Chapter 4

Products of Eigenforms

4.1 Introduction

The space of modular forms of a fixed weight on the full modular group $SL_2(\mathbb{Z})$ has a basis of simultaneous eigenvectors for all Hecke operators. A modular form is called an eigenform if it is a simultaneous eigenvector for all Hecke operators. A natural question to ask is that whether the product of two eigenforms (may be of different weights) is an eigenform. This question was considered by W. Duke [8] and E. Ghate [10]. They proved that there are only finitely many cases where this phenomenon happens. Later, a more general question in the case of Rankin-Cohen bracket of two eigenforms was studied by D. Lanphier and R. Takloo-Bighash [25]. In this case also, they showed that except for a finitely many cases the Rankin-Cohen brackets of two eigenforms is not an eigenform. Recently, J. Beyerl, K. James, C. Trentacoste and H. Xue [4] have proved that this phenomenon extends to a certain class of nearly holomorphic modular forms. More explicitly, they have proved that there is only one additional case apart from the cases listed in [8] and [10] for which the product of two nearly holomorphic eigenforms of certain type is a nearly holomorphic eigenform.

In this chapter, we extend this result for a few more types of modular forms. First, we consider the case of quasimodular eigenforms and then the case of Rankin-Cohen brackets of almost everywhere (in short a.e.) eigenforms, the latter case generalizes the work of Ghate [11] on products of a.e. eigenforms. Finally, we consider the case of products of nearly holomorphic eigenforms. The results of this chapter are contained in [34].
4.2 Preliminaries

(a) Quasimodular forms:

**Definition 4.2.1.** (Quasimodular form of weight \( k \) and depth \( s \)) Let \( k \geq 2, s \geq 0 \) be integers. A holomorphic function \( f : \mathcal{H} \to \mathbb{C} \) is defined to be a quasimodular form of weight \( k \), depth \( s \) on \( SL_2(\mathbb{Z}) \), if there exist holomorphic functions \( f_0, f_1, \ldots, f_s \) on \( \mathcal{H} \) such that

\[
(cz + d)^{-k} f \left( \frac{az + b}{cz + d} \right) = \sum_{j=0}^{s} f_j(z) \left( \frac{c}{cz + d} \right)^j,
\]

for all \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \) and \( f_s \) is holomorphic at infinity and not identically vanishing. By convention, the zero function is a quasimodular form of depth 0 for each weight.

The concept of quasimodular form was first introduced by M. Kaneko and D. Zagier in [14].

**Remark 4.2.1.** It is a fact that if \( f \) is a quasimodular form of weight \( k \) and depth \( s \) on \( SL_2(\mathbb{Z}) \), not identically zero, then \( k \) is even and \( s \leq k/2 \).

A fundamental quasimodular form is the Eisenstein series of weight 2 defined by

\[
E_2(z) = 1 - 24 \sum_{n \geq 1} \sigma_1(n) q^n, \quad q = e^{2\pi i z}.
\]

It is a quasimodular form of weight 2 and depth 1 on \( SL_2(\mathbb{Z}) \). It is also easy to see that any modular form of weight \( k \) is a quasimodular form of weight \( k \) and depth 0.

The vector space of all quasimodular forms of weight \( k \) and depth \( \leq s \) on \( SL_2(\mathbb{Z}) \) is denoted by \( \widetilde{M}_k^s \). Let \( \widetilde{M}_k = \bigcup_s \widetilde{M}_k^s \) and \( \widetilde{M}_s = \oplus_k \widetilde{M}_k \) denote the space of quasimodular forms of weight \( k \) and the graded ring of all quasimodular forms on \( SL_2(\mathbb{Z}) \) respectively. Then it is known that \( \widetilde{M}_s = \mathbb{C}[E_2, E_4, E_6] \) and the differential operator \( D = \frac{1}{2\pi i} \frac{d}{dz} \) takes \( \widetilde{M}_k \) to \( \widetilde{M}_{k+2} \). It can also be seen that if \( f \) is a modular form, then \( Df \) is not a modular form. But using the Eisenstein series \( E_2 \) and \( Df \), one can construct a new modular form.

In fact, if \( f \in M_k \) then we see that

\[
Df - \frac{k}{12} E_2 f \in M_{k+2}.
\]

It is a cusp form iff \( f \) is so. For details on quasimodular forms we refer to [6] and [14].

For an integer \( n \geq 1 \), The Hecke operator \( T_n \) on \( \widetilde{M}_k \) is defined in a similar way as we define in the case of modular forms. For any \( f \in \widetilde{M}_k \), \( T_n \) acts on \( f \) by

\[
(T_n f)(z) = n^{k-1} \sum_{d|n} d^{-k} \sum_{b=0}^{d-1} f \left( \frac{n z + bd}{d^2} \right).
\] (4.2.1)
It is a fact that $T_n$ preserves $\tilde{M}_k$. Let $f \in \tilde{M}_k$. We say that $f$ is an eigenform if it is an eigenvector for all Hecke operators $T_n$, $n \geq 1$.

