CHAPTER VI

NON-IDEAL INTEGER SOLUTIONS OF SECOND DEGREE PROUHET TARRY ESCOTT PROBLEM

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Diophantine equations of the form

\[ \sum_{i=1}^{n} a_i = \sum_{j=1}^{n} b_j \]  \hspace{1cm} (6.1)  

specific examples of (6.1) are considered, namely the equations
which have \( s, m (=n) = 4,4; 3,2; 5,4; 4,2; 7,4; 5,3; \) and \( 6,3. \)

Further, the notation

\[
A_1, A_2, ..., A_p^{n_1, n_2, ..., n_p} = B_1, B_2, ..., B_q
\]  

(6.2)

designates a so-called multidegree equality and means that the sum of the
numbers on the left equals the sum of the numbers on the right for each of
the \( r(n_1, n_2, ..., n_p) \) positive integral powers of the numbers. In [58] A.Glodin
(1948), parametric solutions of the two multi-degreed equalities

\[
A_1, A_2, A_3^{1,4} = B_1, B_2, B_3, \quad A_3 \neq A_1 + A_2, \quad B_3 \neq B_1 + B_2
\]

and

\[
C_1, C_2, ..., C_7^{1,2,4,6,8} = D_1, D_2, ..., D_7
\]

are obtained. A special case of (6.2) is the \( (k, s) \) multigrade Diophantine
equation of the form

\[
\sum_{i=1}^{s} x_i^{j} = \sum_{j=1}^{s} y_i^{j} \quad \text{,} \quad (j=1,2, ..., k)
\]  

(6.3)

The above equation, conveniently denoted by the symbol

\[
x_1, x_2, ..., x_s = y_1, y_2, ..., y_s
\]  

(6.4)

In this chapter, a general form of non-ideal non-trivial parametric integral solutions of the system (6.4), with $k = 2$, $s = 4$ distinct from that of [9] J.Choubey (1991), has been obtained. By considering an appropriate choice of a non ideal integral solution of the 5th degree equation of the form

$$x_1, x_2, x_3, x_4, -x_1, -x_2, -x_3, -x_4 = y_1, y_2, y_3, y_4, -y_1, -y_2, -y_3, -y_4$$

a sequence of integral solutions of the system

$$\sum_{i=1}^{4} (x_i^2)' = \sum_{i=1}^{4} (y_i^2)', \quad j = 1, 2$$
has been determined, from which, after performing some algebra, solutions of the systems

\[ \sum_{i=1}^{3} (x_i^2)^j = \sum_{i=1}^{3} (y_i^2)^j \quad \text{and} \]
\[ \sum_{i=1}^{10} (x_i^2)^j = \sum_{i=1}^{10} (y_i^2)^j , \text{where } j = 1,2,3 \text{ are deduced.} \]

**Method of solving** \( x_1, x_2, x_3, x_4 = y_1, y_2, y_3, y_4 \)

Choosing distinct integers \( h, g, \alpha, \) such that

\( x_1 = y_1 - h, \quad x_2 = y_2 + \alpha, \quad x_3 = y_3 - g, \) \( \text{the first relation of} \)

\[ x_1, x_2, x_3, x_4 = y_1, y_2, y_3, y_4, \] \( (6.5) \)

gives

\[ x_4 = y_4 + h - \alpha + g. \]

Let \( h = GB, \quad \alpha = GA, \quad g = GC \) so that \( g.c.d(A, B, C) = 1 \) \( (6.6) \)

Taking \( GA - y_4 = B_2, \quad GC - x_4 = B_2, \) \( \text{the second relation of} \ (6.5) \text{is satisfied,} \)

provided

\[ x_3 = \frac{y_3}{C} A + \frac{B_2}{C} (A - B - C) + \frac{B_2}{C} B. \]

If \( g.c.d(A, C) = g.c.d(B, C) = 1 \) and in view of \( (6.6) \),
the above equation is written as

\[ x_3 = lA + m(A - B - C) + nB \]

wherein \( \frac{y_2}{C} = l, \frac{B_2}{C} = m, \frac{B_3}{C} = n \).

