CHAPTER IV

THE TERNARY DIOPHANTINE EQUATION

\[ Z^{2n} = DX^2 + Y^2 \]

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The Ternary Diophantine equation of the title

\[ Z^{2n} = DX^2 + Y^2 \quad (4.1) \]

arises while attempting to find the square roots of a complex binomial quadratic surd of form \((Y + iX\sqrt{D})\), \(D\) being a square free integer and whose integral solutions for \(n = 1, 2\) are well known. In this chapter, we present a method to obtain an infinite number of nontrivial integral solutions of equation (4.1) for all positive integral values of \(n\). Also, we present a few interesting relations among the solutions. Some numerical examples are also given.

Employing the solutions \((X_0, Y_0, Z_0)\) of the equation (4.1) when \(n = 1\) and suitably repeatedly applying the lemma of BrahmaGupta, one arrives at
an integral solution of (4.1) denoted by \( (x_{n-1}^{(1)}, y_{n-1}^{(1)}, z_{n-1}^{(1)}) \) and represented through the recurrence relations

\[
x_{n-1}^{(1)} = x_0 y_{n-2} - y_0 x_{n-2}, \quad n \geq 2
\]

\[
y_{n-1}^{(1)} = y_0 y_{n-2} + D x_0 x_{n-2},
\]

\[
z_{n-1}^{(1)} = z_0
\]

(4.2)

where

\[
x_0 = 2pq, \quad y_0 = q^2 - Dp^2, \quad z_0 = q^2 + Dp^2,
\]

and

\[
y_{n-2}^2 + D x_{n-2}^2 = z_{n-2}^{(0,4)}
\]

(4.3)

The second solution \( (x_{n-1}^{(2)}, y_{n-1}^{(2)}, z_{n-1}^{(2)}) \) of equation (4.1) is obtained by using the following transformations

\[
X_{n-1}^{(2)} = \mu n X_{n-1}^{(1)}, \quad Y_{n-1}^{(2)} = mh - \mu n Y_{n-1}^{(1)}, \quad Z_{n-1}^{(2)} = \mu Z_{n-1}^{(1)}
\]

(4.4)

where \( \ell, m, \mu \) are non-zero constants of our choice and \( h \) is an arbitrary non-zero constant to be determined.

Substituting (4.4) in (4.1) we get

\[
h = \frac{2\mu n (D\ell x_{n-1}^{(1)} + m y_{n-1}^{(1)})}{D\ell^2 + m^2}
\]

(4.5)

and hence
The repetition of the above process leads to a sequence of integral solution of the equation (4.1) expressed as

$$
\begin{bmatrix}
X_{n+1}^{(2)} \\
Y_{n+1}^{(2)} \\
Z_{n+1}^{(2)}
\end{bmatrix} =
\begin{bmatrix}
\mu^*(D\ell^2 - m^2) & 2\mu^*\ell m & 0 \\
2\mu^* Dlm & \mu^*(m^2 - D\ell^2) & 0 \\
0 & 0 & \mu(D\ell^2 + m^2)
\end{bmatrix}
\begin{bmatrix}
Y_{n-2} & -X_{n-2} & 0 \\
DX_{n-2} & Y_{n-2} & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
X_0 \\
Y_0 \\
Z_0
\end{bmatrix},
$$

(4.6)

where \( s = 0, 1, 2, 3, \ldots \).

For the sake of simplicity and brevity, we present the solutions of equation (4.1) for some particular values of \( D \) and \( n \).

**Table (4a)**

<table>
<thead>
<tr>
<th>Equations</th>
<th>((X_0, Y_0, Z_0), n = 1)</th>
<th>((X_1, Y_1, Z_1), n = 2)</th>
<th>((X_2, Y_2, Z_2), n = 3)</th>
<th>((X_3, Y_3, Z_3), n = 4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(Z^{2n} = 3X^2 + Y^2)</td>
<td>((2, -2, 4))</td>
<td>((-8, -8, 4))</td>
<td>((-32, -32, 4))</td>
<td>((-128, -128, 4))</td>
</tr>
<tr>
<td>(Z^{2n} = 8X^2 + Y^2)</td>
<td>((2, -7, 9))</td>
<td>((-28, 17, 9))</td>
<td>((-162, -567, 9))</td>
<td>((-2268, 1377, 9))</td>
</tr>
<tr>
<td>(Z^{2n} = 10X^2 + Y^2)</td>
<td>((2, -9, 11))</td>
<td>((-36, 41, 11))</td>
<td>((-242, 1089, 11))</td>
<td>((-4356, 4961, 11))</td>
</tr>
<tr>
<td>(Z^{2n} = 11X^2 + Y^2)</td>
<td>((2, -10, 12))</td>
<td>((-40, 56, 12))</td>
<td>((-288, 1440, 12))</td>
<td>((-5760, 8064, 12))</td>
</tr>
</tbody>
</table>
From the solutions (4.2) we obtain the following relations:

