Chapter 4

Breaking of longitudinal Akhiezer-Polovin waves

It is well known that breaking amplitude of longitudinal Akhiezer-Polovin (AP) waves approaches to infinity when their phase velocity is close to speed of light [119]. However, Infeld and Rowlands [107] have shown that relativistic plasma oscillations break at arbitrarily small amplitude as frequency acquires a spatial dependence for almost all initial conditions. In order to show a connection between both the theories, we first obtain the initial conditions which excite traveling AP waves, once substituted in the exact solution of Infeld and Rowlands [107]. Later, we demonstrate using the 1-D simulation based on Dawson sheet model, that AP waves break at arbitrarily small amplitude through the process of phase mixing when subjected to very small perturbation. Results from the simulation show a good agreement with the Dawson phase mixing formula for inhomogeneous plasma. This result may be of direct relevance to the laser/beam plasma wakefield experiments.

4.1 Introduction

The problem of propagation of relativistically intense nonlinear plasma waves traveling close to the speed of light has been a problem of great interest from the viewpoint of plasma methods which may be used for accelerating particles to very high energies. An exact solution for relativistic traveling waves in cold plasma was
first reported by Akhiezer and Polovin [119]. It is to be noted here that Akhiezer and Polovin demonstrated the existence of these waves without worrying about how they can be excited in real plasma. Later it has been shown analytically [1] that when an ultra-short, ultra-intense laser pulse propagates through underdense plasma, the waves which get excited in the wake of the laser pulse are nothing but AP waves. One of the important properties of these waves is that their frequency depends on their amplitude in such a way that larger the amplitude, smaller the frequency will be. This happens due to increase in the mass of the electrons because of relativistic effects. Second important property of these waves is that their breaking amplitude is very high and can be expressed as $eE_{wb}/(m\omega_{pe}c) = \sqrt{2(\gamma_{ph} - 1)}^{1/2}$, here $\gamma_{ph} = 1/\sqrt{1 - v_{ph}^2/c^2}$ is the relativistic factor associated with the phase velocity $v_{ph}$ of the AP waves. Note here that as $v_{ph} \to c$, $\gamma_{ph} \to \infty$, this implies $E_{wb} \to \infty$. In other words we can say that for highly relativistic plasma waves breaking amplitude becomes too high to break. Breaking property of the relativistic plasma oscillations has also been studied by Infeld and Rowlands by obtaining an exact space and time dependent solution for the relativistic fluid equations in Lagrange coordinates. In contrast to the wave breaking criteria as suggested by Akhiezer and Polovin [119], the authors [107] have shown that their solution shows an explosive behavior (wave breaking) for almost all initial conditions. The authors have shown that relativistic effects bring position dependence in the plasma frequency, as a result plasma oscillations phase mix away and break at arbitrarily small amplitude. On the other hand, the authors accepted that AP waves are very special case of their solution as these waves do not show explosive behavior and need a special set of initial conditions to set them up. However, the authors have not shown how their solution leads to class of traveling waves (do not show explosive behavior), i.e., what initial conditions should be chosen in their solution so as to get AP waves. Elucidation of initial conditions leading to AP traveling waves might show the connection between the theories of Akhiezer & Polovin [119] and Infeld & Rowlands [107]. Besides one may need to worry about sensitivity to initial conditions because the manner AP waves are excited in the plasma may introduce some noise (due to group velocity dispersion of the pulse, thermal effects etc) along with the AP waves.

Therefore, in this chapter we first show what initial conditions we should choose in the solution of Infeld and Rowlands such that we get AP wave solution. We
next perform relativistic sheet simulation [108] in order to study the sensitivity of large amplitude AP waves due to small perturbations. We find that AP waves are very sensitive and their breaking criterion as given in ref.[119] does not really hold in the presence of perturbations. Physically, it happens due to phase mixing effect [108] as frequency of the system which is constant for pure AP waves, acquires a position dependence in the presence of perturbations.