(b) Almost everywhere (a.e.) eigenforms:

Let $M_k(\Gamma_1(N))$ (respectively $M_k(N, \chi)$), $S_k(\Gamma_1(N))$, (respectively $S_k(N, \chi)$) and $E_k(\Gamma_1(N))$ (respectively $E_k(N, \chi)$) be the spaces of modular forms, cusps forms and Eisenstein series of weight $k \geq 1$ on $\Gamma_1(N)$ (respectively on $\Gamma_0(N)$ with character $\chi$) respectively. The space $E_k(\Gamma_1(N))$ is orthogonal to $S_k(\Gamma_1(N))$ in the space $M_k(\Gamma_1(N))$ with respect to the Petersson inner product. In other words,

$$M_k(\Gamma_1(N)) = S_k(\Gamma_1(N)) \oplus E_k(\Gamma_1(N)).$$

Each of the above spaces can also be decomposed as follows.

$$M_k(\Gamma_1(N)) = \bigoplus_{\chi} M_k(N, \chi), \tag{4.2.2}$$

$$S_k(\Gamma_1(N)) = \bigoplus_{\chi} S_k(N, \chi), \tag{4.2.3}$$

$$E_k(\Gamma_1(N)) = \bigoplus_{\chi} E_k(N, \chi), \tag{4.2.4}$$

where the direct sum varies over all Dirichlet characters $\chi$ modulo $N$.

Each of the spaces in the above decomposition has an explicit basis consisting of modular forms that are eigenvectors of all the Hecke operators $T_n$ with $n$ coprime to $N$. For $S_k(N, \chi)$, this basis is constructed from newforms of level $M, M|N$. Let $M$ be a positive integer divisible by the conductor $c(\chi)$ of $\chi$. We say that $f \in S_k(M, \chi)$ is primitive if it is a normalized newform, i.e., it is an eigenvector of all the Hecke operators of level $M$, it is in the newform space $S^{new}_k(M, \chi)$ and the first coefficient of the Fourier expansion of $f$ is 1. Then the set of cusp forms

$$\bigcup_{M} \bigcup_{Q} \{f(Qz) \mid f \text{ is a primitive form of } S_k(M, \chi)\},$$

where $M$ varies through all positive integers satisfying $M|N$ and $c(\chi)|M$, and $Q$ varies through all positive integers dividing $N/M$, forms a basis of $S_k(N, \chi)$ consisting of common eigenforms of all Hecke operators $T_n$ with $n$ coprime to $N$. A basis for $E_k(N, \chi)$ consisting of modular forms that are eigenvectors of all Hecke operators $T_n$ with $n$ coprime to $N$ is given in the following theorem. We refer to Theorems 4.7.1 and 4.7.2 of [35] for more details.
Theorem 4.2.1. Let $\chi_1$ and $\chi_2$ be Dirichlet characters modulo $M_1$ and $M_2$ respectively, such that $\chi_1\chi_2(-1) = (-1)^k$ with $k \geq 1$. Also assume that

- if $k = 2$ and both $\chi_1$ and $\chi_2$ are principal characters, then $M_1 = 1$ and $M_2$ is a prime number,
- otherwise, $\chi_1$ and $\chi_2$ are primitive characters.

Put $M = M_1M_2$ and $\chi = \chi_1\chi_2$. Then there is an element $f = f_k(z, \chi_1, \chi_2) = \sum_{n=0}^{\infty} a_nq^n$ such that $L(s, f) = L(s, \chi_1)L(s-k+1, \chi_2)$,

$$a_o = \begin{cases} 0 & \text{if } k = 1, \text{ and } \chi_1 \text{ is non-principal,} \\ (M - 1)/24 & \text{if } k = 2, \text{ and both } \chi_1 \text{ and } \chi_2 \text{ are non-principal,} \\ -B_{k,\chi}/(2k) & \text{otherwise,} \end{cases}$$

where $B_{k,\chi}$ is the generalized Bernoulli number associated with a primitive Dirichlet character $\chi$ of conductor $c(\chi)$, defined by

$$\sum_{a=1}^{c(\chi)} \frac{\chi(a)e^{at}}{e^{c(\chi)t} - 1} = \sum_{k=0}^{\infty} \frac{B_{k,\chi}t^k}{k!},$$

and $a_n = \sum_{d|n} \chi_1(n/d)\chi_2(d)d^{k-1}$, for $n \geq 1$.

