1}{\rho \gamma^2} \left( \frac{\partial H}{\partial t} \right) \]  

(or)

\[ \frac{\partial H}{\partial t} = S(H) \frac{\partial^2 H}{\partial x^2} \]  

(2.41)

where \( S(H) = \frac{\rho (\gamma + |H|)^2}{\alpha \gamma} \)

In general the finite difference scheme to diffusion equation is given by

\[ H^{n+1} = H^n + \Delta t \left[ \theta F^{n+1} + (1 - \theta) F^n \right] \]  

(2.42)

Where \( F^n \) is the second derivative of \( H \) at the \( n^{th} \) time-step.

The above equation (2.42) gives different schemes depending upon the value of \( \theta \).

(i) If \( \theta = 0 \), Explicit scheme

(ii) \( \theta = 1 \), Fully implicit scheme, and

(iii) \( \theta = 0.5 \), Crank Nicholson scheme

In the implicit scheme of Crank-Nicholson Method, as shown in fig 2.4, the finite difference approximation to equation (2.41) is
\[
\frac{H_{i+1,j} - H_{i,j}}{\Delta T} = \frac{S(H)}{2} \left( \frac{H_{i+1,j+1} - 2H_{i+1,j} + H_{i+1,j-1}}{\Delta x^2} + \frac{H_{i,j+1} - 2H_{i,j} + H_{i,j-1}}{\Delta x^2} \right) \tag{2.43}
\]

After rearrangement, the above equation (2.43) becomes

\[
(2 + 2K)H_{i+1,j} - K(H_{i+1,j+1} - H_{i+1,j-1}) - K(H_{i,j+1}) + (2 - 2K)(H_{i,j} + K(H_{i,j+1})
\]

\[
\tag{2.44}
\]

Where \( K = \frac{\Delta T}{\Delta x^2} S(H) \)

It should be noted that in equation (2.44), the LHS contains three pivotal values of \( H \), which are all unknown, whereas the RHS values of \( H \) are all known. If there are \( N-1 \) internal mesh points, then for \( i = 1 \) and \( j = 2, 3, \ldots, N \), equation (2.44) gives a system of \((N-1)\) linear equations in \((N-1)\) unknown pivotal values of \( H \) along the second time-row. These are in terms of known initial values at first time-row and boundary values. Similarly, for \( i = 2 \), \( j = 2, 3, \ldots, N \), equation (2.44) gives a system of \((N-1)\) linear equations in \((N-1)\) unknown \( H \)'s along the third time-row in terms of the computed \( H \)'s along the second time-row and so on. The coefficient matrix formed by these \((N-1)\) linear simultaneous equations is tri-diagonal. Consequently Thomas algorithm is implemented to solve the equations. It may be noted that the number of operations (only multiplications and divisions) with \('N'\) grid points are \(5(N-2)-4\). Hence, the reduction in computation time.

2.5.1 Modified Crank-Nicholson Method

A modification is done to the standard Crank-Nicholson Method. The modification is given below in a sequence of steps:
(i) The nodal values are computed for one full cycle of surface excitation $h_s = H_s \cos(T)$, using the initial values (zeros) at the first-time-row $T = 0$ and boundary conditions;

(ii) The initial values at $T = 0$ are replaced by the nodal values at $T = 2\pi$;

(iii) The new values at the second time-row are computed using the revised initial values and slopes. The slopes are determined using the average of revised initial values and old values at the second time-row at each layer;

(iv) The new values at the second time-row are compared with the old values. If the difference is more, the old values will be replaced by the new values at each value of $x$ and step (iii) will be repeated. This is done till the error criteria is satisfied, say $e = 0.1$;

(v) In the same way the nodal values at all other rows are computed;

(vi) To achieve symmetry of the penetrating wave, the nodal values at $T = \pi/2$ are compared with the values at $T = 3\pi/2$ at all layers, if the difference is more than 5, then whole procedure starting from step (ii) will be repeated.