In the section.(4.2) we obtain traveling AP wave solution from space time dependent solution of Infeld and Rowlands [107]. Section.(4.5) contains an alternative derivation of AP wave solution. In section.(4.3) we present results from the simulation. In section.(4.4) we show a good match between analytical and numerical scaling of phase mixing time. Finally in section(4.5) we summarize all the results.

4.2 Relativistic fluid equations and Lagrange solution

The basic equations describing the evolution of an arbitrary electrostatic perturbation in an unmagnetized cold homogeneous plasma with immobile ions are

\[
\left( \frac{\partial}{\partial t} + v \frac{\partial}{\partial x} \right) p = -eE \tag{4.1}
\]

\[
\frac{\partial n}{\partial t} + \frac{\partial (nv)}{\partial x} = 0 \tag{4.2}
\]

\[
\frac{\partial E}{\partial x} = 4\pi e(n_0 - n) \tag{4.3}
\]

\[
\frac{\partial E}{\partial t} = 4\pi env \tag{4.4}
\]

where \( p = \gamma mv \) is momentum and \( \gamma = 1/\sqrt{1-v^2/c^2} \) is relativistic factor, \( n_0 \) is the background ion density and other symbols have their usual meaning.

We now introduce Lagrange coordinates \((x_{eq},\tau)\) which are related to Euler coordinates as

\[
x = x_{eq} + \xi(x_{eq},\tau), \quad t = \tau \tag{4.5}
\]

where \( \xi \) is the the displacement from the equilibrium position \( x_{eq} \) of the electron.
fluid (sheet). Using Eq.(4.5), set of Eqs.(4.1)-(4.4) again can be combined as

$$\frac{d^2 p}{d\tau^2} + \omega_{pe}^2 \left[1 + p^2/(m^2 c^2)\right]^{1/2} = 0 \quad (4.6)$$

Integrating once Eq.(4.6) we get

$$\frac{dp}{d\tau} = \pm \sqrt{2} m \omega_{pe} c \left[ a(x_{eq}) - \left[1 + p^2/(m^2 c^2)\right]^{1/2} \right]^{1/2} \quad (4.7)$$

Here ‘a’ is the first integration constant which is a function of position ‘$x_{eq}$’. Let us again substitute

$$a - \left[1 + p^2/(m^2 c^2)\right]^{1/2} = (a - 1) \sin^2 \alpha \quad (4.8)$$

Solution of Eq.(4.7) can be expressed as

$$\omega_{pe} \tau = \sqrt{2(a+1)} E(\alpha, \kappa) - \sqrt{\frac{2}{a+1}} F(\alpha, \kappa) + \Phi(x_{eq})$$

$$\omega_{pe} \tau = \frac{2}{\kappa'} E(\alpha, \kappa) - \kappa F(\alpha, \kappa) + \Phi(x_{eq}) \quad (4.9)$$

Here $\Phi$ is the second integration constant which is also a function of $x_{eq}$ and

$$\kappa = \left[\frac{a - 1}{a + 1}\right]^{1/2}, \quad \kappa' = \sqrt{1 - \kappa^2} \quad (4.10)$$

Thus set of Eqs.(4.8)-(4.10) together with Eq.(4.5) gives the space-time evolution of relativistic Langmuir waves initiated by an arbitrary perturbation. This exact solution was first obtained by Infeld and Rowlands [107]. The frequency of the wave is obtained by integrating equation (4.9) over “$\alpha$” from 0 to $\pi/2$, as

$$\omega = \omega_{pe} \frac{\pi}{2} \frac{\kappa'}{2E(\kappa) - \kappa' K(\kappa)} \quad (4.11)$$

Note here that the equations (4.10)-(4.11) together with equation (4.5) give an exact dependence of frequency on the initial spatial position of sheets. We are now going to construct plane wave solution from this non-trivial space-time dependent
solution. We know that
\[ p = \gamma mv = \frac{\dot{\xi}}{\sqrt{1 - \dot{\xi}^2/c^2}} \]  
(4.12)

