FOUR-DIMENSIONAL FINSLER SPACE WITH

CONSTANT UNIFIED MAIN SCALAR

1. Introduction.

In 1984 ([20]) F. Ikeda had studied the properties of Finsler spaces satisfying the condition $L^2C^2 = f(x)$, where $L$ is the fundamental function and $C$ is the length of the torsion vector $C_i$. In 1991 ([21]) he considered the condition: $L^2C^2 =$ non-zero constant, which is stronger than the corresponding condition considered in [20]. A two-dimensional Berwald space is an example of such a Finsler space with constant function LC. A theory of intrinsic orthonormal frame field on n-dimensional Finsler space has been studied by Matsumoto and Miron ([38]) and is called ‘Miron frame’ by Matsumoto.

The three-dimensional Finsler space with constant unified main scalar has been studied by Ikeda [22], U.P. Singh and Bindu Kumari [52] and other researchers. In three-dimensional Finsler space there are three main scalars $H, I, J$ in which the sum of $H$ and $I$ is $LC$, called
unified main scalar, whereas in four-dimensional Finsler space there are eight main scalars $H, I, J, K, H', I', J', K'$, in which the sum of $H$, $I$ and $K$ is LC. Also there are three $h$-vector fields and three $v$-vector fields in four-dimensional Finsler space. The orthonormal frame field $(l', m', n', p')$, called the Miron frame of $F^4$, plays an important role in four-dimensional Finsler space.

The purpose of the present chapter is to discuss the theory of C-reducible, semi-C-reducible and C2-like four-dimensional Finsler spaces with constant unified main scalar.

2. **Scalar components in the Miron's frame.**

Let $F^4$ be a four-dimensional Finsler space with the fundamental function $L(x, y)$. The metric tensor $g_{ij}$ and C-torsion tensor $C_{ijk}$ of $F^4$ are given by

$$g_{ij} = (1/2)\partial_i \partial_j L^2, \quad C_{ijk} = (1/4)\partial_i \partial_j \partial_k L^2.$$ 

The frame $\{e^i_\alpha\}; \alpha = 1, 2, 3, 4$ is called Miron frame of $F^4$, where

$$e^i_1 = l^i = \frac{y^i}{L}$$ is the normalized supporting element, $e^i_2 = m^i = \frac{C^i}{C}$ is the
normalized torsion vector; \( e_{i}^{'} = n^{'} \), \( e_{3}^{'} = p^{'} \) are constructed by
\[
g_{ij} e_{a_{1}}^{'} e_{a_{2}}^{'} = \delta_{a_{3}a_{4}}.\]
Here \( C \) is the length of torsion vector \( C = C_{ijk} g^{ik} \). The Greek letters \( \alpha, \beta, \gamma, \delta \) vary from 1 to 4 throughout the chapter. Summation convention is applied for both the Greek and Latin indices.

In the Miron’s frame an arbitrary tensor can be expressed by scalar components along the unit vectors \( l^{'} , m^{'} , n^{'} , p^{'} \). For instance let \( T = T_{j}^{i} \) be a tensor field of (1,1) type. Then the scalar components \( T_{a\beta} \) of \( T \) are defined as \( T_{a\beta} = T_{j}^{i} e_{a_{1}}^{'} e_{a_{2}}^{'} \), and the components \( T_{j}^{i} \) of the tensor \( T \) are expressed as \( T_{j}^{i} = T_{a\beta} e_{a_{1}}^{'} e_{a_{2}}^{'} \). From the equations \( g_{ij} e_{a_{1}}^{'} e_{a_{2}}^{'} = \delta_{a_{3}a_{4}} \), we have

\[
(2.1) \quad g_{ij} = l_{i}l_{j} + m_{i}m_{j} + n_{i}n_{j} + p_{i}p_{j}.
\]