2.6 PSEUDO-SPECTRAL METHOD (PSM)

A new class of methods for obtaining numerical solutions of partial differential equations are known as spectral methods [17]. Spectral methods may be viewed as an extreme development of class of discretization schemes for differential equations known generically as the Method of Weighted Residuals (MWR). The key elements of the MWR are the trial functions (also called expansion or approximate functions) and test functions (also known as weight functions). The trial functions are used as the basis functions for a truncated series expansion of the solution. The test
functions are used to ensure that the differential equation is satisfied as closely as possible by the truncated series expansion. This is achieved by minimizing the residual. An equivalent requirement is that the residual should satisfy a suitable orthogonality condition with respect to each of the test functions.

The trial functions for the spectral methods are infinitely differentiable global functions. In the case of finite-element methods, the domain is divided into small elements and trial function is specified in each element. Thus the trial functions are local in character and well suited for handling complex geometries [18,19]. The finite difference trial functions are likewise local.

The choice of test functions distinguishes three most commonly used spectral schemes. Namely, the Galerkin, Collocation, and Tau versions. In the Galerkin method, the test functions are the same as the trial functions. In the Collocation method, the test functions are translated Dirac delta functions or unit impulse functions \( \delta(x - x_i) \), centered at special, so called collocation points. This approach requires that the differential equation to be satisfied exactly at the collocation points. Spectral tau methods in the way that the differential equation is enforced. However, none of the test functions satisfy the boundary conditions. Hence, a supplementary set of equations are used to apply the boundary conditions.

The commonly used trial functions are

(a) Fourier \( e^{i k x} \)

(b) Chebyshev \( T_k (x) \)

(c) Legendre \( L_k (x) \)

(d) Spherical Harmonics \( Y_n^m(x,y) \)
The Fourier functions can be used only for periodic problems, whereas, Chebyshev functions can be used for the expansion of periodic as well as non-periodic functions. All the trial functions are orthogonal functions. Chebyshev functions are defined in the interval \(-1 \leq x \leq 1\). However, a function can be approximated in the region \(a \leq y \leq b\), employing linear mapping.

\[
x = \frac{2(y - a)}{(b - a)} - 1 \quad (2.45)
\]

If \(a = 0\) and \(b = 1\), the span of space is reduced to half the total value and the equation (2.45) reduces to

\[
x = 2y - 1 \quad \text{(or)} \quad \partial x = 2 \, \partial y \quad (2.46)
\]

The above equation (2.46) can be written as

\[
\frac{\partial u}{\partial x} = \frac{\partial u}{2 \, \partial y}
\]

Further,

\[
\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x} \left( \frac{1}{2} \frac{\partial u}{\partial y} \right) = \frac{\partial}{\partial y} \left( \frac{1}{2} \frac{\partial u}{\partial y} \right) = \frac{1}{4} \frac{\partial^2 u}{\partial y^2}
\]

\[
\text{(or)} \quad \frac{\partial^3 u}{\partial y^2 \partial x} = 4 \frac{\partial^3 u}{\partial x^2} \quad (2.47)
\]

In general

\[
\frac{\partial^2 u}{\partial y^2} = K^2 \frac{\partial^2 u}{\partial x^2} \quad (2.48)
\]

Where \(K\) is the factor by which the span is compressed.
The approximation of a function $f(x)$ is given by a finite Chebyshev series with $N$ terms as

$$f_N(x) = \sum_{k=0}^{N} a_k T_k(x)$$  \hfill (2.49)

Where $a_k$'s are spectral coefficients. These are computed like Fourier coefficients by making use of the orthogonal property of functions. The orthogonal property of the Chebyshev trial functions is

$$\int_{-1}^{1} \frac{T_m(x)T_n(x)dx}{\sqrt{1-x^2}} = \left( \frac{\pi}{2} \right) C_m \delta_{mn}$$ \hfill (2.50)

Where, $\delta_{mn}$ is the Kronecker delta function defined by

$$\delta_{mn} = \begin{cases} 
1, & m = k \\
0, & \text{otherwise}
\end{cases}$$

and

$$C_m \ (\text{or} \ C_k) = \begin{cases} 
2, & m = 0 \\
1, & \text{otherwise}
\end{cases}$$

