CHAPTER II

INTEGRAL SOLUTIONS OF BINARY QUADRATIC

DIOPHANTINE EQUATION $y^2 = dx^2 + n$

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CHAPTER II

INTEGRAL SOLUTIONS OF BINARY QUADRATIC DIOPHANTINE EQUATION

\[ Y^2 = DX^2 + N \]

In this chapter, we illustrate an elegant way of solving binary quadratic Diophantine equation represented by

\[ Y^2 = DX^2 + N \]  \hspace{1cm} (2.1)

where D is any positive non square integer and N a given positive integer and exhibit a method to generate a sequence of nontrivial integral solutions of equation (2.1). Also for a given D, a procedure is provided to get a sequence of values of N such that, the equation (2.1) has nontrivial integral solutions. In addition, some special cases of interest of equation (2.1) have been considered.

At the outset, we note that the equation (2.1) is symmetric in X and Y and thus, it is enough to obtain nontrivial integer solutions. Given D, N choose a non zero integer K so that,

\[ D + K = U^2, \] \hspace{0.5cm} \[ D + NK = V^2. \]
It is seen that

\[(X_0, Y_0) = \left( \frac{U \pm V}{K}, \frac{UV \pm D}{K} \right)\]  \hspace{1cm} (2.2)

represents a solution of (2.1) in which, one is fractional and the other is an integral. In what follows, we exhibit a method of generating the second solution of equation (2.1) knowing its integral solution \((X_0, Y_0)\).

Let \((X_1, Y_1)\) be the second solution of (2.1) represented by the transformations

\[X_1 = l_1 X_0 + m_1 Y_0, \quad Y_1 = lX_0 + mY_0\]  \hspace{1cm} (2.3)

where \(l_1, m_1, l, m\) are parameters to be determined.

Since,

\[Y_1^2 - DX_1^2 = Y_0^2 - DX_0^2\]

we, on comparing the coefficients of like terms, obtain the following relations

\[lm = Dl_1 m_1\]  \hspace{1cm} (2.4)

\[l^2 - Dl_1^2 = -D\]  \hspace{1cm} (2.5)

\[m^2 - Dm_1^2 = 1\]  \hspace{1cm} (2.6)
The equations (2.4) & (2.5) are equivalent to the equation (2.6) for the choice \( l_i = m \) and the equation (2.6) is the Pellian equation whose integral solutions are well known. Thus, the second solution \((x_1, y_1)\) of equation (2.1) can be obtained from equation (2.3), which is written in the matrix form as

\[
\begin{bmatrix}
X_1 \\
Y_1
\end{bmatrix} = M \begin{bmatrix}
X_0 \\
Y_0
\end{bmatrix}
\]

where

\[
M = \begin{bmatrix}
m & m_1 \\
m_1 & m
\end{bmatrix}
\]

(2.7)

The repetition of the above process leads to the general solution \((x_n, y_n)\) of equation (2.1) given by

\[
\begin{bmatrix}
x_n \\
y_n
\end{bmatrix} = M^n \begin{bmatrix}
x_0 \\
y_0
\end{bmatrix}
\]

we, now present a derivation of the formula for \(n^{th}\) power of a \(2 \times 2\) matrix \(A\).

By Cayley- Hamilton theorem, the matrix \(A\) satisfies its characteristic equation

\[
(A - \alpha I)(A - \beta I) = 0
\]
where $\alpha, \beta$ are eigen values of matrix $A$ and $I$ is the unit matrix of order 2. If $\alpha$ and $\beta$ are distinct, define matrices $X, Y$ given by

$$X = \frac{A - \beta I}{\alpha - \beta}, \quad Y = \frac{A - \alpha I}{\beta - \alpha}$$

(2.8)

so that

$$\alpha X + \beta Y = A$$

(2.9)

Note that

$$\{ X^t = X, \quad Y^t = Y, \quad \text{for } K \geq 2 \}
\quad
\{ XY = YX = 0 \}$$

(2.10)