Thus, it follows, after some algebra, that

\[ x_1 = C(G - n) \quad y_1 = C(G - n) + GB \]

\[ x_2 = Cl + GA \quad y_2 = Cl \]

\[ x_3 = Al + (A - B - C)m + Bn \quad y_3 = Al + Bn + (A - B - C)m + GC \]

\[ x_4 = GC + GB - Cm \quad y_4 = GA - Cm \]

exhibiting non ideal, nontrivial, 7 parametric \((A, B, C, l, m, n, G)\) integral solutions of (6.5).

Typical solutions are exhibited below:
Table (6a(i))

<table>
<thead>
<tr>
<th>S.No</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>m</th>
<th>n</th>
<th>G</th>
<th>x₁</th>
<th>x₂</th>
<th>x₃</th>
<th>x₄</th>
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<tbody>
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<td>i</td>
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<td>13</td>
<td>5</td>
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<td>17</td>
<td>5</td>
<td>7</td>
<td>9</td>
<td>19</td>
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<td>275</td>
<td>27</td>
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<td>ix</td>
<td>12</td>
<td>17</td>
<td>19</td>
<td>7</td>
<td>9</td>
<td>11</td>
<td>13</td>
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<td>19</td>
<td>41</td>
<td>726</td>
<td>1167</td>
<td>179</td>
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</table>
It may be noted that through the application of the theorem due to Frolov ([13] H.L.Dorwart & O.E.Brown (1937)) one may obtain equivalent
solutions for each of the solutions presented above. Equivalent typical solutions for each of the above solutions are exhibited below:

Table (6b)

<table>
<thead>
<tr>
<th>S.No</th>
<th>M</th>
<th>K</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
<th>$y_1$</th>
<th>$y_2$</th>
<th>$y_3$</th>
<th>$y_4$</th>
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<td>-5</td>
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<td>5</td>
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<td>-67</td>
<td>17</td>
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<td>93</td>
<td>21</td>
<td>-103</td>
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<td>-59</td>
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<td>865</td>
<td>57</td>
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<td>756</td>
<td>481</td>
<td>726</td>
<td>1121</td>
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<td>xi</td>
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<td>14</td>
<td>1249</td>
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<td>-2820</td>
<td>573</td>
<td>2796</td>
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<td>-1091</td>
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<td>-153</td>
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<td>925</td>
<td>2941</td>
<td>-783</td>
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<tr>
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<td>17</td>
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<td>7553</td>
<td>-479</td>
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<td>6465</td>
<td>4193</td>
<td>6481</td>
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<td>19</td>
<td>1135</td>
<td>12187</td>
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<td>xv</td>
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<td>41098</td>
<td>37963</td>
<td>8171</td>
<td>29128</td>
<td>5264</td>
</tr>
</tbody>
</table>
As a special case, to solve system (6.5) wherein $\sum x_i = \sum y_i = 0$, we write

\[
\begin{align*}
x_1 &= -\alpha_1 + \alpha_2 + \alpha_3 \\
x_2 &= \alpha_1 - \alpha_2 + \alpha_3 \\
x_3 &= \alpha_1 + \alpha_2 - \alpha_3 \\
x_4 &= -\alpha_1 - \alpha_2 - \alpha_3
\end{align*}
\]

\[
\begin{align*}
y_1 &= -\beta_1 + \beta_2 + \beta_3 \\
y_2 &= \beta_1 - \beta_2 + \beta_3 \\
y_3 &= \beta_1 + \beta_2 - \beta_3 \\
y_4 &= -\beta_1 - \beta_2 - \beta_3
\end{align*}
\]

in (6.5), reducing it to the equation

\[
\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = \beta_1^2 + \beta_2^2 + \beta_3^2
\]  