Upon multiplying the equation (2.49) by $T_m(x)/\sqrt{(1-x^2)}$ and integrating from $x = -1$ to $1$, yields an expression for the spectral coefficients as

$$a_m = \frac{2}{\pi C_m} \int_{-1}^{1} \frac{f(x) T_m(x)dx}{\sqrt{1-x^2}}$$ \hfill (2.51)
The grid points or collocation points are so chosen that the accurate evaluation of integral of equation (2.51) is achieved. The Gaussian quadratures are very accurate means of numerical integration, wherein the choice of weights and collocation points are found, so as to get the best approximation of an integral. The integral is estimated by using a weighted sum of the function values at the collocation points. That is

\[ \int_a^b W(x) f(x) dx = \sum_{j=0}^N \alpha_j f_N(x_j) \]  

(2.52)

with \( W(x) = \frac{1}{\sqrt{1-x^2}} \)

Where

- \( W(x) \) - a non-negative weight function
- \( f_N(x_j) \) - the function values at the collocation points \( x_j \). These values are written as \( f_j; j = 0,1,2,\ldots,N \).
- \( \alpha_j \) - the weights assigned to this function value.

For the Chebyshev, the collocation points (These are called as Chebyshev-Gauss-Lobatto points) are defined by

\[ x_j = \cos \left( \frac{\pi j}{N} \right), \quad j = 0,1,\ldots,N. \]  

(2.53)

This set has the end points \( (x = \pm 1) \), which makes it easier to impose boundary conditions. Moreover, not only does this choice of uneven spacing produce highly accurate approximations, but also enables the fast Fourier transforms to be implemented.

For the Chebyshev - Gauss - Lobatto quadratures, the weights are
Replacing the RHS of equation (2.51) by the quadratures \{equation (2.52)}), gives

\[ a_k = \left( \frac{2}{\pi C_k} \right) \sum_{j=0}^{N} \alpha_j \Phi(x_j), \quad k = 0, 1, \ldots, N. \] (2.55)

Where \( T_k(x_j) \) are the Chebyshev polynomials.

The equation (2.55) can be evaluated by fast Fourier transforms.

The Chebyshev polynomials are defined by

\[ T_k(x_j) = \cos[k \cos^{-1}(x)] \] (2.56)

With the uneven spacing, given by the equation (2.53), the equation (2.56) reduces to

\[ T_k(x_j) = \cos\left(\frac{k \pi j}{N}\right) \] (2.57)

The equation (2.55) can be written in the matrix form

\[
\begin{pmatrix}
  a_0 \\
  a_1 \\
  a_2 \\
  \vdots \\
  a_N
\end{pmatrix}
= \begin{pmatrix}
  f_0 \\
  f_1 \\
  f_2 \\
  \vdots \\
  f_N
\end{pmatrix}
\]

(2.58)
Where the elements of $A$ are given by

$$A(j, k) = \frac{2}{N} \frac{1}{c_k} \frac{1}{c_j} \cos\left(\frac{\pi j k}{N}\right); \quad j, k = 0, 1, \ldots, N.$$  \hfill (2.59)

with $c_0 = c_N = 2; c_i = 1, 1 \leq i \leq N-1$.

One of the methods [17, 20] to evaluate derivatives of function $f(x)$ at the collocation points is as follows:

The second derivative of equation (2.49) is given by

$$f^{(2)}_N(x_j) = \sum_{k=0}^{N} a_k T^{(2)}_k(x_j); \quad j = 0, 1, \ldots, N$$  \hfill (2.60)

Also

$$f^{(2)}_N(x_j) = \sum_{k=0}^{N} T^{(2)}_k(x_j) a_k; \quad j = 0, 1, \ldots, N$$  \hfill (2.61)

Where the superscript '2' within brackets indicate the second derivative.