From Eqs.(4.8) and (4.12) we can easily get an expression for \( \dot{\xi} \) and \( \xi \) as
\[ \dot{\xi} = c \frac{(2\kappa/c^2) \cos \alpha [1 - \kappa^2 \sin^2 \alpha]^{1/2}}{[1 + 2(\kappa^2/c^2) \cos^2 \alpha]} \]  
(4.13)
\[ \xi = c \frac{2\kappa}{\omega_{pe} \kappa'} \sin \alpha \]  
(4.14)

Now subtract \( \omega_{pe} x/\beta \) on both sides of Eq.(4.9) and using Eqs.(4.5),(4.14) we get
\[ \omega_{pe} (t - x/\beta) = \frac{2}{\kappa'} E(\alpha, \kappa) - \kappa' F(\alpha, \kappa) \]
\[ -\omega_{pe} \frac{x_{eq}}{\beta} - \frac{c}{\beta} \frac{2\kappa}{\kappa'} \sin \alpha + \Phi(x_{eq}) \]  
(4.15)

Now we first choose ‘\( a \)’ to be independent of ‘\( x_{eq} \)’ and \( \Phi(x_{eq}) \) as follows
\[ \Phi(x_{eq}) = \omega_{pe} \frac{x_{eq}}{\beta} \]  
(4.16)

then Eq.(4.15) becomes
\[ \omega_{pe} (t - x/\beta) = \frac{2}{\kappa'} E(\alpha, \kappa) - \kappa' F(\alpha, \kappa) \]
\[ -\frac{c}{\beta} \frac{2\kappa}{\kappa'} \sin \alpha \]  
(4.17)

Thus we have obtained longitudinal plane wave solution from non-trivial space-time dependent solution with special choice of integration constants. One can note from Eq.(4.9) that, the frequency of the system will have a position dependence if “\( a \)” is a function of position and the solution will show an explosive behavior according to Infeld et al. [107]. However, we know that AP waves do not show explosive behavior, therefore “\( a \)” must be independent of ‘\( x_{eq} \)’ to excite AP wave. Moreover to make ‘\( a \)’ (or momentum) a function of (\( t - x/\beta \)) alone restricts us to choose \( \Phi(x_{eq}) \) as \( \omega_{pe} \frac{x_{eq}}{\beta} \). Thus, assuming ‘\( a \)’ to be independent of ‘\( x_{eq} \)’, Eqs.(4.9) and (4.16) together with Eqs.(4.13) and (4.14) give the velocity and displacement.
profiles of particles in the transcendental form which can be loaded easily in the relativistic PIC or sheet code to excite class of AP waves. On the other hand if initial conditions are not perfectly loaded but have some perturbations on them, the frequency may become a function of $x_{eq}$ giving the possibility of bursty solutions and phase mixing effect. This is what we need to explore next.

4.3 Results from the simulation

In this section we perform relativistic sheet simulation based on Dawson sheet model in order to study the sensitivity of large amplitude AP waves to small amplitude longitudinal perturbations. For this purpose, we have used a relativistic sheet simulation code [108] where we solve the relativistic equation of motion for $\sim 10000$ sheets, using fourth-order Runge-Kutta scheme for a specific choice of initial conditions (pure AP waves and AP waves with perturbations). Ordering of the sheets for sheet crossing is checked at each time step. Phase mixing/wave breaking time is measured as the time taken by any two of the adjacent sheets to cross over.

We first load AP type initial conditions (as discussed earlier in this chapter) in the relativistic sheet code and have seen smooth traveling structures in all physical variables (quantities) up to 1000’s of plasma periods. Then we add a very small amplitude perturbation to the nonlinear AP wave and find that the structure breaks at a time decided by the $u_m$ and $\delta$ (where $u_m$ and $\delta$ are respectively the amplitude of the AP wave and the perturbation). In this way we show that AP waves are very sensitive to longitudinal perturbations and the wave breaking criterion does not hold in the presence of perturbations. In all the simulation runs we keep the phase velocity of AP waves close to speed of light i.e. $v_{ph} \sim 0.9995c$. Moreover, time is normalized to $\omega_{pe}^{-1}$ and distances are normalized to $c\omega_{pe}^{-1}$. Figure (4.1) shows the space-time evolution for the density profile of AP wave with maximum velocity amplitude $u_m \sim 0.81$. Thus we see that there is no numerical dissipation in our code and relativistic shift in the frequency is clearly visible. Now we add a sine wave (very small amplitude AP wave) to this large amplitude AP wave with wavelength same as the large amplitude AP wave and maximum velocity amplitude $\delta = 0.001$. Figure(4.2) shows the space and time evolution of the density
Figure 4.1: Space time evolution of electron density for pure AP wave of maximum velocity amplitude ($u_m = 0.81$) up to 1000’s of plasma periods.