In view of (2.10), we obtain from (2.9)

$$A^n = (\alpha X + \beta Y)^n$$

$$= \alpha^n X^n + \beta^n Y^n$$

$$= \alpha^n X + \beta^n Y,$$

$$A^n = \alpha^n \left( \frac{A - \beta I}{\alpha - \beta} \right) + \beta^n \left( \frac{A - \alpha I}{\beta - \alpha} \right), \quad \alpha \neq \beta$$

(2.11)

In case $\alpha = \beta$, define a matrix $Z$ given by

$$Z = A - \alpha I$$
so that

\[ Z^k = 0, \quad K \geq 2 \]

and hence

\[ A^n = (Z + KI)^n = \alpha^n I + n\alpha^{n-1}Z \]
\[ A^n = \alpha^{n-1}(n\alpha - (n-1)I), \quad \alpha = \beta \quad (2.12) \]

If the matrix A is invertible \((\alpha \neq 0, \beta \neq 0)\), it is easy to see that (2.11) & (2.12) holds for all integral values of \(n\).

If A is real but its eigen values \(\alpha = p + iq\) and \(\beta = p - iq\) are non real \((q \neq 0)\) with some power of them real, say \((p + iq)^m = (p - iq)^m = r\), then by (2.11), we have

\[ A^m = rI \]

Now, for the matrix M given by (2.7), the eigen values \(\alpha, \beta\) are

\[ \alpha = m + \sqrt{Dm_1}, \quad \beta = m - \sqrt{Dm_1} \]

which are real and distinct. By employing the formula presented in equation (2.11), we have
\[ M^n = \left( m + m \sqrt{D} \right)^n \left[ \frac{1}{2\sqrt{D}} \begin{pmatrix} \sqrt{D} & 1 \\ D & \sqrt{D} \end{pmatrix} + \left( m - \sqrt{D} m_1 \right)^n \left[ \frac{1}{2\sqrt{D}} \begin{pmatrix} \sqrt{D} & -1 \\ -D & \sqrt{D} \end{pmatrix} \right] \]

\[ = \begin{pmatrix} \alpha_n & \beta_n \\ \sqrt{D} \beta_n & \alpha_n \end{pmatrix}, \]

in which \( \alpha_n, \beta_n = \frac{1}{2} \left( m + \sqrt{D} m_1 \right)^n \pm \left( m - \sqrt{D} m_1 \right)^n \)

Thus, a sequence \( \{(X_n, Y_n), n = 1, 2, 3, \ldots \} \) of integral solutions of (2.1)

is obtained from

\[
\begin{pmatrix} X_n \\ Y_n \end{pmatrix} = \begin{pmatrix} \alpha_n & \beta_n \\ \sqrt{D} \beta_n & \alpha_n \end{pmatrix} \begin{pmatrix} X_0 \\ Y_0 \end{pmatrix} \quad (2.13)
\]

\[ = \begin{pmatrix} \alpha_n X_0 + \frac{\beta_n}{\sqrt{D}} Y_0 \\ \sqrt{D} \beta_n X_0 + \alpha_n Y_0 \end{pmatrix} \]

For the sake of simplicity and brevity, we present the first three solutions of equation (2.1) for some particular values of D and N.
Results of Interest:

(1) It is verified that \((X_n, Y_n)\) satisfies the higher order Diophantine equation

\[ Y_{n+1}^{*} = Y_{n}^{*\pm 1}(DX^{2} + N) \]
By considering suitable choices of linear combination of \( (X_n, Y_n) \) of equation (2.1), one may obtain solutions of various forms of Diophantine equations. A few examples are presented below:

**Table (2b)**

<table>
<thead>
<tr>
<th>Equations:</th>
<th>Solutions:</th>
</tr>
</thead>
<tbody>
<tr>
<td>( Y^2 = DX^2 - (D^{2x+1})^k N )</td>
<td>( (D^k Y_n, D^{k+1} X_n) )</td>
</tr>
<tr>
<td>( Y^2 = DX^2 + \left( 1 - \lambda^2 D^{2r-1} \right)^k N )</td>
<td>( \left( \frac{1}{\lambda D}, \frac{\lambda D - 1}{1} \right) \begin{pmatrix} X_n \ Y_n \end{pmatrix} )</td>
</tr>
<tr>
<td>( Y^2 = DX^2 + \left( D^{2a} - D^{2r-1} \right)^k N )</td>
<td>( (D^a X_n + D^b Y_n, D^c Y_n + D^d X_n) )</td>
</tr>
<tr>
<td>( Y^2 = DX^2 + (1 - D)^k \left( 1 - D^3 \right)^k N )</td>
<td>( \left( \frac{1}{D}, \frac{1}{D^2} \right) \begin{pmatrix} X_n \ Y_n \end{pmatrix} )</td>
</tr>
</tbody>
</table>