(6.7)

whose general form of integral solutions may be taken as

\[
\begin{align*}
\alpha_1 &= m^2 - n^2, & \alpha_2 &= 2mn, & \alpha_3 &= p^2 + q^2 \\
\beta_1 &= p^2 - q^2, & \beta_2 &= 2pq, & \beta_3 &= m^2 + n^2
\end{align*}
\]

wherein $m$, $n$, $p$ and $q$ are distinct non zero integers and thus, the solutions of (6.5) are given by

\[
\begin{align*}
x_1 &= -m^2 + n^2 + 2mn + p^2 + q^2 \\
x_2 &= m^2 - n^2 - 2mn + p^2 + q^2 \\
x_3 &= m^2 - n^2 + 2mn - p^2 - q^2 \\
x_4 &= -m^2 + n^2 - 2mn - (p^2 + q^2)
\end{align*}
\]

\[
\begin{align*}
y_1 &= -p^2 + q^2 + 2pq + m^2 + n^2 \\
y_2 &= p^2 - q^2 - 2pq + m^2 + n^2 \\
y_3 &= p^2 - q^2 + 2pq -(m^2 + n^2) \\
y_4 &= -p^2 + q^2 - 2pq - (m^2 + n^2)
\end{align*}
\]
Typical solutions are presented below:

\[

d_1 = 89, 33, -57, -65 \\
\begin{array}{c}
89 \\
33 \\
-57 \\
-65
\end{array}
\]

\[

d_2 = 91, -51, 29, -69
\]

\[

d_1 = 128, 72, -96, -104 \\
\begin{array}{c}
128 \\
72 \\
-96 \\
-104
\end{array}
\]

\[

d_2 = 144, -104, 48, -88
\]

\[

d_1 = 88, 60, -76, -72 \\
\begin{array}{c}
88 \\
60 \\
-76 \\
-72
\end{array}
\]

\[

d_2 = 104, -84, 36, -56
\]

\[

d_1 = 68, 54, -44, -78 \\
\begin{array}{c}
68 \\
54 \\
-44 \\
-78
\end{array}
\]

\[

d_2 = 84, -58, 36, -62
\]

\[

d_1 = 181, 147, -157, -171 \\
\begin{array}{c}
181 \\
147 \\
-157 \\
-171
\end{array}
\]

\[

d_2 = 209, -183, 111, -137
\]

\[

d_1 = 288, 162, -216, -234 \\
\begin{array}{c}
288 \\
162 \\
-216 \\
-234
\end{array}
\]

\[

d_2 = 324, -234, 108, -198
\]

\[

d_1 = 783, 491, -603, -671 \\
\begin{array}{c}
783 \\
491 \\
-603 \\
-671
\end{array}
\]

\[

d_2 = 939, -727, 237, -449
\]

\[

d_1 = 310, 126, -246, -190 \\
\begin{array}{c}
310 \\
126 \\
-246 \\
-190
\end{array}
\]

\[

d_2 = 370, -234, -6, -130
\]

\[

d_1 = 187, 153, -163, -177 \\
\begin{array}{c}
187 \\
153 \\
-163 \\
-177
\end{array}
\]

\[

d_2 = 239, -213, 69, -95
\]

\[

d_1 = 561, 355, -465, -451 \\
\begin{array}{c}
561 \\
355 \\
-465 \\
-451
\end{array}
\]

\[

d_2 = 635, -489, 249, -395
\]

\[

d_1 = 843, 185, -323, -705 \\
\begin{array}{c}
843 \\
185 \\
-323 \\
-705
\end{array}
\]

\[

d_2 = 843, -305, 177, -715
\]

\[

d_1 = 1021, 139, -301, -859 \\
\begin{array}{c}
1021 \\
139 \\
-301 \\
-859
\end{array}
\]

\[

d_2 = 1013, -275, 139, -877
\]