The profile of the resultant structure up to the breaking point. As time progresses, the density peak becomes more and more spiky as energy is going irreversibly into the higher harmonics (a signature of phase mixing leading to wave breaking [108]) and the time at which neighboring sheets cross (wave breaking point) density burst can be seen.

Figure (4.3) shows the Fourier spectrum of pure AP wave and AP wave at the time of breaking. It is clear from the figure that a significant amount of energy has gone to the higher harmonics which is another signature of wave breaking. Thus we see that though breaking amplitude of AP wave for our choice of parameters is very high i.e., $eE_{wb}/(m\omega_{pe}c) \sim 7.8$, it breaks at a lower amplitude when perturbed slightly. Figure (4.4) contains the space-time evolution of electric field which clearly shows that maximum amplitude of the electric field is much less than 7.8 even at the time of breaking. In order to get a dependence of phase mixing time on the amplitude of the perturbation we repeat the numerical experiment such that maximum velocity amplitude of AP wave is kept fixed at $u_m = 0.55$ and amplitude of the perturbation $\delta$ is varied. In figure (4.5) points (‘*’) represent the results from the simulation which clearly indicate that as amplitude of the perturbation is...
Figure 4.2: Space time evolution of electron density for AP wave of maximum velocity amplitude ($u_m = 0.81$) with perturbation amplitude $\delta = 0.001$.

Figure 4.3: Fourier spectrum of pure AP wave ($u_m = 0.81$) and AP wave at the time of breaking due to perturbations ($\delta = 0.001$). Here “$k_L$” is the lowest wave number.

increased, phase mixing time of AP wave decreases. We perform another numerical
Figure 4.4: Space time evolution of electric field for AP wave of maximum velocity amplitude ($u_m = 0.81$) with perturbation amplitude $\delta = 0.001$.

Figure 4.5: Theoretical (‘o’) and numerical (‘*’) scaling of phase mixing time for a finite amplitude AP wave ($u_m = 0.55$) as a function of perturbation amplitudes $\delta$.

eperiment where amplitude of the perturbation is kept fixed at $\delta = 0.01$ and amplitude of the AP wave is varied. This case is presented in figure(4.6) by points
which shows that for a finite longitudinal perturbation, smaller the amplitude of AP wave, longer is the phase mixing time.

Figure 4.6: Theoretical (‘-o’) and numerical (‘*’) scaling of phase mixing time as function of amplitude of AP waves ($u_m$) in the presence of finite perturbation amplitude $\delta = 0.01$.

We know that relativistic plasma waves cannot accelerate particles indefinitely, but give us maximum acceleration only up to dephasing length or dephasing time. If phase mixing time is longer than dephasing time, phase mixing would not affect the acceleration process significantly. However, if the phase mixing time is shorter, maximum acceleration cannot be achieved as the wave gets damped before reaching the dephasing time because of phase mixing leading to breaking. Note here that numerical scaling in figure(4.5) gives a clue that we have to reduce the noise in the particle acceleration experiments in order to get maximum acceleration and the scaling presented in figure(4.6) gives an indication that if one cannot reduce the noise below a threshold, one must use smaller amplitude AP waves in order to avoid phase mixing effect and to gain maximum energy.
4.4 Match between theory and simulation

Along with the choice of initial conditions as made above, equations (4.13) and (4.14) together with equation (4.9) at \( t = 0 \) respectively give velocity and displacement profiles of sheets which are loaded in the code so as to excite a AP wave. We note here that with this choice of “\( a \)” (equation (4.11)) becomes independent of position and therefore no phase mixing occurs as shown in figure(4.1).