Where \( a, b, r, k, \lambda = 1, 2, 3 \ldots \ldots \)

In the matrix representation presented by (2.13) denoting \( \left( \frac{\beta_n}{\sqrt{D}}, \alpha_n \right) \), \( (\alpha_n, \sqrt{D} \beta_n) \) and \( (X_n, Y_n) \) by \( H_n, G_n, R_n \) respectively, it is seen that \( H_n, G_n \) satisfy the Diophantine equations \( Y^2 = DX^2 + 1 \), \( Y^2 = D(X^2 - 1) \) respectively. Also observe that the areas of the triangles \( OG_n R_n \) and \( OG_n R_n \) are same, where O is the origin.
(4) Let \( D = K^2 + 1 \), in equation (2.1). The triplet obtained from the solution of (2.1) namely
\[
(2X_n(KX_n + Y_n), (K^2 - 1)X_n^2 + 2KX_nY_n + Y_n^2, (K^2 + 1)X_n^2 + 2KX_nY_n + Y_n^2)
\]
is a Pythagorean triangle in which one leg exceeds \( K \) times the other leg by \( N \).

(5) Let \( D = K^2 - 1,(K \neq 1) \) in equation (2.1). In this case, we obtain the same Pythagorean triangle given above which is such that the hypotenuse exceeds \( K \) times a leg by \( N \).

**Special cases of equation (2.1)**

In what follows, we present tables exhibiting non-trivial integral solutions of a few Diophantine equations of interest obtained from (2.1):
Table (2c)

D = a square free integer, \( N = K \alpha^2 \). (\( K \) = an arbitrary constant)

<table>
<thead>
<tr>
<th>( D )</th>
<th>( K )</th>
<th>( X )</th>
<th>( Y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a^2 + \alpha )</td>
<td>( n^2 + a(2n-1) )</td>
<td>( \alpha )</td>
<td>( (n+a) \alpha )</td>
</tr>
<tr>
<td></td>
<td>( 3a + 4 )</td>
<td>( -3 \alpha )</td>
<td>( -(3a+2) \alpha )</td>
</tr>
<tr>
<td>( a^2 + 2\alpha - 1 )</td>
<td>( 2\alpha + 5 )</td>
<td>( -(\alpha + 2) \alpha )</td>
<td>( -(\alpha^2 + 3\alpha + 1) \alpha )</td>
</tr>
<tr>
<td></td>
<td>( n^2 + 2a(n-1) + 1 )</td>
<td>( -\alpha )</td>
<td>( -(n+a) \alpha )</td>
</tr>
<tr>
<td>( a(a+2) )</td>
<td>( n^2 + a(2n-2) )</td>
<td>( \alpha )</td>
<td>( (n+a) \alpha )</td>
</tr>
<tr>
<td></td>
<td>( n^2 + 2n(a+1) + 1 )</td>
<td>( \alpha )</td>
<td>( (n+a+1) \alpha )</td>
</tr>
<tr>
<td>( a^2 + 1 )</td>
<td>( n^2 + 2n\alpha - 1 )</td>
<td>( (n+2\alpha) \alpha )</td>
<td>( 2a^2 + na + 1 )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( -n \alpha )</td>
<td>( (1-an) \alpha )</td>
</tr>
<tr>
<td>( a^2 - 1 )</td>
<td>( n^2 - 2n\alpha + 1 )</td>
<td>( n \alpha )</td>
<td>( (1-an) \alpha )</td>
</tr>
<tr>
<td>( 4n^2 - 1 )</td>
<td>( 4n^2 - 2 )</td>
<td>( (2n-1) \alpha )</td>
<td>( (1+2n-4n^2) \alpha )</td>
</tr>
<tr>
<td>( 4n^2 + 4n )</td>
<td>( -(4n^2 + 4n - 1) )</td>
<td>( 2n \alpha )</td>
<td>( (1-4n^2 - 2n) \alpha )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( 2(n+1) \alpha )</td>
<td>( -(4n^2 + 6n + 1) \alpha )</td>
</tr>
</tbody>
</table>
### Table (2d)