Next the C-tensor \( C_{ijk} = \frac{1}{2} \delta_{k} g_{ij} \) satisfies \( C_{ijk} l_{i} = 0 \) and is symmetric in \( i,j,k \). Therefore if \( C_{a\beta\gamma} \) are scalar components of \( LC_{ijk} \), that is

\[
(2.2) \quad LC_{ijk} = C_{a\beta\gamma} e_{a_{1}}^{'} e_{a_{2}}^{'} e_{\beta_{1}}^{'} e_{\gamma_{1}}^{'} ,
\]

then we have
\[(2.3) \quad LC_{ijk} = C_{222} m_j m_k + C_{333} \pi_{(ijk)} \{m_j m_k\} + C_{444} \pi_{(ijk)} \{m_j p_k\} + C_{444} \pi_{(ijk)} \{m_j p_k\} + C_{444} \pi_{(ijk)} \{n_j p_k\} + C_{444} p_j p_k + C_{234} \pi_{(ijk)} \{n_i (p_j + n_k p_j)\},\]

where \( \pi_{(ijk)} \{\} \) denotes the cyclic interchange of \(i, j, k\) and summation.

For instance

\[\pi_{(ijk)} \{A_i B_j C_k\} = A_i B_j C_k + A_i B_k C_j + A_k B_j C_i,\]

Contracting (2.3) by \(g^{jk}\), we get \(LC_{mi} = C_{aibc} e_{aj},\). Thus if we put

\[(2.4) \quad C_{222} = H, \quad C_{233} = I, \quad C_{333} = K, \quad C_{333} = J,\]

\[C_{444} = J', \quad C_{444} = H', \quad C_{433} = I', \quad C_{234} = K',\]

then we have ([45])

\[(2.5) \quad H + I + K = LC, \quad C_{332} = -(J + J'), \quad C_{422} = -(H' + I').\]

Hence equation (2.3) may be written as

\[(2.6) \quad LC_{ijk} = H m_j m_k + J n_j n_k + H' p_j p_k + I \pi_{(ijk)} \{m_j m_k\} + K \pi_{(ijk)} \{m_j p_k\} + J' \pi_{(ijk)} \{n_j p_k\} + J' \pi_{(ijk)} \{n_j p_k\},\]

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\[-(J + J')\pi_{(jk)} \{m_m n_k\} + I'\pi_{(jk)} \{n_n p_k\}\]

\[-(H' + I')\pi_{(jk)} \{m_m p_k\} + K'\pi_{(jk)} \{m_p n_k + n_k p_k\}\].

The eight scalars \(H, I, J, K, H', I', J', K'\) are called the main scalars of a four-dimensional Finsler space. We shall use Cartan's connection \(\mathcal{C}\Gamma = (\Gamma_{jk}^i, G_{k}^i, C_{jk}^i)\). The \(h\)-and \(v\)-covariant derivatives of the frame field \(e_{\alpha\beta}\) are given by ([32]):

\[e_{\alpha\beta\gamma} = H_{\alpha\beta}\gamma e_{\beta\gamma\gamma} e_{\gamma\gamma},\]

\[L e_{\alpha\beta\gamma} = V_{\alpha\beta\gamma} e_{\beta\gamma\gamma} e_{\gamma\gamma},\]

where \(H_{\alpha\beta}\gamma\) and \(V_{\alpha\beta}\gamma\), \(\gamma\) being fixed, are given by

\[
H_{\alpha\beta}\gamma = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & h_{\gamma} & J_{\gamma} \\ 0 & -h_{\gamma} & 0 & k_{\gamma} \\ 0 & -J_{\gamma} & -k_{\gamma} & 0 \end{bmatrix},
\]

\[
V_{\alpha\beta}\gamma = \begin{bmatrix} 0 & \delta_{2\gamma} & \delta_{3\gamma} & \delta_{4\gamma} \\ -\delta_{2\gamma} & 0 & u_{\gamma} & v_{\gamma} \\ -\delta_{3\gamma} & -u_{\gamma} & 0 & w_{\gamma} \\ -\delta_{4\gamma} & -v_{\gamma} & -w_{\gamma} & 0 \end{bmatrix},
\]

where we have put

\[H_{233\gamma} = -H_{323\gamma} = h_{\gamma}, \quad V_{233\gamma} = -V_{323\gamma} = u_{\gamma},\]

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\( H_{24} = -H_{42} = J, \quad V_{24} = -V_{42} = v, \)
\( H_{34} = -H_{43} = k, \quad V_{34} = -V_{43} = w. \)