(i) \[ Y_{n-1}^{(1)} X_{n-2} - DX_{n-2} X_{n-1}^{(1)} = Y_0 \left( Z_{n-2}^{(n-1)} \right) \]

(ii) \[ X_{n-1}^{(1)} - Y_{n-1}^{(1)} = Y_{n-2} \left( X_0 - Y_0 \right) - X_{n-2} \left( Y_0 + DX_0 \right) \]

(iii) \( \left( Z_{n-1}^{(0)} \right)^n = Z_0 \left( Z_{n-2}^{(n-1)} \right) \)

(iv) \[ X_{n-1}^{(0)} Y_{n-1}^{(0)} = X_0 Y_0 \left( Y_{n-2}^2 - DX_{n-2}^2 \right) + X_{n-2} Y_{n-2} \left( DX_0^2 - Y_0^2 \right) \]

Besides the above results, from the solutions (4.7) one may observe the following relations:

(i) \( \frac{X_6}{X_0} = \frac{X_2}{X_1} = \left( \frac{X_2}{X_0} \right)^3 = \left( \frac{X_3}{X_1} \right)^3 = \left( \frac{X_4 X_3}{X_0 X_1} \right) = \left( \frac{X_5 X_4}{X_0 X_1} \right) = \left( \frac{X_6}{X_0} \right)^3 = \left( \frac{X_6}{X_1} \right)^3 = \left( \frac{X_6}{X_0} \right)^3 = \left( \frac{X_6}{X_1} \right)^3 = \left( \frac{X_6}{X_0} \right)^3 = \left( \frac{X_6}{X_1} \right)^3 = \left( \frac{X_6}{X_0} \right)^3 = \left( \frac{X_6}{X_1} \right)^3 \)

(ii) \( \sum_{f=x,y,z}^{\infty} f_{S+2} = 3 \text{ cube root of } \prod_{f=x,y,z}^{\infty} \frac{f_{S+2}}{f_{S}} \)

(iii) \( \frac{f_{S+2}}{f_0} = \frac{f_{S+1}}{f_1} = \left( \frac{f_2}{f_0} \right)^s = \left( \frac{f_3}{f_1} \right)^s \)

(iv) \( f_{S+2k} = \frac{(f_{S+2})^2}{f_s^2} \)

(v) \( \prod_{S=1}^{n} \left( \frac{f_{2k}}{f} \right)^{(2k)f=S_1} = (D+1)^{2^k S_k}, \quad k = 1,2,3 \)

\( (D+1)^{n}, \quad k = 0 \)

where \( S_k = 1^k + 2^k + 3^k + \ldots + n^k \)

in which 'f' represents either X or Y or Z.
Further, define a matrix $M$ given by

$$M = \begin{pmatrix} a & \pm b \sigma^q \\ Db \sigma^q & \mp a \end{pmatrix}$$

where $a$, $b$ and $q$ are non-zero integral constants.

We write

$$M^\beta \begin{pmatrix} \tilde{X}^{(s+1)}_{\beta(n-1)} \\ \tilde{Y}^{(s+1)}_{\beta(n-1)} \end{pmatrix} = \begin{pmatrix} \tilde{X}^{(s+1)}_{\beta(n-1)} \\ \tilde{Y}^{(s+1)}_{\beta(n-1)} \end{pmatrix}^t,$$

where in $\beta \geq 1$ and $t$ is the transpose.

It is seen that

$$\begin{pmatrix} \tilde{X}^{(s+1)}_{\beta(n-1)} \\ \tilde{Y}^{(s+1)}_{\beta(n-1)} \\ Z^{(s+1)}_{\beta} \end{pmatrix}$$

is an integral solution of the equation

$$\left(a^2 + Db^2 \sigma^{2q}\right) Z^{2n} = DX^2 + Y^2 \quad \text{(4.8)}$$

As an application of (4.1), we determine the positive square root of a binomial quadratic surd $Y_{n-1}^{(l)} + iX_{n-1}^{(l)} \sqrt{D}$.