\[ D = m^2, \quad (m > 1), \quad N \text{ is a positive square free integer} \]

<table>
<thead>
<tr>
<th>N</th>
<th>X</th>
<th>Y</th>
</tr>
</thead>
<tbody>
<tr>
<td>( C^2 \pm 2maC )</td>
<td>( \pm \alpha )</td>
<td>( C \pm ma )</td>
</tr>
<tr>
<td>( C^2 \mp 2maC )</td>
<td>( \pm \alpha )</td>
<td>( \pm ma - C )</td>
</tr>
</tbody>
</table>

The letters \( C, \alpha \) are arbitrary constants.

### Table (2e)

\[ D = m^2, (m > 1), \quad N = -N_1, \quad N_1 \text{ is a positive square free integer}. \]

<table>
<thead>
<tr>
<th>( N_1 )</th>
<th>X</th>
<th>Y</th>
</tr>
</thead>
<tbody>
<tr>
<td>( (2m-1)C^2 )</td>
<td>(-C)</td>
<td>((1-m)C)</td>
</tr>
<tr>
<td>( -(2m+1)C^2 )</td>
<td>(-C)</td>
<td>(-(m+1)C)</td>
</tr>
</tbody>
</table>

The letters \( C, \alpha \) are arbitrary constants.
Table (2f)

\[ D = m^2, \quad N = K\alpha^2, \quad (K > 1), \quad K = 2mn + 1, \quad m = (r + 1), \quad r, n = 1,2,3 \ldots \]

<table>
<thead>
<tr>
<th>( X )</th>
<th>( Y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \alpha n )</td>
<td>( (nr + n + 1)\alpha )</td>
</tr>
</tbody>
</table>

As a special case, when \( D = m^2, N = \alpha^2 \) the solutions of (2.1) for particular values of \( m \) and \( \alpha \) are given in Table (2g) below:

Table (2g)

<table>
<thead>
<tr>
<th>( m )</th>
<th>( \alpha )</th>
<th>( X )</th>
<th>( Y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( (r + 2) )</td>
<td>( (2r + 2)n )</td>
<td>( r n )</td>
<td>( n(r^2 + 2r + 2) )</td>
</tr>
<tr>
<td>( (r + 1)^2 )</td>
<td>( (4r + 2r^2)n )</td>
<td>( (r^2 + 2r - 1)n )</td>
<td>( (r^3 + 4r^3 + 4r^2 + 1)n )</td>
</tr>
</tbody>
</table>

The equation (2.13) gives nontrivial integral solutions of equation (2.1) for known values of \( D \) and \( N \). Now for a given value of \( D \), we present a method to get a sequence of values of the parameter \( N \ (\in \mathbb{Z} \sim \{0\}) \) such that the equation (2.1) has nontrivial solutions:
Choose non zero positive integers \( \gamma, \delta, \alpha \) such that

(i) \( \frac{\gamma^2 - D}{\delta} = 1 \)

(ii) \( Da^2 + \delta \) is a perfect square say \( b^2 \). Then, it is verified that

\[
\left( \frac{a\gamma + b}{\delta}, \frac{by + Da^2}{\delta} \right)
\]

satisfies the equation \( Y^2 = DX^2 + 1 \).

It is noted that, by replacing \( \gamma \) by \( \gamma + \alpha \delta \), \( \gamma = \pm 1, \pm 2, \ldots \), in (i)

we have \( \frac{(\gamma + \alpha \delta)^2 - D}{\delta} \) is an integer \( > 1 \).