Thus in four-dimensional Finsler space there exist three h-connection vectors \( h, J, k \), whose scalar components with respect to the frame \( \{e_i^a\} \) are \( h, J, k; \) i.e.,

\[(2.10) \quad h_i = h_r e_{r i}, \quad J_i = J_r e_{r i}, \quad k_i = k_r e_{r i}. \]

Also there exist three v-connection vectors \( u, v, w \), whose scalar components with respect to the frame \( \{e_i^a\} \) are \( u, v, w; \) i.e.,

\[(2.11) \quad u_i = u_r e_{r i}, \quad v_i = v_r e_{r i}, \quad w_i = w_r e_{r i}. \]

In view of equations (2.9), (2.10) and (2.11), the equations (2.7) and (2.8) respectively may be explicitly written as

\[(2.12) \quad \begin{cases} l_{ij} = 0, \\ m_{ij} = n_i h_j + p_i J_j, \\ n_{ij} = p_i k_j - m_i h_j, \\ p_{ij} = -m_i J_j - n_i k_j \end{cases} \]

\[(2.13) \quad \begin{cases} L_{l} = m_j m_j + n_j n_j + p_i p_j = h_{ij}, \\ L m_{ij} = -l_m h_j + n_i u_j + p_i v_j, \\ L n_{ij} = -l_i n_j - m_i u_j + p_i w_j, \\ L p_{ij} = -l_j p_j - m_j v_j - n_j w_j. \end{cases} \]
Since $m_i, n_i, p_i$ are homogeneous functions of degree zero in $y'$, we have $Lm_i|_{y'} = Ln_i|_{y'} = Lp_i|_{y'} = 0$, which in view of equations (2.11) and (2.13), give $u_i = 0, v_i = 0$ and $w_i = 0$.

The h-scalar derivative of the adopted components $T_{\alpha\beta}$ of the tensor $T_{y'}^i$ of type (1,1) is defined as ([32])

$$T_{\alpha\beta, r} = (\delta_k T_{\alpha\beta}) e^k_\gamma + T_{\mu\alpha} H_{\mu\gamma r} + T_{\alpha\mu} H_{\mu r},$$

where $\delta_k = \partial_k - G_k^\xi \hat{\tau}_\xi$. Similarly the v-scalar derivative of the adopted components $T_{\alpha\beta}$ of $LT_{y'}^i$ is defined as ([32])

$$T_{\alpha\beta, r} = L(\dot{\delta}_k T_{\alpha\beta}) e^k_\gamma + T_{\mu\alpha} V_{\mu\gamma r} + T_{\alpha\mu} V_{\mu r}. $$

Thus $T_{\alpha\beta, r}$ and $T_{\alpha\beta, r}$ are adopted components of $T_{y'i}^j$ and $T_{y'i}^j |_{k}$ respectively, i.e.,

$$T_{y'i}^j |_{k} = T_{\alpha\beta, r} e^i_{\alpha} e^j_{\beta} e_{r} |_{k},$$

$$LT_{y'i}^j |_{k} = T_{\alpha\beta, r} e^i_{\alpha} e^j_{\beta} e_{r} |_{k}. $$

From (2.2) it follows that

$$L^2 C_{hij} |_{k} + LC_{hij} |_{k} = C_{\alpha\beta\delta\epsilon} e^i_{\alpha} e^j_{\beta} e^k_{\epsilon} |_{\delta},$$

which implies that
(2.18) \[ L \mathcal{C}_{h_{\delta}} \mid = (C_{_{a_{\delta}r_{\delta}}} - C_{_{a_{\delta}p_{\delta}}} \delta_{_{a_{\delta}r_{\delta}}}) e_{_{a_{\delta}y_{\delta}}} e_{_{y_{\delta}l_{\delta}}} e_{_{l_{\delta}l_{\delta}}} e_{_{l_{\delta}l_{\delta}}} \]

The explicit form of \( C_{_{a_{\delta}r_{\delta}}} \) is easily obtained:

(2.19)(a) \( C_{_{1_{\delta}r_{\delta}}} = -C_{_{r_{\delta}r_{\delta}}} \),

(b) \( C_{_{22_{\delta}}} = H_{_{\delta}} + 3(J + J')u_{_{\delta}} + 3(H' + I')v_{_{\delta}} \),