By substituting equation (2.58), the equation (2.61) can be brought into matrix form, where the second derivatives are in terms of function values themselves.

$$
\begin{pmatrix}
  f^{(1)}_0 \\
  f^{(1)}_1 \\
  f^{(1)}_2 \\
  \vdots \\
  f^{(1)}_N
\end{pmatrix}
= 
\begin{pmatrix}
  A \\
  (N+1) \times (N+1)
\end{pmatrix}
\begin{pmatrix}
  f^{(2)}_0 \\
  f^{(2)}_1 \\
  f^{(2)}_2 \\
  \vdots \\
  f^{(2)}_N
\end{pmatrix}
= 
\begin{pmatrix}
  B \\
  (N+1) \times (N+1)
\end{pmatrix}
\begin{pmatrix}
  f^{(1)}_0 \\
  f^{(1)}_1 \\
  f^{(1)}_2 \\
  \vdots \\
  f^{(1)}_N
\end{pmatrix}
$$  \hfill (2.62)

The entries of the matrix $B$ are obtained by differentiating equation (2.56) with respect to 'x' twice. The elements are given by
with $x = \cos(\pi j/N)$, $j = 1, 2, \ldots, N-1$ and $k = 0, 1, 2, \ldots, N$

And at $x = 1$ i.e. for $j = 0$

$B(0, k) = (k^4 - k^2)/3; k = 0, 1, 2, \ldots, N$

At $x = -1$ i.e. for $j = N$

$B(N, k) = (-1)^k (k^4 - k^2)/3; k = 0, 1, 2, \ldots, N$

The simplified form of equation (2.62) is

$$
\begin{pmatrix}
    f_0^{(2)} \\
    f_1^{(2)} \\
    f_2^{(2)} \\
    \vdots \\
    f_N^{(2)}
\end{pmatrix} =
\begin{pmatrix}
    (N+1) \times (N+1)
\end{pmatrix}
\begin{pmatrix}
    C \\
    f_0 \\
    f_1 \\
    f_2 \\
    \vdots \\
    f_N
\end{pmatrix}
$$

It has been verified that the second derivative matrix ‘$C$’ of equation (2.64) with the square of first derivative matrix given in [Canuto et al. (1988)] and found correct.

2.6.1 Implicit method

Making use of equation (2.64), the implicit time stepping scheme to equation (2.41) is

$$
\begin{pmatrix}
    H_0 \\
    \vdots \\
    H_N
\end{pmatrix}^{n+1} =
\begin{pmatrix}
    H_0 \\
    \vdots \\
    H_N
\end{pmatrix}^n + \Delta T \times
\begin{pmatrix}
    C \\
    (N+1) \times (N+1)
\end{pmatrix}
\begin{pmatrix}
    H_0 \\
    \vdots \\
    H_N
\end{pmatrix}^{n+1}
$$

(2.65)
The above matrix equation (2.65) is written as

\[
\begin{pmatrix}
H_0 \\
\vdots \\
H_N
\end{pmatrix}^{n+1} =
\begin{pmatrix}
H_0 \\
\vdots \\
H_N
\end{pmatrix}^n + \Delta T \begin{pmatrix}
C_i \\
\vdots \\
C_i
\end{pmatrix} \begin{pmatrix}
(H + 1) \times (N + 1) \\
(H + 1) \times (N + 1) \\
\vdots \\
(H + 1) \times (N + 1)
\end{pmatrix} \times 
\begin{pmatrix}
H_0 \\
\vdots \\
H_N
\end{pmatrix}^{n+1}
\]

(2.66)

where \( C_i(i, j) = S(H_{x}) \times C(i, j), \quad i, j = 0, 1, ..., N \)

Finally the matrix equation (2.66) is simplified as

\[
\begin{pmatrix}
D \\
(N + 1) \times (N + 1)
\end{pmatrix} \begin{pmatrix}
H_0 \\
\vdots \\
H_N
\end{pmatrix}^{n+1} = 
\begin{pmatrix}
H_0 \\
\vdots \\
H_N
\end{pmatrix}^n
\]

(2.67)