Therefore, the sequence of values of \( N \) for a given \( D \) are given by

\[
N = \frac{(\gamma + \alpha \delta)^2 - D}{\delta}, \quad (2.14)
\]

\[\alpha = \pm 1, \pm 2, \ldots\]

Thus, for a given value of \( D \), the non trivial integral solutions of \( (2.1) \),

for values of \( N \) represented by the equation \( (2.14) \) are seen to be

\[
\left( \frac{\alpha + 2\gamma}{\delta}, \frac{\gamma \alpha + \alpha^2 + D}{\delta} \right) \quad \text{and} \quad \left( -\alpha, \frac{\gamma^2 - D}{\delta} \right)
\]
in which one is fractional and the other is integral.

In Table (2h) below we exhibit the above scheme with some illustrations.

\[ r^2 = DX^4 + N \] 

(2.15)
are obtained after performing some algebra. The following Table (2i) exhibits a few illustrations.

**Table (2i)**

<table>
<thead>
<tr>
<th>D</th>
<th>N</th>
<th>X</th>
<th>Y</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>(2(a+1)^2 - 4(a+1)^2 + 1)</td>
<td>(a+1)</td>
<td>(2(a+1)^2 - 1)</td>
</tr>
<tr>
<td>3</td>
<td>(1-6\alpha^2 + 6\alpha^4)</td>
<td>(\alpha)</td>
<td>(3\alpha^2 - 1)</td>
</tr>
<tr>
<td>5</td>
<td>(4\alpha^4 + 16\alpha^2 + 18\alpha^2 + 4\alpha - 1)</td>
<td>(a+1)</td>
<td>(3\alpha^2 + 6\alpha + 2)</td>
</tr>
<tr>
<td>8</td>
<td>(8\alpha^4 - 8\alpha^2 + 1)</td>
<td>(\alpha)</td>
<td>(4\alpha^2 - 1)</td>
</tr>
<tr>
<td>12</td>
<td>(4\alpha^4 + 16\alpha^2 + 16\alpha^2 - 3)</td>
<td>(a+1)</td>
<td>(4\alpha^2 + 8\alpha + 3)</td>
</tr>
<tr>
<td>61</td>
<td>(3\alpha^4 + 12\alpha^2 + 2\alpha^2 - 20\alpha - 12)</td>
<td>(a+1)</td>
<td>(8\alpha^2 + 16\alpha + 7)</td>
</tr>
</tbody>
</table>

In Table (2j) below, we exhibit the representation of prime N for the values of D and \(\sigma\) presented in the Table (2h).
### Table (2j)

<table>
<thead>
<tr>
<th>D</th>
<th>( \delta )</th>
<th>Representation of Prime N</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>( 2n + 4 )</td>
<td>( 8g + 17 ) (or) ( 8g + 31 )</td>
</tr>
<tr>
<td>3</td>
<td>( 6n - 3 )</td>
<td>( 12g + 1 ) (or) ( 12g + 13 )</td>
</tr>
<tr>
<td>5</td>
<td>( 4n + 1 )</td>
<td>( 20g + 1 ) (or) ( 20g + 9 ) (or) ( 20g + 11 ) (or) ( 20g + 19 )</td>
</tr>
<tr>
<td>8</td>
<td>( 8n - 4 )</td>
<td>( 32g + 1 ) (or) ( 32g + 17 )</td>
</tr>
<tr>
<td>12</td>
<td>( 4n )</td>
<td>( 48g + 1 ) (or) ( 48g + 13 ) (or) ( 48g + 45 )</td>
</tr>
<tr>
<td>17</td>
<td>( 8n + 3 )</td>
<td>( 68g + 21 ) (or) ( 68g + 25 ) (or) ( 68g + 43 )</td>
</tr>
<tr>
<td>61</td>
<td>( 3n - 2 )</td>
<td>( 224g + 25 )</td>
</tr>
</tbody>
</table>

**Remarks:**

\( (I) \) The fundamental solution \( p_o(x_o, y_o) \) of the Pellian equation

\[
y^2 = DX^2 + 1 \quad \text{(2.16)}
\]
where $D$ is a non-square positive integer, is obtained by developing the value of $\sqrt{D}$ into a continued fraction and the equation (2.16) always has an infinity of solutions in [7]Robert D.Carmichael (1915), [26] T.Nagell (1964). The aim of this remark is to bring out rather interesting results of the integral solutions.