(c) \( C_{_{22_{\delta}}} = -(J + J')_{_{\delta}} + (H - 2I)u_{_{\delta}} - 2K'v_{_{\delta}} + (H' + I')w_{_{\delta}} \),

(d) \( C_{_{333_{\delta}}} = I_{_{\delta}} - (3J + 2J')u_{_{\delta}} - I'v_{_{\delta}} - 2K'w_{_{\delta}} \),

(e) \( C_{_{224_{\delta}}} = -(H + H')_{_{\delta}} - 2K'u_{_{\delta}} + (H - 2K)v_{_{\delta}} - (J + J')w_{_{\delta}} \),

(f) \( C_{_{234_{\delta}}} = K'_{_{\delta}} - (H' + 2I')u_{_{\delta}} - (J + 2J')v_{_{\delta}} + (I - K)w_{_{\delta}} \),

(g) \( C_{_{244_{\delta}}} = K'_{_{\delta}} - J'u_{_{\delta}} - (3H' + 2I')v_{_{\delta}} + 2K'w_{_{\delta}} \),

(h) \( C_{_{333_{\delta}}} = J_{_{\delta}} + 3Ju_{_{\delta}} - 3I'w_{_{\delta}} \),

(i) \( C_{_{334_{\delta}}} = I'_{_{\delta}} + 2K'u_{_{\delta}} + Iv_{_{\delta}} + (J - 2J')w_{_{\delta}} \),

(j) \( C_{_{444_{\delta}}} = J'_{_{\delta}} + Ku_{_{\delta}} + 2K'v_{_{\delta}} + (2I' - H')w_{_{\delta}} \),

(k) \( C_{_{444_{\delta}}} = H'_{_{\delta}} + 3Kv_{_{\delta}} + 3J'w_{_{\delta}} \),

where \( H'_{_{\delta}} \), for instance, is the v-scalar derivative of the single scalar \( H \), namely \( H'_{_{\delta}} = L(\dot{\delta}, H) e_{_{\delta}} \).
The tensor \( C_{\alpha\beta\gamma} | k \) is completely symmetric. Accordingly

(2.18) yields

(2.20) \[ C_{\alpha\beta\gamma} \delta_{\delta\sigma\tau} = C_{\alpha\beta\gamma} \delta_{\delta\sigma\tau} - C_{\alpha\beta\delta} \delta_{\gamma\tau}. \]

This is explicitly written as

(2.21) (a) \[ -(J + J')_2 + (H - 2I)u_2 - 2K'v_2 + (H' + I')w_2 \]

\[ = H_3 + 3(J + J')u_3 + 3(H' + I')v_3, \]

(b) \[ I_2 - (3J + 2J')u_2 - I'v_2 - 2K'w_2 \]

\[ = -(J + J')_2 + (H - 2I)u_3 - 2K'v_3 + (H' + I')w_3, \]

(c) \[ K'_2 - (H' + 2I)u_2 + (J + 2J')v_2 + (I - K)w_2 \]

\[ = -(H' + I')_3 - 2K'u_3 + (H - 2K)v_3 - (J + J')w_3, \]

\[ = -(J + J')_4 + (H - 2I)u_4 - 2K'v_4 + (H' + I')w_4, \]

(d) \[ J_2 + 3Ju_2 - 3I'w_2 = I_2 - (3J + 2J')u_3 - I'v_3 - 2K'w_3, \]

(e) \[ I'_2 + 2K'v_2 + I'v_2 + (J - 2J')w_2 \]

\[ = K'_3 - (H' + 2I')u_3 + (J + 2J')v_3 + (I - K)w_3, \]

\[ = I'_4 - (3J + 2J')u_4 - I'v_4 - 2K'w_4, \]

(f) \[ J'_2 + Ku_2 + 2K'v_2 + (2I' - H')w_2 \]