Where the superscript denotes the time step. The boundary conditions given by equation (2.5) are incorporated in to the coefficient matrix ‘D’, by changing the elements of first-row and last-row accordingly. That is

(i) \( D(1,1) = 1, D(1,i) = 0, i = 2, 3, ..., N \)

(ii) \( D(N,N) = 1, D(N,i) = 0; i = 1, 2, ..., N-1 \)

(iii) \( H_0^n \) is equal to the surface value and \( H_N^n = 0 \)

The solution of equation (2.67) with the boundary conditions. is the solution of the problem. It may be noted here that the derivative boundary condition \( (dH/dx) = 0 \), at \( x = x_N \) of equation (2.5) can also be implemented by making the following;

(i) \( D(N,N) = -D(N,N-1) = 1; \)

(ii) \( D(N,i) = 0; i = 1, 2, ..., N-2. \)

(iii) \( H_N^n = 0 \)
Since \[ \frac{H(N,N) - H(N,N - 1)}{\Delta x} = 0 \]

2.6.2 Features

In most practical applications, the benefit of the spectral method is

(i) Not the extraordinary accuracy available for large \( N \), but rather the small size of \( N \) (necessary) for a moderately accurate solution,

(ii) It is well suited to transform techniques,

(iii) The uneven spacing is not only economical but also helps to study the field very closely at the surface.

2.7 EXPERIMENTAL TOROID OF REFERENCE [3] AND ITS CHARACTERISTICS

2.7.1 The details of Toroid

<table>
<thead>
<tr>
<th>Material</th>
<th>: Mild Steel</th>
</tr>
</thead>
<tbody>
<tr>
<td>Resistivity</td>
<td>: 18.5 \times 10^{-8} , \Omega\cdot m</td>
</tr>
<tr>
<td>B-H Curve</td>
<td>: Fig.2.5</td>
</tr>
<tr>
<td>External diameter</td>
<td>: 0.3366 m</td>
</tr>
<tr>
<td>Internal diameter</td>
<td>: 0.2921 m</td>
</tr>
<tr>
<td>cross-section</td>
<td>: 0.0222x0.0143 m</td>
</tr>
<tr>
<td>Perimeter of section</td>
<td>: 0.0717 m</td>
</tr>
<tr>
<td>Mean circumference</td>
<td>: 0.987 m</td>
</tr>
<tr>
<td>Number of turns of magnetizing winding</td>
<td>: 800</td>
</tr>
<tr>
<td>Number of secondary turns</td>
<td>: 500</td>
</tr>
</tbody>
</table>

2.7.2 Characteristics

The Magnetization curve of toroid is given in figure 2.5 and it is represented by equation (2.39), with \( \alpha = 1.803 \, T \) and \( \gamma = 935 \, A/m \).
2.8 EVALUATION OF EDDY CURRENT LOSSES BY DIFFERENT METHODS

For the purpose of comparison of eddy current losses of three different methods, the surface excitation \( H_s = 16000 \) A/m is chosen.

2.8.1 Liming Nonlinear Theory (LNT)

The loss per unit surface area is calculated using equation (2.35)

\[
P_l = \frac{\rho B_m \omega}{3\sqrt{2}} H_s^{3/2} \text{ watts/m}^2
\]

(2.68)

Where \( B_m = 1.705 \) T at \( H_s = 16000 \) A/m, from the B-H curve, shown in fig. 2.5.

![B-H Curve of the specimen](image)
2.8.2 Modified Crank-Nicholson Method (CNM)

The outcome of the sequential procedure discussed in section 2.5.1 is given in Table 2-1. Knowing the grid values for one full cycle of surface excitation, the Fundamental Components of Magnetizing Force (FCMF) at the specified layers are computed. Also the current density at the same layers is determined using the finite difference formula (2.69). Both results are given in Table 2-2.