\[

d_1 = 1268, 262, -436, -1094 \\
\begin{array}{c}
1268 \\
262 \\
-436 \\
-1094
\end{array}
\]

\[

d_2 = 1298, -448, 214, -1064
\]
Alternatively, the general solution of (6.7) may be written as

\[ \alpha_1 = pq, \quad \alpha_2 = rs, \quad \alpha_3 = ps + qr \]
\[ \beta_1 = pq + rs, \quad \beta_2 = ps, \quad \beta_3 = rq \]

where \( p, q, r, s \) are arbitrary non zero integers and thus we now get the following parametric solutions of system (6.5)
\[
x_1 = -pq + ps + rs + rq \\
x_2 = pq + ps + rq - rs \\
x_3 = qp - qr - sp + sr \\
x_4 = -qp - qr - sp - sr
\]

\[
y_1 = qr - qp - sr + sp \\
y_2 = pq - ps + rs + rq \\
y_3 = pq + ps + rs - rq \\
y_4 = -qp - qr - ps - sr
\]

Typical examples are presented below:

\[
\begin{align*}
54, & \quad -10, \quad 16, \quad -60 \quad \overset{2}{=} \quad -16, \quad 46, \quad 30, \quad -60 \\
52, & \quad 4, \quad 8, \quad -64 \quad \overset{2}{=} \quad -8, \quad 28, \quad 44, \quad -64 \\
20, & \quad 0, \quad 4, \quad -24 \quad \overset{2}{=} \quad -4, \quad 16, \quad 12, \quad -24 \\
80, & \quad 0, \quad 16, \quad -96 \quad \overset{2}{=} \quad -16, \quad 64, \quad 48, \quad -96 \\
118, & \quad -30, \quad 42, \quad -130 \quad \overset{2}{=} \quad -42, \quad 90, \quad 82, \quad -130 \\
180, & \quad 0, \quad 36, \quad -216 \quad \overset{2}{=} \quad -36, \quad 144, \quad 108, \quad -216 \\
500, & \quad 0, \quad 100, \quad -600 \quad \overset{2}{=} \quad -100, \quad 400, \quad 300, \quad -600 \\
157, & \quad 7, \quad 25, \quad -189 \quad \overset{2}{=} \quad -25, \quad 137, \quad 77, \quad -189 \\
114, & \quad -28, \quad 40, \quad -126 \quad \overset{2}{=} \quad -40, \quad 82, \quad 84, \quad -126 \\
352, & \quad -42, \quad 90, \quad -400 \quad \overset{2}{=} \quad -90, \quad 298, \quad 192, \quad -400
\end{align*}
\]
Method of solving $x_1^2, x_2^2, x_3^2, x_4^2 = y_1^2, y_2^2, y_3^2, y_4^2$

To determine the solution set of system (6.5), wherein each member of the set is a perfect square, conjecture a solution of the 5th degree of the form

$$x_1, x_2, x_3, x_4, -x_1, -x_2, -x_3, -x_4 = y_1, y_2, y_3, y_4, -y_1, -y_2, -y_3, -y_4 \quad (6.8)$$
Assume a solution of (6.8) given by

\[ (x_i) = (a, a+k, a+2k, b) \]
\[ (y_i) = (k, 2k, 3k, b+2r) \]  \hspace{1cm} (6.9)

This is equivalent to the system

(i) \[ a^2 + (a+k)^2 + (a+2k)^2 + (b)^2 = k^2 + (2k)^2 + (3k)^2 + (b+2r)^2 \]

(ii) \[ a^4 + (a+k)^4 + (a+2k)^4 + (b)^4 = k^4 + (2k)^4 + (3k)^4 + (b+2r)^4 \]  \hspace{1cm} (6.10)

In consequence of (6.10(i)), we may write

\[ a^2 + 2ak - 3k^2 = \frac{4}{3} rM, \]

where \( b = M-r \).  \hspace{1cm} (6.11)