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\[ = K_3 - J'u_2 - (3H' + 2I')v_3 + 2K'w_3, \]
\[ = K'_3 - (H' + 2I')u_4 - (J + 2J')v_4 + (I - K)w_4, \]
\[ (g) \quad -(H' + I')_2 - 2K'u_2 + (H - 2K)v_2 - (J + J')w_2 \]
\[ = H_3 + 3(J + J')u_4 + 3(H' + I')v_4, \]
\[ (h) \quad K_2 - J'u_2 - (3H' + 2I')v_2 + 2K'w_2 \]
\[ = -(H' + I')_4 - 2K'u_4 + (H - 2K)v_4 - (J + J')w_4, \]
\[ (i) \quad H'_3 + 3Kv_2 + 3J'w_3 = K_4 - J'u_4 - (3H' + 2I')v_4 + 2K'w_4, \]
\[ (j) \quad L'_3 + 2K'u_3 + Iv_3 + (J - 2J')w_3 = J'_4 + 3u_4 - 3I'w_4, \]
\[ (k) \quad J'_3 + Ku_4 + 2K'v_3 + (2I' - H')w_3 \]
\[ = L'_4 + 2K'u_4 + Iv_4 + (J - 2J')w_4, \]
\[ (l) \quad H'_3 + 3Kv_3 + 3J'w_3 = J'_4 + Ku_4 + 2K'v_4 + (2I' - H')w_4. \]

The \( v \)-curvature tensor \( S_{hijk} \) of \( C\Gamma \) is defined by

\[ S_{hijk} = C_{hkr}C'_{ij} - C_{hji}C'_{rk}. \]

Therefore the scalar components \( S_{abcde} \) of \( L^2S_{hijk} \) are given by

\[ (2.22) \quad L^2S_{hijk} = S_{abcde} e_\alpha e_\beta e_\gamma e_\delta e_\epsilon. \]

Then from (2.2) it follows that
(2.23) \[ S_{\alpha\beta\gamma\delta} = C_{\alpha\delta\gamma} C_{\beta\gamma\delta} - C_{\alpha\beta\gamma} C_{\gamma\delta\delta}. \]

The v-Ricci tensor \( S_{\alpha} \) is defined as \( S_{\alpha} = s_{\alpha k} g^{k} \).

The scalar components of \( L^2 S_{\alpha} \) are given by \( S_{\alpha\beta\gamma\delta} \).

Hence in view of (2.23) it follows that the v-scalar curvature \( L^2 S = L^2 S_{\alpha} g^{\alpha} \) is given by

\[ L^2 S = C_{\alpha\beta\gamma} C_{\alpha\beta\gamma} - C_{\alpha\alpha\gamma} C_{\beta\beta\gamma}. \]

In terms of main scalars this equation gives

(2.24) \[ L^2 S = 2(K^2 + I^2 - HI - HK - KI) + 4(J^2 + H^2) \]

\[ + 6(J^2 + K^2 + I^2 + JJ + HT). \]

**Theorem (2.1).** In terms of main scalars the v-scalar curvature \( L^2 S \) is given by (2.24).

3. **The constant unified main scalar.**

In a four-dimensional Finsler space \( H + I + K = LC \) is called unified main scalar. We now consider a four-dimensional Finsler space with non-zero constant unified main scalar. We have

(3.1) \[ (H + I + K)_{\alpha} = (LC)_{\alpha} = 0 \text{ for } \alpha = 1, 2, 3, 4. \]
Adding equations (2.21) (a), (2.21) (d) and first part of (2.21) (f) and applying equation (3.1) we get \( u_2 = 0 \). Similarly adding equations (2.21)(g), (2.21)(i) and last part of (2.21)(e) and applying equation (3.1) we get \( v_2 = 0 \). Again adding (2.21) (j), (2.21) (l) and last equation of (2.21)(c) and applying equation (3.1) we get \( v_3 = u_4 \). Hence we have the following:

**Theorem (3.1).** In a four-dimensional Finsler space with non-zero constant unified main scalar, the scalar components of v-connection vectors \( u_i \) and \( v_i \) are given by

\[
    u_i = u_i n_i + u_4 p_i, \quad v_i = v_i n_i + v_4 p_i, \quad \text{where} \quad v_3 = u_4.
\]