In figure(4.7) we plot frequency of the system as a function of position for both pure AP wave and AP wave with perturbations. From this figure we clearly see that for pure AP wave frequency shows a flat dependence on position, i.e., each sheet oscillates with the same frequency and hence no phase mixing occurs. However, for nonzero ‘\( \delta \)’, frequency of the system acquires a position dependence which gets stronger for larger value of ‘\( \delta \)’. Since frequency here becomes function of position, phase mixing happens [102, 103, 106, 108] which is responsible for the breaking of AP waves at arbitrary amplitude.

![Figure 4.7](image)

Figure 4.7: Frequency of the system as a function of position for fixed AP wave (\( u_m = 0.81 \)) along with various perturbation amplitudes \( \delta \).

It is well known that plasma oscillations/waves phase mix away when the plasma frequency for some physical reason acquires a spatial dependence. In our
case also frequency acquires a position dependence in the presence of perturbations. We therefore expect that scaling of phase mixing can be interpreted from Dawson’s formula \cite{102} for phase mixing in inhomogeneous plasma, which is

$$\omega_{pe}\tau_{mix} \sim \frac{1}{\frac{d\omega}{dx_{eq}} \xi}$$

From equation (4.11) we numerically evaluate $d\omega/dx_{eq}$ and use Dawson’s formula \cite{102} to get theoretical scaling of phase mixing time. In figures (4.5) and (4.6) solid lines represent the theoretical scaling of phase mixing time with “$\delta$” and “$u_m$” which clearly show a good match between numerical experiments and theory.

4.5 Summary

In summary, we have first obtained the initial conditions to excite AP waves from the exact space time dependent solution of Infeld and Rowlands. Later, in order to validate our code we have used these initial conditions in the relativistic sheet code to show their propagation upto thousands of plasma periods in all physical variables. We have further added a small perturbation (sine wave) to the larger amplitude AP wave and found that AP wave show an explosive behavior (wave breaking) after finite time. In order to get a dependence of wave breaking time on the amplitude of the perturbation and amplitude of the AP wave, we have repeated the numerical experiment and found two results which are as follows. For a finite amplitude AP wave, larger the amplitude of the perturbation, smaller the wave breaking time is. Thus one has to reduce noise in the experiment in order to get maximum acceleration. Also for finite perturbation amplitude, smaller the amplitude of AP wave, larger the wave breaking time is. Thus one needs to work at smaller amplitude AP waves to gain maximum energy if the noise can not be reduced below a threshold. In order to gain an insight into the physics behind these results we have plotted plasma frequency with respect to the position and found that for pure AP wave, frequencies of the sheets shows a flat dependence on the position. On the other hand, if we add a small perturbation to the AP wave, frequency of the system acquires a spatial dependence which is responsible for phase mixing leading to wave breaking. We have also shown that scaling of

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phase mixing for both the results discussed above can be interpreted from Dawson’s formula [102]. Thus we have shown that although breaking amplitude of AP waves is very high, they break at arbitrary amplitude via the process of phase mixing when perturbed slightly.

Therefore all those experiments/simulation which use AP wave breaking formula may require revisiting. For example, in a recent particle acceleration experiment [28] a maximum gain in the energy up to 200 MeV was observed. The authors used the the old formula for energy gain [2] (which is valid as long as \( eE/(m\omega_{pe}c) \leq 1 \)) to interpret their observation. However, in their case \( eE/(m\omega_{pe}c) \) was approximately 3.8 which is much greater than unity and therefore one has to use the energy gain expression for nonlinear waves [27]. If we do so, energy gain would have been approximately 975 MeV. This much energy can be obtained if the plasma wave accelerates particles upto the full dephasing length. However, from the experimental observation [28] it seems that wave is not able to travel up to the dephasing time. We believe that it may be the phase mixing effect due to noise in the system which is preventing electrons to gain energy greater than 200 MeV in the above mentioned experiment [28].