Solving (6.11) as a quadratic in ‘\( a \)’, we get

\[ a + k = 2 \sqrt{k^2 + M \left( \frac{r}{3} \right)} \]  \hspace{1cm} (6.12)

Then (6.10(ii)) reduces to

\[ 3M^2 - 2Mr + 3r^2 = 18k^2, \]  \hspace{1cm} (6.13)

a solution of which is

\[ M = 36mn + 3m^2 - 54n^2 \]
\[ r = 36mn - 3m^2 + 54n^2 \]
\[ k = 2m^2 + 36n^2 \]  \hspace{1cm} (6.14)
Using (6.11) and (6.12), we get

\[ a = - \left(2m^2 + 36n^2\right) \pm 2\sqrt{648m^2n^2 + (m^2 + 18n^2)^2} \]

\[ b = 6(m^2 - 18n^2) \]

Since our interest centers on finding integral solutions of system (6.8), choose \( m \) and \( n \) so that the value of \( 'a' \) is an integer. Thus, knowing the integer values of \( a, k, b \) and \( r \), the solution set of the system

\[ x_1^2, x_2^2, x_3^2, x_4^2 = y_1^2, y_2^2, y_3^2, y_4^2 \] (6.15)

readily follows. For example, we present below two different choices of \( m \) and \( n \) so that the values of \( 'a' \) are integers and the corresponding solutions of equation (6.15).

**Table (6c(i))**

<table>
<thead>
<tr>
<th>m</th>
<th>n</th>
<th>( x_1 )</th>
<th>( x_2 )</th>
<th>( x_3 )</th>
<th>( x_4 )</th>
</tr>
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<tbody>
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<td>( 32\alpha^2 )</td>
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<td>( 324\alpha^2 )</td>
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</tr>
<tr>
<td>( 3\alpha + 3 )</td>
<td>( \alpha + 1 )</td>
<td>( 108(\alpha + 1)^2 )</td>
<td>( 162(\alpha + 1)^2 )</td>
<td>( 216(\alpha + 1)^2 )</td>
<td>(-54(\alpha + 1)^2 )</td>
</tr>
<tr>
<td>( 12\alpha )</td>
<td>( 2\alpha )</td>
<td>( 864\alpha^2 )</td>
<td>( 1296\alpha^2 )</td>
<td>( 1728\alpha^2 )</td>
<td>( 432\alpha^2 )</td>
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<td>( 576\alpha^2 )</td>
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</tr>
</tbody>
</table>
\[ \begin{array}{cccccc}
\text{m} & \text{n} & y_1 & y_2 & y_3 & y_4 \\
\alpha & 2\alpha & 146\alpha^2 & 292\alpha^2 & 438\alpha^2 & 144\alpha^2 \\
3\alpha+3 & \alpha+1 & 54(\alpha+1)^2 & 108(\alpha+1)^2 & 162(\alpha+1)^2 & 216(\alpha+1)^2 \\
12\alpha & 2\alpha & 432\alpha^2 & 864\alpha^2 & 1296\alpha^2 & 1728\alpha^2 \\
36\alpha & \alpha & 2628\alpha^2 & 5256\alpha^2 & 7884\alpha^2 & 2592\alpha^2 \\
\end{array} \]

**Deductions:**

Observe that the system (6.15) leads to the equation

\[
\sum_{i,j=1}^{4} x_i^2 x_j^2 = \sum_{i,j=1}^{4} y_i^2 y_j^2
\]

(6.16)

Equations (6.15) and (6.16) are equivalent to

\[
x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_1^2 + x_2^2 + x_3^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2 + \frac{2}{2} y_1^2 + y_2^2 + y_3^2 + y_4^2 + y_5^2 + y_6^2 + y_7^2 + y_8^2 + y_9^2 + y_{10}^2 + \cdots
\]

\[
y_1^2 + y_2^2 + y_3^2 + y_4^2 + y_5^2 + y_6^2 + y_7^2 + y_8^2 + y_9^2 + y_{10}^2
\]