**Table 2-1: Final Nodal values of Crank-Nicholson Method**

<table>
<thead>
<tr>
<th>$\omega t$</th>
<th>$x$</th>
<th>0</th>
<th>$1 \times 10^{-4}$</th>
<th>$1 \times 10^{-3}$</th>
<th>$2 \times 10^{-3}$</th>
<th>$3 \times 10^{-3}$</th>
<th>$4 \times 10^{-3}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>-546</td>
<td>-2350</td>
<td>-2526</td>
<td>-1846</td>
<td>-684</td>
<td></td>
</tr>
<tr>
<td>0.10 $\pi$</td>
<td>4944</td>
<td>4032</td>
<td>-979</td>
<td>-1687</td>
<td>-1457</td>
<td>-707</td>
<td></td>
</tr>
<tr>
<td>0.20 $\pi$</td>
<td>9405</td>
<td>8536</td>
<td>1147</td>
<td>-1115</td>
<td>-1167</td>
<td>-673</td>
<td></td>
</tr>
<tr>
<td>1.90 $\pi$</td>
<td>-4944</td>
<td>-4972</td>
<td>-4755</td>
<td>-3788</td>
<td>-2269</td>
<td>-158</td>
<td></td>
</tr>
<tr>
<td>2.00 $\pi$</td>
<td>0</td>
<td>-546</td>
<td>-2351</td>
<td>-2528</td>
<td>-1850</td>
<td>-689</td>
<td></td>
</tr>
</tbody>
</table>

**Table 2-2: Variation of FCMF and current density of Crank-Nicholson Method**

<table>
<thead>
<tr>
<th>$X$(m)</th>
<th>0</th>
<th>$1 \times 10^{-4}$</th>
<th>$1 \times 10^{-3}$</th>
<th>$2 \times 10^{-3}$</th>
<th>$3 \times 10^{-3}$</th>
<th>$4 \times 10^{-3}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H_F$ (A/m)</td>
<td>16000</td>
<td>15289</td>
<td>9485</td>
<td>4789</td>
<td>2086</td>
<td>782</td>
</tr>
<tr>
<td>$J$ (A/m²)</td>
<td>$711 \times 10^4$</td>
<td>$708 \times 10^4$</td>
<td>$576 \times 10^4$</td>
<td>$318 \times 10^4$</td>
<td>$181 \times 10^4$</td>
<td>$79 \times 10^4$</td>
</tr>
</tbody>
</table>
\[ J_x = \frac{H_F^k - H_F^{k+1}}{\Delta h} \]  \hspace{1cm} (2.69)

Where the superscript 'k' indicates the layer number

The subscript 'F' stands for fundamental quantity. So

\( H_F^k \) is the FCMF at \( x = 0, 1 \times 10^{-3}, 2 \times 10^{-3}, \ldots, \)

\( H_F^{k+1} \) is the FCMF at \( x = 0 + \Delta h, 1 \times 10^{-3} + \Delta h, 2 \times 10^{-3} + \Delta h, \ldots, \)

With \( \Delta h = 1 \times 10^{-4} \) m

It may be noted that the nodal values at \( 1 \times 10^{-3} + \Delta h, 2 \times 10^{-3} + \Delta h, \ldots, \) are not provided.

The iron loss per unit surface area is calculated using the Poynting equation given below

\[ P_i = 0.5 J_s \rho H_s \sin \psi_s \text{ watts} \]  \hspace{1cm} (2.70)

Where \( \sin \psi_s \) is given by the equation (2.37).

2.8.3 Pseudo-Spectral Method (PSM)

Having taken zero initial values at the first time-row \( T = 0 \), the nodal values at the second time-row are computed using the equation (2.67). With the computed nodal values at the second time-row, the nodal values at the third time-row are computed considering the nodal values at the second time-row as initial values and using the equation (2.67). Similarly, the nodal values at all time-rows are obtained and summarized in table 2-3. The field distribution at various layers is also shown in fig. 2.6. The FCMF and the current density at different layers, as mentioned in table 2-3., are evaluated. The results are given in table 2-4.
### Table 2-3: Distribution of field by Spectral Method