In view of theorem (3.1), the independent equations in (2.21) can be rewritten as

\[
\begin{align*}
    (3.2) \quad (a) & \quad I_2' - 2K'w_2 - (H' + I')w_3 = -(J + J')_3 + (H - 2I)u_3 - 2K'v_3, \\
    (b) & \quad J_2' - 3I'w_2 + 2K'w_3 = I_3' - (3J + 2J')u_3 - I'u_4, \\
    (c) & \quad J_2' + (J - 2J')w_2 - (I - K)w_3 = K_3' - (H' + 2I')u_3 - (J + 2J')u_4, \\
    (d) & \quad J_2' + (2I' - H')w_2 - 2K'w_3 = K_3' - J'u_3 - (3H' + 2I')u_4,
\end{align*}
\]
(e) \( K'_3 + (I - K)w_2 - (H' + I')w_4 = -(J + J')w_4 + (H - 2I)v_3 - 2K'v_4, \)

(f) \( K'_2 + 2K'w_2 - (J + J')w_4 = -(H' + I')w_4 - 2K'v_3 + (H - 2K)v_4, \)

(g) \( I'_2 + (J - 2J')w_2 + 2K'w_4 = I_3 - (3J + 2J')v_3 - I'v_4, \)

(h) \( J'_2 + (2I' - H')w_2 - (I - K)w_4 = K'_4 - (H' + 2I')v_3 - (J + 2J')v_4, \)

(i) \( H'_2 + 3Jw_2 - 2K'w_4 = K'_4 - J'v_3 - (3H' + 2I')v_4, \)

(j) \( J_4 + 2Ju_4 = I'_3 + 2K'u_5 + (J - 2J')w_3 + 3I'w_4, \)

(k) \( I'_4 + Iv_4 = J'_3 + Ku_3 + (2I' - H')w_3 - (J - 2J')w_4, \)

(l) \( J'_4 - 2Kv_3 + 2K'v_4 = H'_3 + 3J'w_3 - (2I' - H')w_4. \)

Now we suppose that the only non-vanishing main scalars are \( H, I, K, i.e., J = H' = I' = J' = K' = J' = 0. \) Then equation (2.21) reduces to

\[ (3.3) \quad \begin{align*}
(a) & \quad H_3 = (H - 2I)u_2, \quad & (b) & \quad I_2 = (H - 2I)u_3, \\
(c) & \quad (H - 2K)v_3 = (H - 2I)u_4 = (I - K)w_2, \quad & (d) & \quad I_3 = 3Ju_2, \\
(e) & \quad I_4 = Iv_2 = (I - K)w_3, \quad & (f) & \quad K_3 = Ku_3 = (I - K)w_4, \\
(g) & \quad H'_3 = (H - 2K)v_2, \quad & (h) & \quad K_2 = (H - 2K)v_4, \\
(i) & \quad K'_4 = 3Kv_2, \quad & (j) & \quad v_3 = 3u_4,
\end{align*} \]
\[(k) \quad K u_3 = lv_4, \quad (l) \quad 3 v_3 = u_4.\]

From (3.3) (c), (j) and (l) it follows that
\[v_3 = u_4 = 0, w_2 = 0 \text{ or } I = K.\] To solve the remaining equations in (3.3) we classify the following five cases:

(1) \[H \neq 2I \neq 2K,\]  
(2) \[H = 2I \neq 2K,\]  
(3) \[H \neq 2I = 2K,\]  
(4) \[H = 2K \neq 2I,\]  
(5) \[H = 2I = 2K.\]

In case (1), using \(H + I + K = LC\) and solving equations in (3.3) for scalar components of \(v\)-connection vectors, we have

\[\text{(3.4)} \quad (a) \quad u_2 = \frac{(LC)_3}{LC}, \quad u_3 = \frac{I_2}{H - 2I}, \quad u_4 = 0,\]

\[\text{(b) } v_2 = \frac{(LC)_4}{LC}, \quad v_3 = 0, \quad v_4 = \frac{KI_2}{I(H - 2I)} = \frac{K_2}{H - 2K},\]

\[\text{(c) } w_2 = 0, \quad w_3 = \frac{I(LC)_4}{(I - K)L C} = \frac{I_4}{I - K}, \quad w_4 = \frac{K(LC)_3}{(I - K)L C} = \frac{K_3}{I - K}.\]