Now, it is seen that

\[
\begin{align*}
(x_1^2 + x_2^2)^6 + (x_3^2 + x_4^2)^6 + (x_5^2 + x_6^2)^6 + (x_7^2 + x_8^2)^6 + (x_9^2 + x_10^2)^6 \\
= 3(x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2 + x_7^2 + x_8^2 + x_9^2 + x_{10}^2)
\end{align*}
\]
In a similar manner, we observe that

\[
(y_1^2 + y_2^2)^3 + (y_3^2 + y_4^2)^3 + (y_5^2 + y_6^2)^3 + (y_7^2 + y_8^2)^3 + (y_9^2 + y_{10}^2)^3 = 3(y_1^2 + y_2^2 + y_3^2 + y_4^2)
\]

In view of (6.15), we have

\[
(x_1^2 + x_2^2)^3 + (x_3^2 + x_4^2)^3 + (x_5^2 + x_6^2)^3 + (x_7^2 + x_8^2)^3 + (x_9^2 + x_{10}^2)^3 = (y_1^2 + y_2^2)^3 + (y_3^2 + y_4^2)^3
\]

\[+ (y_5^2 + y_6^2)^3 + (y_7^2 + y_8^2)^3 + (y_9^2 + y_{10}^2)^3 \]

Thus, we arrive at the following (3,10) multigrade Diophantine equation

\[
a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10} = b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8, b_9, b_{10}
\]

where

\[
a = (x_1^2 + x_2^2, x_3^2 + x_4^2, x_5^2 + x_6^2, x_7^2 + x_8^2, x_9^2 + x_{10}^2, x_1^2 + x_3^2, x_5^2 + x_7^2, x_9^2)
\]

\[
b = (y_1^2 + y_2^2, y_3^2 + y_4^2, y_5^2 + y_6^2, y_7^2 + y_8^2, y_9^2 + y_{10}^2, y_1^2 + y_3^2, y_5^2 + y_7^2, y_9^2)
\]

A few examples are presented in the following Table (6d).
### Table (6d(i))

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<tr>
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### Table (6d(ii))

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<th>(a_9)</th>
<th>(a_{10})</th>
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<td>2916 ((\alpha + 1)^4)</td>
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### Table (6d(iii))

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### Table (6d(iv))

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<td>62157456$\alpha^4$</td>
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</tr>
</tbody>
</table>
Also (6.15) and (6.16) are equivalent to

\[ \frac{-x_i^2 + x_j^2 + x_k^2 + x_l^2}{2}, \frac{x_i^2 - x_j^2 + x_k^2 + x_l^2}{2}, \frac{x_i^2 + x_j^2 - x_k^2 + x_l^2}{2}, \frac{x_i^2 + x_j^2 + x_k^2 - x_l^2}{2}, x_i^2, x_j^2, x_k^2, x_l^2 = \]

\[ \frac{-y_i^2 + y_j^2 + y_k^2 + y_l^2}{2}, \frac{y_i^2 - y_j^2 + y_k^2 + y_l^2}{2}, \frac{y_i^2 + y_j^2 - y_k^2 + y_l^2}{2}, \frac{y_i^2 + y_j^2 + y_k^2 - y_l^2}{2}, y_i^2, y_j^2, y_k^2, y_l^2 \]

