CHAPTER-I

INTRODUCTION

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CHAPTER-1

RELEVANT LITERATURE SURVEY

ON DIOPHANTINE EQUATIONS

Introduction:

Diophantine problems, named after Diophantus of Alexandria (c.250 A.D), are concerned with the integral solutions of polynomial equations with integer coefficients. An equation which has two or more unknowns is called an indeterminate equation. More generally, a system of equations is called indeterminate, if the number of equations is less than that of the unknowns. The theory of indeterminate equations play a significant role in the theory of Higher Arithmetic (Number Theory) and have a marvelous effect on credulous people and always occupy a remarkable position due to unquestioned historical importance. All that is needed is something to arouse interest and get started. As a pleasant and effective stimulus, the historical background of the theory should be presented to have a deep analysis about Diophantine equations.

Diophantus, one of the last Alexandrian Mathematicians of 3rd century devoted himself to Algebra. He proposed many indeterminate problems in
his arithmetic and made systematic use of algebraic symbols. He was the first Mathematician to make such an effort towards developing a symbolism for the powers of algebraic expressions. He was content with a single numerical rational solution although, the problems usually had infinitely many rational solutions and often integral solutions. Typical of Diophantus is that he was only interested in positive rational solutions; he called irrational solutions "impossible" and never accepted as an answer to a problem, a quantity which was negative. He was careful to select his coefficients, so as to get the positive rational solutions he was looking for. Frequently, he followed a method resembling somewhat the Hindu "false position": a preliminary value is assigned to the unknown, which satisfied only one or two of the necessary conditions. This leads to expressions palpably wrong, but nevertheless suggesting some stratagem by which one of the correct values can be obtained. Diophantus knew how to solve quadratic equations, but in the extant books of his *Arithmetica* he nowhere explains the mode of solution. Because he sometimes restricted his solutions to integers, in his honour, his name is attached to the kind of indeterminate equations for which the values of the variables are integers and study of such equations is known as Diophantine Analysis.
Diophantus wrote 3 books: (i) *Arithmetica*, originally in 13 books, of which 6 are extant, (ii) a tract *De polygonis numeris* of which a portion is extant and (iii) a number of propositions under the title of *porisms*. It should be stated that, most ancient manuscript of *Arithmetica* now extant was written in the 13th century - about a thousand years after the original one appeared. Therefore, we are quite uncertain as to the symbols used by Diophantus himself and as to the various interpolations that may have been made by medieval copyists. His collection of problems is of wide variation. Their solutions are often highly ingenious and one should appreciate his power of imagination.

All that we know of Diophantus is that he lived at Alexandria and that most likely he was not a Greek. He may have been a *Hellanized Babylonian*, as it seems probable that some of his algebra was of Babylonian origin, although the connection has yet to be traced. We are ignorant of his parentage, place of nativity and date of his career. The details of life of Diaphanous is reported in a curious problem in the Greek Anthology dating from the fifth century or sixth century described below:
"God granted him to be a boy for the sixth part of his life, and adding a twelfth part to this, He clothed his cheeks with down; He lit him the light of wedlock after a seventh part, and five years after his marriage He granted him a son. Alas! late-born wretched child; after attaining the measure of half his father's life, Chill Fate took him. After consoling his grief by this science of numbers for four years he ended his life ".

(ie) the problem states that his boyhood lasted 1/6 of his life, his beard grew after 1/12 more, after 1/7 more he married, 5 years later his son was born, the son lived to half his father's age, and the father died 4 years after his son. While the statement is obscure at one point, it is generally thought to mean that Diophantus married at 33 and died at 84.

The first Mathematicians to study integral solutions of equations seem to be Brahmagupta (600.A.D), Aryabhata (700 A.D) and Bhaskara (1100 A.D). The modern era in the subject begins with Pierre de Fermat (1601-1665) (generally acknowledged to be the Father of Modern Number Theory). Fermat's interest was aroused by reading Bachet's translation of the surviving books of Diophantus. Fermat solved new problems; posed many challenges to other Mathematicians, invented new methods, and in general was much more advanced than contemporary mathematicians. In
D.E. Smith (1958), in [39] S.G. Telang (1996), [42] Harry N. Wright (1939), and [45] J.W. Young (1923). The subject of Diophantine equations is quite difficult. It is hard to tell if a given equation has solutions or not, and when it does, there may be no method to find all of them. It is difficult to tell which are easily solvable and which require advanced techniques. A Diophantine problem is considered as solved if a method is available to decide whether the problem is solvable or not and, in case of its solvability, to exhibit all integers satisfying the requirements set forth in the problem. A partial solution of Diophantine problem has only a very limited interest. Although the study of indeterminate equation and its solution in integers has had a very important place in the development of Number Theory, there is no well-unified body of knowledge concerning general methods. There are but very few Diophantine problems of a general type in which the complete solution is known and [11] L.E. Dickson (1952) provides an extensive review of the sizable literature. The successful completion of exhibiting all integers satisfying the requirements set forth in the problem add to further progress of Number Theory. It is therefore, towards this end, we wish to develop procedures to find an infinite number of nontrivial integral solutions to a few interesting ternary Diophantine problems of polynomial type. This dissertation consists of six chapters.
While developing procedures to determine nontrivial integer solutions to the ternary Diophantine problems of interest, one naturally encounters a binary quadratic Diophantine equation

\[ Y^2 = D X^2 + N \quad (1.1) \]