Set

$$\sqrt{Y_{n-1}^{(l)}} + i \sqrt{D} X_{n-1}^{(l)} = A + iB \sqrt{D} \quad \text{(4.9)}$$
Since the complex roots occur in pairs we have

\[ \sqrt{y_{n-1}^{(0)} - i x_{n-1}^{(0)} \sqrt{D}} = A - iB \sqrt{D} \]  \hspace{1cm} (4.10)

Multiplying (4.9) and (4.10), we get

\[ A^2 + DB^2 = \sqrt{(y_{n-1}^{(0)})^2 + D(x_{n-1}^{(0)})^2} = (Z_{n-1}^{(0)})^p = Z_0^a \]  \hspace{1cm} (4.11)

we write

\[ A = a - D\bar{\mu}, \quad B = a + \bar{\mu} \]  \hspace{1cm} (4.12)

where \( \bar{\mu} \) is a non-zero constant of our choice and \( a \) is an arbitrary non-zero constant to be determined. From equations (4.11) and (4.12), we get

\[ a = \sqrt{\frac{Z_0^a}{D + 1 - \bar{\mu}^2 D}} \]  \hspace{1cm} (4.13)

It is to be noted here that the square root on the R.H.S of the above equation is evaluated only for the values of \( Z_0 \) obtained from equation (4.3) when \( p \) and \( q \) take the same values, while for other values of \( p \) and \( q \), though we get integral values of ‘\( a \)’ the equation (4.9) is not exactly satisfied.

Thus, in view of (4.12), one can determine the values of \( A \) and \( B \).

A few examples are presented in the following Table (4b).
<table>
<thead>
<tr>
<th>Equations</th>
<th>n</th>
<th>p</th>
<th>q</th>
<th>((X_{n-1}^{(i)}, Y_{n-1}^{(i)}, Z_{n-1}^{(i)}))</th>
<th>(\sqrt[n]{Y_{n-1}^{(i)} + i\sqrt[D]{X_{n-1}^{(i)}}})</th>
<th>(A + iB\sqrt[D]{D})</th>
</tr>
</thead>
<tbody>
<tr>
<td>(Z^{2n} = 2X^2 + Y^2)</td>
<td>4</td>
<td>1</td>
<td>1</td>
<td>((-36, -63, 3)) ((-9216, -16128, 12))</td>
<td>(\sqrt{-63 + i\sqrt{2}(-36)})</td>
<td>(-3 + i6\sqrt{2})</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>2</td>
<td>2</td>
<td>(\sqrt{-16128 + i\sqrt{2}(-9216)})</td>
<td>(-48 + i96\sqrt{2})</td>
<td></td>
</tr>
<tr>
<td>(Z^{2n} = 3X^2 + Y^2)</td>
<td>3</td>
<td>1</td>
<td>1</td>
<td>((-32, -32, 4)) ((-2048, -2048, 16))</td>
<td>(\sqrt{-32 + i\sqrt{3}(-32)})</td>
<td>(-4 + i\sqrt{3}(4))</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>2</td>
<td>2</td>
<td>(\sqrt{-2048 + i\sqrt{3}(-2048)})</td>
<td>(-32 + i\sqrt{3}(32))</td>
<td></td>
</tr>
<tr>
<td>(Z^{2n} = 10X^2 + Y^2)</td>
<td>3</td>
<td>1</td>
<td>1</td>
<td>((-242, -1089, 11)) ((-15488, -69696, 44))</td>
<td>(\sqrt{-1089 + i\sqrt{10}(-242)})</td>
<td>(-11 + i1\sqrt{10})</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>2</td>
<td>2</td>
<td>(\sqrt{-69696 + i\sqrt{10}(-15488)})</td>
<td>(-88 + i88\sqrt{10})</td>
<td></td>
</tr>
<tr>
<td>(Z^{2n} = 11X^2 + Y^2)</td>
<td>4</td>
<td>1</td>
<td>1</td>
<td>((-5760, 8064, 12)) ((-1474560, -2064384, 48))</td>
<td>(\sqrt{8064 + i\sqrt{11}(-5760)})</td>
<td>(-120 + i24\sqrt{11})</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>2</td>
<td>2</td>
<td>(\sqrt{-2064384 + i\sqrt{11}(-1474560)})</td>
<td>(-1920 + i384\sqrt{11})</td>
<td></td>
</tr>
</tbody>
</table>