Now, it is seen that

\[ \left( \frac{-x_i^2 + x_j^2 + x_k^2 + x_l^2}{2} \right)^3 + \left( \frac{x_i^2 - x_j^2 + x_k^2 + x_l^2}{2} \right)^3 + \left( \frac{x_i^2 + x_j^2 - x_k^2 + x_l^2}{2} \right)^3 + \left( \frac{x_i^2 + x_j^2 + x_k^2 - x_l^2}{2} \right)^3 + \]

\[ (x_i^2 + x_j^2 + x_k^2 + x_l^2) = -3(x_i^2 + x_j^2 + x_k^2 + x_l^2)(x_i^2 + x_j^2 + x_k^2 + x_l^2)(x_i^2 + x_j^2 + x_k^2 + x_l^2)(x_i^2 + x_j^2 + x_k^2 + x_l^2) + \]

\[ (x_i^2 + x_j^2 + x_k^2 + x_l^2)^3 + 2 \left( \frac{x_i^2 + x_j^2 + x_k^2 + x_l^2}{2} \right)^3 \]

In a similar manner we observe that

\[ \left( \frac{-y_i^2 + y_j^2 + y_k^2 + y_l^2}{2} \right)^3 + \left( \frac{y_i^2 - y_j^2 + y_k^2 + y_l^2}{2} \right)^3 + \left( \frac{y_i^2 + y_j^2 - y_k^2 + y_l^2}{2} \right)^3 + \left( \frac{y_i^2 + y_j^2 + y_k^2 - y_l^2}{2} \right)^3 + \]

\[ (y_i^2 + y_j^2 + y_k^2 + y_l^2) = -3(y_i^2 + y_j^2 + y_k^2 + y_l^2)(y_i^2 + y_j^2 + y_k^2 + y_l^2)(y_i^2 + y_j^2 + y_k^2 + y_l^2)(y_i^2 + y_j^2 + y_k^2 + y_l^2) \]

\[ + (y_i^2 + y_j^2 + y_k^2 + y_l^2)^3 + 2 \left( \frac{y_i^2 + y_j^2 + y_k^2 + y_l^2}{2} \right)^3 \]
In view of (6.15), we have

\[
\left( -\frac{x_1^2 + x_2^2 + x_3^2 + x_4^2}{2} \right)^3 + \left( -\frac{x_1^2 - x_2^2 + x_3^2 + x_4^2}{2} \right)^3 + \left( -\frac{x_1^2 + x_2^2 - x_3^2 + x_4^2}{2} \right)^3 = \\
\left( -\frac{y_1^2 + y_2^2 + y_3^2 + y_4^2}{2} \right)^3 + \left( -\frac{y_1^2 - y_2^2 + y_3^2 + y_4^2}{2} \right)^3 + \left( -\frac{y_1^2 + y_2^2 - y_3^2 + y_4^2}{2} \right)^3
\]

Thus, we arrive at the following (3,8) multigrade Diophantine equation

\[
a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8 = b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8
\]

where

\[
a_i = \left( -\frac{x_1^2 + x_2^2 + x_3^2 + x_4^2}{2}, -\frac{x_1^2 - x_2^2 + x_3^2 + x_4^2}{2}, -\frac{x_1^2 + x_2^2 - x_3^2 + x_4^2}{2}, \frac{x_1^2 + x_2^2 + x_3^2 - x_4^2}{2}, x_1, x_2, x_3, x_4 \right)
\]

\[
b_i = \left( -\frac{y_1^2 + y_2^2 + y_3^2 + y_4^2}{2}, -\frac{y_1^2 - y_2^2 + y_3^2 + y_4^2}{2}, -\frac{y_1^2 + y_2^2 - y_3^2 + y_4^2}{2}, \frac{y_1^2 + y_2^2 + y_3^2 - y_4^2}{2}, y_1, y_2, y_3, y_4 \right)
\]
\textbf{Table (6e(i))}

<table>
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<th>m</th>
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### Table 6e(iii)

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### Table 6e(iv)

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</tr>
<tr>
<td>$36\alpha$</td>
<td>$\alpha$</td>
<td>$6906384\alpha^4$</td>
<td>$27625536\alpha^4$</td>
<td>$62157456\alpha^4$</td>
<td>$6718464\alpha^3$</td>
</tr>
</tbody>
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