<table>
<thead>
<tr>
<th>( \omega t )</th>
<th>0</th>
<th>( 1 \times 10^4 )</th>
<th>( 1.52 \times 10^3 )</th>
<th>( 2.31 \times 10^3 )</th>
<th>( 3.22 \times 10^3 )</th>
<th>( 4.19 \times 10^3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>16000</td>
<td>15335</td>
<td>6161</td>
<td>1656</td>
<td>-436</td>
<td>-502</td>
</tr>
<tr>
<td>0.1( \pi )</td>
<td>15217</td>
<td>14724</td>
<td>7752</td>
<td>3847</td>
<td>121</td>
<td>-457</td>
</tr>
<tr>
<td>1.4( \pi )</td>
<td>-4944</td>
<td>-4964</td>
<td>-4291</td>
<td>-3350</td>
<td>-1878</td>
<td>-256</td>
</tr>
<tr>
<td>1.5( \pi )</td>
<td>0</td>
<td>-488</td>
<td>-2504</td>
<td>-2335</td>
<td>-1578</td>
<td>-463</td>
</tr>
<tr>
<td>3.4( \pi )</td>
<td>-4944</td>
<td>-4963</td>
<td>-4282</td>
<td>-3337</td>
<td>-1860</td>
<td>-236</td>
</tr>
<tr>
<td>3.5( \pi )</td>
<td>0</td>
<td>-488</td>
<td>-2499</td>
<td>-2326</td>
<td>-1562</td>
<td>-440</td>
</tr>
</tbody>
</table>

### Table 2-4: Variation FCMF and current density of Spectral Method

<table>
<thead>
<tr>
<th>( X(\text{m}) )</th>
<th>0</th>
<th>( 1 \times 10^{-4} )</th>
<th>( 1.52 \times 10^{-3} )</th>
<th>( 2.31 \times 10^{-3} )</th>
<th>( 3.22 \times 10^{-3} )</th>
<th>( 4.19 \times 10^{-3} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( H_F(\text{A/m}) )</td>
<td>16000</td>
<td>15257</td>
<td>6549</td>
<td>3574</td>
<td>1586</td>
<td>564</td>
</tr>
<tr>
<td>( J(\text{A/m}^2) )</td>
<td>(742 \times 10^4)</td>
<td>(706 \times 10^4)</td>
<td>(377 \times 10^4)</td>
<td>(221 \times 10^4)</td>
<td>(105 \times 10^4)</td>
<td>(39 \times 10^4)</td>
</tr>
</tbody>
</table>
The iron loss per unit surface area is calculated using the equation (2.70). The computed values of Eddy Current losses by three different methods along with practical value [3] at $H_s = 16000 \text{ A/m}$ are tabulated in table 2-5.

**Table 2-5: Comparison of eddy current losses by three different methods with experimental value [10]**

<table>
<thead>
<tr>
<th>Expt. Value</th>
<th>LNT</th>
<th>CNM</th>
<th>PSM</th>
</tr>
</thead>
<tbody>
<tr>
<td>9150 watts</td>
<td>9779 watts</td>
<td>8592 watts</td>
<td>8966 watts</td>
</tr>
</tbody>
</table>
2.9 COMPARISION OF RESULTS

The simulated results of Pseudo-Spectral Method are compared with that of Crank - Nicholson Method and experimental results of the same toroid. It is clear from the figures 2.7 and 2.8, that the simulated results of Pseudo-Spectral Method are close to the practical values compared to that of Crank-Nicholson Method. It may be noted that the value of N for Pseudo-Spectral Method is 16, whereas for the Crank-Nicholson method is 80, but the $\Delta T/\Delta x$ ratio is same for both methods.

![Variation of Iron losses with Field Intensity](image)

**Fig. 2.7: Variation of Iron losses with Field Intensity**
2.10 CONCLUSIONS

Different methods of evaluating eddy current losses have been compared. It is observed that the proposed Pseudo-Spectral Method yields results of reasonable accuracy with less number of nodes. Moreover, this method is suitable to use Fast Fourier Transforms. Therefore, the proposed method is suggested as an alternative to the existing finite difference and finite element methods.