In case (2) we have \(I_{\alpha} = H_{\alpha} = 0\) for \(\alpha = 2, 3\) and \(v\)-connection vectors are such that

\[\text{(3.5)} \quad (a) \quad u_2 = 0, \quad u_3 = \frac{IK_2}{K(H - 2K)}, \quad u_4 = 0,\]

\[\text{(b) } v_2 = \frac{(LC)_4}{LC}, \quad v_3 = 0, \quad v_4 = \frac{K_2}{(H - 2K)},\]

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(c) \( w_2 = 0, \quad w_3 = \frac{I(LC)_4}{(I-K)L'C}, \quad w_4 = 0. \)

In case (3) we have \( I_{,\alpha} = K_{,\alpha} = 0 \) for \( \alpha = 3, 4 \) and \( H_3 = 0 \). The v-connection vectors are such that \( u_2 = u_4 = v_2 = v_3 = 0, \) and \( u_3, v_4 \) are the same as in case (1), whereas \( w_2, w_3 \) and \( w_4 \) are arbitrary.

In case (4) we have \( H_{,\alpha} = K_{,\alpha} = 0 \) for \( \alpha = 2, 4 \) and \( I_{,4} = 0 \). The v-connection vectors are such that

\[
u_4 = v_2 = v_3 = w_2 = w_3 = 0, \quad v_4 = \frac{KI_2}{I(H-2I)},\]

and \( u_2, u_3, w_4 \) are the same as in case (1).

In case (5) we have \( H_{,\alpha} = I_{,\alpha} = K_{,\alpha} = 0 \) for \( \alpha = 1, 2, 3, 4 \). Hence the main scalars \( H, I, K \) are functions of position only. Furthermore we have \( u_i = u_3 n_i, \) \( v_i = v_4 p_i \) and \( w_i \) is arbitrary.

We may summarize these results in the following theorem.

**Theorem (3.2).** Let \( F^4 \) be a four-dimensional Finsler space with non-zero constant unified main scalar. If the main scalars are such that \( J = H' = I' = K' = J' = 0, \) then
(1) \( for \ H \neq 2I \neq 2K, \) the \( v \)-connection vectors vanish if and only if \( I_{,2} = 0, \)

(2) \( for \ H = 2I \neq 2K, \) the \( v \)-connection vectors vanish if and only if \( K_{,2} = 0, \)

(3) \( for \ H \neq 2I = 2K, \) the \( v \)-connection vectors \( u_i \) and \( v_i \) vanish if and only if \( I_{,2} = 0, \) whereas \( w_i \) is arbitrary,

(4) \( for \ H = 2K \neq 2I, \) the \( v \)-connection vectors vanish if and only if \( I_{,2} = 0, \)

(5) \( for \ H = 2I = 2K, \) the main scalars are functions of position only and \( u_i = u_i n_i, \ v_i = v_n p_i \) and \( w_i \) is arbitrary.

Remark. In the above theorem we have assumed that \( J = H' = I' = K' = J' = 0. \) The question arises: Does there exist any four-dimensional Finsler space in which these conditions hold? To answer this question we give some examples of special Finsler spaces in the next article.
4. **C-reducible, semi-C-reducible and C2-like Finsler spaces.**

A Finsler space of dimension n (n > 2) is called C-reducible if $C_{ijk}$ is written as ([32])

\[
(4.1) \quad C_{ijk} = \left( C_i h_{jk} + C_j h_{ki} + C_k h_{ij} \right) / (n+1)
\]

where $h_{ij} = g_{ij} - l_{ri} l^r_j$ is the angular metric tensor. Since $\delta_{a\beta} - \delta_{1a} \delta_{1\beta}$ are scalar components of $h_{ij}$ with respect to the Miron's frame $\{ e'_a \}$ of $F^4$, therefore for a four-dimensional C-reducible Finsler space we have

\[
(4.2) \quad C_{a\beta r} = LC \left\{ \delta_{2a} (\delta_{r\beta} - \delta_{1a} \delta_{1\beta} ) + \delta_{2\beta} (\delta_{a\alpha} - \delta_{1a} \delta_{1\alpha} ) + \delta_{2r} (\delta_{a\beta} - \delta_{1a} \delta_{1\beta} ) \right\} / 5
\]