(1) Let $P_{n_1}(X_{n_1}, Y_{n_1})$, $i = 0, 1, 2, 3, \ldots$ denote the integral solutions of the equation (2.16) for a given value of $D$. Introducing the transformation

$$\frac{X_{n_1}}{X_i} = \alpha$$

$$Y_{n_1} + 1 = \beta$$

the equation (2.16) is reduced to $\alpha^2 = 2\beta$, a parabola with vertex at the origin, whereas, considering $y_{n_1} - 1 = \delta$ then the equation (2.16) is reduced to $\alpha^2 = 2(\delta + 2)$ which represents a parabola with vertex at $(0, -2)$.

(2) Representing the point $(y_{n_1}, Dx_{n_1})$ by $Q_{n_1}$, we obtain the following results:

(i) For prime $D \equiv 1 \pmod{4}$, $x_0 = 0 \pmod{4}$ and $x_0 \neq 0 \pmod{D}$
(ii) For prime $D = 3 \pmod{4}$, $x_0 = 1 \pmod{2}$

(iii) The points $P_0, P_1, Q_1$ lie on a straight line when $D = 2, 3$.

(iv) The area of the triangle $OQ_1, P_{r+1} = \frac{1}{2}$

(v) The area of the triangle $OP_{r+2}P_{r+1} = \frac{1}{2}|x_{r+1}|$

(vi) The area of the triangle $OQ_{r+2}P_{r+1} = \frac{1}{2}|y_{r+1}|$

(3) The integral solutions of the equation $Z^3 = X^2 + Y^2$ are well known in [3] David M. Burton (1995). Alternatively, to solve the above equation,

let us consider the equation

$$Y^2 = m^2X^2 + 1$$

(2.17)

Multiply the equation (2.17) by $y^{2(n+2)}$ and define

$$(X,Y,Z) = (\lambda (x^2 + 1)^{2n+1}, (x^2 + 1)^{n+1}, (x^2 + 1)^{2n+1})$$

(2.18)

where $\lambda (= mx)$ is an arbitrary parameter. Thus, the equation $Z^3 = X^2 + Y^2$ is seen to be birationally equivalent [62] David W. Boyd (1990) to the Pell
equation (2.16) where in $D = m^2$. It is to be noticed here that the solution
(2.18) is not unique, since, multiplying (2.17) by $m^6 x^6 y^4$, we get

$$(X, Y, Z) = (x'(x' + 1)^2, y'(x' + 1)^2, z'(x' + 1)^2)$$

As a special case, it is found that the Diophantine equation

$$Y^3 = X^2 + Y^2$$

(2.19)

is birationally equivalent to the Pythagorean equation (2.17).

The significance of the equation (2.19) is that, it represents the sum of the
squares of two integrals $X$ and $Y$ to be equal to the integral $Y$ raised to the
power 3. A generalization of this result is given by

$$Y^{2k+1} = \alpha^2 + \beta^2,$$

$$\alpha = \lambda(\lambda^2 + 1)^k, \quad \beta = (\lambda^2 + 1)^k$$

(4) Any linear combination between $X_{r+1}$ and $Y_{r+1}$ of the form

$$(D^n Y_{r+1} \pm X_{r+1}, \pm Y_{r+1} + D^{n+1} X_{r+1})$$

will be a solution of the Diophantine equation

$$Y^2 = DX^2 + 1 - D^{2n+1}$$
In a similar manner, one can consider other forms of linear combination between \( X_{\ast} \) and \( Y_{\ast} \). A few examples are given in the following table wherein the letter 't' represents the transpose and \( M, N \) denote respectively the matrices

\[
\begin{pmatrix}
D^t & \pm 1 \\
\pm 1 & D^{t+1}
\end{pmatrix}, \quad \begin{pmatrix}
\pm 1 & D^{t+1} \\
D^t & \pm 1
\end{pmatrix}, \quad r = 0, 1, 2, \ldots
\]