The studies presented upto this chapter assume that ions are static. In the next chapter we going to study the effect of ion motion on plasma oscillations.

**Appendix : Relativistic wave frame solution**

In this appendix we are going to present a new derivation of longitudinal AP waves. We assume that wave is quasi-static i.e. all quantities are functions of the single variable \( \psi = t - x/\beta \), here \( \beta = \omega/k \) is the phase velocity the plane wave. This permits the substitutions \( \partial/\partial x = -(1/\beta) d/d\psi \) and \( \partial/\partial t = d/d\psi \) and therefore set of Eqs.(4.1)-(4.4) can be combined as

\[
(1 - \frac{v}{\beta}) \frac{d}{d\psi} \left(1 - \frac{v}{\beta}\right) \frac{d}{d\psi} p + \frac{\omega_{pe}^2}{1 + p^2/(m^2c^2)} \frac{p}{\left[1 + p^2/(m^2c^2)\right]^{1/2}} = 0
\]

(4.18)

with \( p = \gamma mv \) and

\[
n = \frac{n_0}{1 - v/\beta}
\]

(4.19)

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\[ v = \frac{(p/m)}{\left[1 + p^2/(m^2c^2)\right]^{1/2}} \] \hspace{1cm} (4.20)

Now we use the following transformation as

\[ (1 - v/\beta) \frac{d}{d\psi} = \frac{d}{d\phi} \] \hspace{1cm} (4.21)

and Eqn. (4.18) becomes

\[ \frac{d^2p}{d\phi^2} + \omega_{pe}^2 \left[1 + p^2/(m^2c^2)\right]^{1/2} = 0 \] \hspace{1cm} (4.22)

Integrating once, we obtain

\[ \frac{dp}{d\phi} = \pm \sqrt{2} \omega_{pe} c \left[ A - [1 + p^2/(m^2c^2)]^{1/2} \right]^{1/2} = 0, \text{ here } A = \text{cons.} \] \hspace{1cm} (4.23)

We now substitute

\[ A - [1 + p^2/(m^2c^2)]^{1/2} = (A - 1) \sin^2 \theta \] \hspace{1cm} (4.24)

Then the solution of Eq.(4.25) can be expressed in terms of Elliptic integrals \textit{i.e.}

\[ \omega_{pe}\psi = \sqrt{2(A + 1)} E(\theta, r) - \sqrt{\frac{2}{A + 1}} F(\theta, r) + B \] \hspace{1cm} (4.25)

Here \( E(\theta, r), F(\theta, r) \) are incomplete elliptic integrals of second and first kind respectively, \( B \) is the second integration constant and

\[ r = \left[\frac{A - 1}{A + 1}\right]^{1/2} \] \hspace{1cm} (4.26)

Now integrating Eq.(4.21) we get

\[ \psi = \phi - \frac{1}{\beta} \int v d\phi \] \hspace{1cm} (4.27)

Using Eqs.(7.20),(4.24) and (4.25), Eq.(4.27) becomes

\[ \omega_{pe}\psi = \sqrt{2(A + 1)} E(\theta, r) - \sqrt{\frac{2}{A + 1}} F(\theta, r) - \frac{c}{\beta r} 2r \sin \theta + B \]
Since the constant ‘B’ affects only phase of the wave, it can be chosen to be zero i.e.

\[
\omega_{pe}(t - x/\beta) = \frac{2}{r'} E(\theta, r) - r' F(\theta, r) - \frac{c}{\beta} \frac{2r}{r'} \sin \theta
\]  \hspace{1cm} (4.28)

Here \( r' = \sqrt{1 - r^2} \).

Thus Eq.(4.28) together with (4.24) gives the plane wave solution for relativistic Langmuir waves for an arbitrary phase velocity in cold plasma. Note here that Eq.(4.28) is exactly similar to the Eq.(4.17). These solutions are identical to the plane wave solution obtained by Akhiezer and Polovin [119].