In view of notation given in (2.4), this equation gives ([45])

\[
(4.3) \quad J = H' = I' = K' = J' = 0, \quad H = 3I = 3K = (3/5)LC.
\]

If the unified main scalar is constant, LC is constant. Therefore $H, I, K$ are constants and we have the following:

**Theorem (4.1).** In a four-dimensional C-reducible Finsler space with non-zero constant unified main scalar, the main scalars $H, I$ and $K$ are non-zero constants and all the remaining main scalars vanish.
Making use of equations (4.3) and theorem (4.1) in equation (3.3) we have $u_\alpha = v_\alpha = 0$ for $\alpha = 1, 2, 3, 4$. Furthermore from (2.24) and (4.3) we get

$$L^3S = -2L^2C^2/5.$$ 

Hence we have the following:

**Theorem (4.2).** In a four-dimensional C-reducible Finsler space with non-zero constant unified main scalar, the v-connection vectors $u_\alpha$ and $v_\alpha$ vanish identically and the v-scalar curvature $(L^3S)$ is constant.

We should remark here that the metric function $L$ of a C-reducible Finsler space is either of Randers type or of Kropina type ([39]), i.e., $L = \alpha + \beta$ (Randers metric) or $L = \alpha^2 / \beta$ (Kropina metric), where $\alpha$ is Riemannian metric and $\beta$ is one-form. Thus we have

**Theorem (4.3).** In a four-dimensional Randers or Kropina space with non-zero constant unified main scalar, the v-connection vectors $u_\alpha$ and $v_\alpha$ vanish identically and the v-scalar curvature $(L^3S)$ is constant.

An $n$-dimensional semi-C-reducible Finsler space is characterized by
(4.4) \[ C_{y_k} = p \left( C_i h_{j_k} + C_j h_{k_i} + C_k h_{j_i} \right) \left/ \left( n+1 \right) \right. \left( q/C^2 \right)^{C_i C_j C_k}, \]

where \( p + 1 = 1 \). If \( p \) and \( q \) are constants, it is called semi-C-reducible Finsler space with constant coefficients ([39]). In terms of scalar components the equation (4.4) may be written for a four-dimensional space as

\[ (4.5) \quad C_{a \beta y} = \frac{p LC \left\{ \delta_{2a} \left( \delta_{\beta r} - \delta_{1 \beta} \delta_{1 r} \right) + \delta_{2 \beta} \left( \delta_{\gamma a} - \delta_{1 \gamma} \delta_{1 a} \right) + \delta_{2 r} \left( \delta_{\alpha \beta} - \delta_{1 \alpha} \delta_{1 \beta} \right) \right\}}{5} \]

\[ + q LC \delta_{2a} \delta_{2 \beta} \delta_{2 r}, \]

which gives

\[ (4.6) \quad J = H = I = K' = J' = 0, \quad H = LC \left\{ (3p/5) + q \right\}, \]

\[ I = p LC / 5, \quad K = p LC / 5. \]

From these equations we have:

**Theorem (4.4).** In a four-dimensional semi-C-reducible Finsler space with constant coefficients and constant unified main scalar, the main scalars \( H, I \) and \( K \) are non-zero constants and all the remaining main scalars vanish.
Making use of equations (4.6) and theorem (4.4) in equation (3.3) we have \( u_\alpha = v_\alpha = 0 \) for \( \alpha = 1, 2, 3, 4 \). Furthermore from (2.24) and (4.6) we get

\[
L^2 S = -2p(p + q)L^2 C^2 / 5.
\]

Hence we have the following theorem:

**Theorem (4.5).** In a four -dimensional semi-C-reducible Finsler space with constant coefficients and constant unified main scalar, the v-connection vectors \( u_i \) and \( v_i \) vanish identically and the v-scalar curvature \( (L^2 S) \) is constant.

An n-diemensional (n>2) Finsler space is called C2-like ([39]) if \( C_{ijk} = (1/C^2)C_iC_jC_k \). Thus for a four-dimensional C2-like Finsler space we have \( H = LC, \quad I = K = J = H' = I' = K' = J' = 0 \). Hence we have:

**Theorem (4.6).** In a four-dimensional C2-like Finsler space all the main scalars vanish except the main scalar \( H \) which is equal to the unified main scalar \( LC \).