**Table (2k)**

<table>
<thead>
<tr>
<th>S.No.</th>
<th>Diophantine equations</th>
<th>Linear combinations</th>
</tr>
</thead>
<tbody>
<tr>
<td>(i)</td>
<td>( Y^2 = DX^2 + (1 - D^{2\omega}) )</td>
<td>( M, N_1^{\omega-1} P )</td>
</tr>
<tr>
<td>(ii)</td>
<td>( Y^2 = DX^2 + (1 - D^3)^\prime )</td>
<td>( M, N_0^{\omega-1} P )</td>
</tr>
<tr>
<td>(iii)</td>
<td>( Y^2 = DX^2 + \prod_{k=1}^{n} (1 - D^{2k-1}) )</td>
<td>( M, N_1^{\omega-1} P )</td>
</tr>
</tbody>
</table>

**(2)** The integral solutions of the Diophantine equation

\[
Y^2 = DX^2 + \sigma^2
\]

are expressed through difference equations,
In [7] Robert D. Carmichael (1915), the most general form of integral solutions \((X_n, Y_n)\) of the equation

\[
Y^2 = DX^2 + \sigma^2
\]  

(2.20)

is written as

\[
Y_n + \sqrt{D}X_n = \sigma(b + \sqrt{D}a)^n,
\]  

(2.21)

where \(D\) is not a perfect square and \((a, b)\) is the least non-zero positive integer solution of the Pellian

\[
Y^2 = DX^2 + 1
\]  

(2.22)

Expressing \((b + \sqrt{D}a)\) in the form of a square and by properties of surds involving conjugation, one obtains

\[
Y_n = \left[\left(b_0 + \sqrt{Da_0}\right)^{2n} + \left(b_0 - \sqrt{Da_0}\right)^{2n}\right]/\left(2C_0^n\right)
\]  

(2.23)

where \(a_0, b_0, c_0\) are non-zero constants such that

\[
b = \left(b_0^2 + Da_0^2\right)/c_0, \ a = 2a_0b_0/c_0
\]  

(2.24)

Now,
Also, from (2.25) and (2.26), we have

\[ Y_{n+1} = \sigma \left[ \frac{(b_0 + \sqrt{D}a_0)^{2(n+2)}}{(b_0 + \sqrt{D}a_0)^4} + \frac{(b_0 - \sqrt{D}a_0)^{2(n+2)}}{(b_0 - \sqrt{D}a_0)^4} \right] \left( \frac{1}{2c_0^{n+1}} \right) \]

(2.25)

Also,

\[ Y_{n+1} = \sigma \left[ \frac{(b_0 + \sqrt{D}a_0)^{2(n+2)}}{(b_0 + \sqrt{D}a_0)^4} + \frac{(b_0 - \sqrt{D}a_0)^{2(n+2)}}{(b_0 - \sqrt{D}a_0)^4} \right] \left( \frac{1}{2c_0^n} \right) \]

(2.26)

From (2.25) and (2.26), we have

\[ 2c_0 (b_0^2 + Da_0^2) Y_{n+1} - (b_0^2 - Da_0^2)^2 Y_n = c_0^2 Y_{n+2} \]

(2.27)

Likewise, is seen that \( X_n \) satisfies a similar difference equation. With \( X_0 = 0, Y_0 = \sigma, X_1 = \sigma a, Y_1 = \sigma b \) and substituting \( n = 0, 1, 2... \) in the above difference equations, we can deduce infinite number of solutions.
**Examples:**

(1) If $A_n$ is the area of the triangle with sides $3Y - 1, 2Y, 3Y + 1$ corresponding to $Y = Y_n$, then by invoking the procedure presented above, it can be shown that $A_n$ satisfies the recurrence relations:

$$A_{n+2} - 34A_{n+1} + A_n = 0, \quad n = 0, 1, 2, \ldots$$

$$A_0 = 0, \quad A_1 = 24 \tag{2.28}$$

(2) The most general form of integral solution $(X_n, Y_n)$ of the equation

$$(x + y)^3 + (x + y)^3 = 2(x^3 + y^3)$$

Satisfies the recurrence relations:

$$f_{n+2} - 14f_{n+1} + f_n - 6 = 0, \quad n = 0, 1, 2, \ldots$$

$$x_0 = 0, \quad y_0 = 0, \quad x_1 = 5, \quad y_1 = 1 \tag{2.29}$$

(3) A significant application of the Diophantine equation of the title is presented here.