where \( D \) is any positive non-square integer and \( N \) is a nonzero integer. In Chapter II of this dissertation, a method is provided in a finite number of steps by developing an idea, not being noticed previously, by [1] William W. Adams (1976), [5] Bibhotibhusan Batta and Avadhesh Narayan Singh (1938), [14] Daniel Shanks (1971), [24] L.J. Mordell (1969), [25] R.A. Mollin (1998), [26] T. Nagell (1964), [37] B. Stolt (1952), [40] J.V. Uspensky (1939) to get a sequence of values of parameter \( N \in \mathbb{Z} - \{0\} \) such that the equation (1.1) has non-trivial solutions. Note that the value of \( N \) is a quadratic residue of every prime, which divides \( D \). A general form exhibiting all integral solutions of equation (1.1) is presented along with the representation of prime \( N \) for a few values of \( D \). Various relationships between variables in the equation are found excluding the possibility of trivial solutions. In addition, some special cases of interest of the equation (1.1) are considered. A part of the above work has been published in [16] M.A. Gopalan & R. Srikant (2001).
The most widely known indeterminate equation of degree 2 in three variables is the ternary quadratic Legendre's equation

\[ AX^2 + BY^2 + CZ^2 = 0 \]  

(1.2)

Given A, B, C to be non zero integers having no prime factors \( \equiv 3 \pmod{4} \), not all of the same sign, ABC square free, equation (1.2) is proved to be solvable in non zero integers \( x, y, z \) iff \(-BC, -CA, -AB\) are quadratic residues of A, B, C respectively in [24] L.J. Mordell (1969), [26] T. Nagell (1964), [27] I. Niven, H. Zuckerman & H. Montgomery (1991), [36] Nigel P. Smart (1999), [43] A. Weil (1983), [44] Kenneth S. William & Richard H. Hudson (1983). Clearly without loss of generality, we may suppose throughout that \( A>0, B>0, C<0 \) in equation (1.2) and for which integral solutions have been obtained by various authors [11] L. E. Dickson (1952), [24] L. J. Mordell (1969), [36] Nigel P. Smart (1999), [43] A. Weil (1983), [44] Kenneth S. William & Richard H. Hudson (1983), using specific values for A, B and C and when A, B, C are pairwise co prime and square free integer which can be seen in [36] Nigel P. Smart (1999). However, quite often, we come across the above type of equation with various values of A, B, C and as such one may require an integral solution of the equation (1.2) in its most general form. Consequently, a new method of approach becomes
essential which indeed has motivated our interest in the present topic. Thus in Chapter (III), a procedure is presented for finding all integral solutions of equation (1.2) which are uniformly valid for all integral values of A, B and C. As immediate application, we determine the positive square root of a complex binomial quadratic surd of the form $(X \sqrt{M} + iY \sqrt{N})$, G.C.D $(M, N) = C$. A part of the work has been published in [17] M.A.Gopalan & R.Srikanth (2002).

A special case of equation (1.2) is the equation of the form

$$AX^2 + BY^3 = Z^2$$

(1.3)

which has been investigated for integral solutions by several authors [11] L.E.Dickson (1952), [43] A.Weil (1983) and most of them were primarily concerned with specific values of A and B only. Although, Lagrange proposed a method of finding integral solutions of the equation (1.3), in its most general form, yet the method suggested is found to be not simple. It is towards this end, we, by employing the analysis similar to that of equation (1.2), obtain the general form of integral solutions of (1.3) in [18] M.A.Gopalan, R.Ganapathy & R.Srikanth (2000).
A ternary Diophantine equation of interest represented by

$$DX^2 + Y^2 = Z^{2n}, \quad n > 0 \quad (1.4)$$

is arrived at while making an attempt to find the positive square root of a complex binomial quadratic surd of the form $(Y + iX\sqrt{D})$. The integral solutions of (1.4) when $n=1,2$ are given in [8] F.L.Carmichael (1916), [11]L.E.Dickson (1952). Thus, in Chapter (IV) we investigate (1.4) for integral solutions for all values of $n \geq 2$ and present a few relations among them. A part of the work has been published in [19] M.A.Gopalan, R.Srikanth, a remark on the square root of a complex binomial quadratic surd.

Another ternary equation of interest which is of degree four is of the form

$$aX^4 + bX^2Y^2 + cY^4 = dZ^2$$

where $a$, $b$, $c$, $d$ are non zero integers and it has a long history going back to Fermat and Euler as can be seen in [7] Robert D.Carmichael (1915), [11] L.E.Dickson (1952), [24] L.J.Mordell (1969), [26]T.Nagell (1964), [41] T.Venkatesh (1995). In order to come upon the more general problem, we
consider in Chapter (V) of this dissertation, the ternary equation of powers (≥ 2) of the form
\[ aX^{2m} - bX^mY^m + cY^{2m} = dZ^2 \] (1.5)
and propose a method in [20] M.A. Gopalan, R. Srikanth & M.G. Sankaranarayanan, to generate a sequence of integral solutions, knowing a solution of the equation (1.5).

In chapter (VI), non ideal non trivial parametric integral solutions of the system of double equations represented by
\[ x_1, x_2, x_3, x_4 = y_1, y_2, y_3, y_4, \]
\[ x_1, x_2, x_3, x_4 = y_1, y_2, y_3, y_4, -y_1, -y_2, -y_3, -y_4 \]
a sequence of integral solutions of the system
\[ \sum_{i=1}^{5} (x_i^j)' = \sum_{j=1}^{5} (y_i^j)' \quad j = 1, 2 \]
has been determined, from which, after performing some algebra, solutions
of the systems

\[ \sum_{i=1}^{8} (x_i^2)' = \sum_{i=1}^{8} (y_i^2)' \]

and

\[ \sum_{i=1}^{10} (x_i^2)' = \sum_{i=1}^{10} (y_i^2)' \], where \( j = 1, 2, 3 \) are deduced.