Consider the equation

$$X^2 + 4XY + Y^2 - 2X + 2Y - 8 = 0 \tag{2.30}$$
The above equation represents a hyperbola whose center is (-1,1). By shifting the origin to the center (-1,1), the equation of the hyperbola can be reduced to

\[ x^2 + y^2 + 4xy - 6 = 0 \quad (2.31) \]

where

\[ X = x - 1 \quad \text{and} \quad Y = y + 1 \quad (2.32) \]

Again, setting

\[ x = M \pm N, \quad y = M \mp N \quad (2.33) \]

the above equation becomes

\[ N^2 = 3(M^2 - 1) \quad (2.34) \]

By employing the analysis presented in this chapter II, the integral solutions is given by

\[ M_n = \frac{1}{2} \left[ (2 + \sqrt{3})^n + (2 - \sqrt{3})^n \right] \]

\[ N_n = \pm \frac{3}{2\sqrt{3}} \left[ (2 + \sqrt{3})^n - (2 - \sqrt{3})^n \right] \quad (2.35) \]

where \( n = 0, 1, 2, \ldots \).
Taking advantage of the equations (2.33) and (2.32), the sequence of integral solutions of the hyperbola can be written as

\[ X_n = M_n + N_n - 1, \quad Y_n = M_n - N_n + 1 \] 

(i.e.)

\[
X_n = \frac{1}{2^{n+1}} \left\{ (\sqrt{3} + 1)^{2n+1} - (\sqrt{3} - 1)^{2n+1} \right\} - 1
\]

\[
Y_n = \frac{1}{2^{n+1}} \left\{ (\sqrt{3} + 1)^{2n+1} + (\sqrt{3} - 1)^{2n+1} \right\} + 1
\] 

(or)

\[ X_n = M_n - N_n - 1, \quad Y_n = M_n + N_n + 1 \] 

(i.e.)

\[
X_n = \frac{1}{2^{n+1}} \left\{ (\sqrt{3} + 1)^{2n+1} + (\sqrt{3} - 1)^{2n+1} \right\} - 1
\]

\[
Y_n = \frac{1}{2^{n+1}} \left\{ (\sqrt{3} + 1)^{2n+1} - (\sqrt{3} - 1)^{2n+1} \right\} + 1
\] 

From (2.37) and (2.39) we can observe that both \( X_n \) and \( Y_n \) are even.

Their sum is even and their product is represented as the difference of two squares.
From (2.37), the solutions $X_n$ and $Y_n$ satisfy the following recurrence relations:

(i) $X_{n+2} - 4X_{n+1} + X_n = 2, \quad X_0 = 0, \quad X_1 = 4.$

(ii) $Y_{n+2} - 4Y_{n+1} + Y_n = -2, \quad Y_0 = 2, \quad Y_1 = 0.$

(iii) $(Y_n - 1)^2 - (X_n + 1)^2 = 2.$

(iv) $2^{2(n-1)}[X_n^2 - (Y_n - 1)^2 + 2^2] = 1 - 2^{n-2}[X_n + 1].$

From (2.39) the solutions $X_n$ and $Y_n$ satisfy the following recurrence relations:

(i) $X_{n+2} - 4X_{n+1} + X_n = 2, \quad X_0 = 0, \quad X_1 = -2.$

(ii) $Y_{n+2} - 4Y_{n+1} + Y_n = -2, \quad Y_0 = 2, \quad Y_1 = 6.$

(iii) $(X_n + 1)^2 - (Y_n - 1)^2 = 2.$

(iv) $2^{2n-2} \left[ X_n^2 - 2^2(X_n + 1)^2 \right] + 2^{n-2}[X_n + 1] = 1.$