The modular form $f_k(z, \chi_1, \chi_2) \in \mathcal{E}_k(M, \chi)$ is an eigenvector for the Hecke operators of level $M$. Modulo the relation $f_1(z, \chi_1, \chi_2) = f_1(z, \chi_2, \chi_1)$ when $k = 1$, the set of elements $f_k(Qz, \chi_1, \chi_2)$, where $QM_1M_2|N$ form a basis of $\mathcal{E}_k(N, \chi)$ consisting of common eigenforms of all the Hecke operators $T_n$ of level $N$, with $(n, N) = 1$.

Let $\mathcal{B}$ denote the explicit basis of $M_k(\Gamma_1(N))$ obtained as above. An element of $M_k(\Gamma_1(N))$ is called an almost everywhere eigenform (or a.e. eigenform in short) if, up to a scalar multiple, it is an element of $\mathcal{B}$.

For $k > 2$ and $\psi$, a Dirichlet character modulo $N$, we define the Eisenstein series

$$E_k^{(N,\psi)}(z) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(N)} \overline{\psi}(d)(cz + d)^{-k}, \quad (4.2.5)$$

where $z \in \mathcal{H}$ and the sum varies over all $\gamma = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \Gamma_0(N)$ modulo $\Gamma_\infty = \left\{ \left( \begin{array}{cc} 1 & n \\ 0 & 1 \end{array} \right) | n \in \mathbb{Z} \right\}$. It is known that $E_k^{(N,\psi)} \in \mathcal{E}_k(N, \psi)$ ([35]).
Atkin-Lehner Involutions

Let $N \geq 1$ be an arbitrary integer and let $Q|N$ be such that $(Q, N/Q) = 1$. Then the Atkin-Lehner involution $W_Q$ is defined by

$$W_Q = \begin{pmatrix} Qx & y \\ Nv & Qw \end{pmatrix},$$

where $x, y, v, w \in \mathbb{Z}$ satisfying $x \equiv 1 \pmod{N/Q}$, $y \equiv 1 \pmod{Q}$ and the determinant of $W_Q$ is $Q$. $W_Q$ acts on the space $M_k(N, \chi)$ in the usual way, i.e., if $f \in M_k(N, \chi)$, then

$$f|W_Q(z) = \det(W_Q)^{k/2}(Nvz + Qw)^{-k}f\left(\frac{Qxz + y}{Nvz + Qw}\right).$$

Now, let us decompose $\chi = \chi_Q\chi_{N/Q}$ into its $Q$ and $N/Q$ parts. It is well known (see Proposition 1.1 of [3]) that $W_Q$ maps

$$W_Q : M_k(N, \chi_Q\chi_{N/Q}) \to M_k(N, \overline{\chi}_Q\chi_{N/Q})$$

and that it takes cusp forms to cusp forms. Moreover, since $W_Q$ commutes, up to a constant, with $T_n$ for all $n$ with $(n, N) = 1$, it takes a.e. eigenforms to a.e. eigenforms (see Proposition 1.2 of [3]). In fact $W_Q$ takes primitive cusp forms to primitive cusp forms (up to multiplication by a constant).

(c) Nearly holomorphic modular forms:

**Definition 4.2.2.** (Nearly holomorphic modular form) A nearly holomorphic modular form $F(z)$ of weight $k$ on $SL_2(\mathbb{Z})$ is a polynomial in $\frac{1}{Im(z)}$ whose coefficients are holomorphic functions on $\mathcal{H}$ such that

$$(cz + d)^{-k}F\left(\frac{az + b}{cz + d}\right) = F(z),$$

for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$.

The function $E_2^*$ defined by

$$E_2^*(z) = E_2(z) - \frac{3}{\pi Im(z)}$$

is a nearly holomorphic modular form of weight 2. The space of nearly holomorphic modular forms of weight $k$ on $SL_2(\mathbb{Z})$ is denoted by $\widehat{M}_k$. Any modular form of weight $k$ is trivially a nearly holomorphic modular form of weight $k$. 
Definition 4.2.3. (Maass-Shimura operator) The Maass-Shimura operator $\delta_k$ on $f \in \hat{M}_k$ is defined by

$$\delta_k(f) = \left( \frac{1}{2\pi i} \left( \frac{k}{2i Im(z)} + \frac{\partial}{\partial z} \right) f \right)(z).$$

It is known that $\delta_k$ maps $\hat{M}_k$ to $\hat{M}_k + 2$. In particular $\delta_k$ maps $M_k$ to $\hat{M}_k + 2$. For details on nearly holomorphic modular forms we refer to [43]. The Hecke operators $T_n$ on $\hat{M}_k$ is defined in the same way as in (4.2.1). A nearly holomorphic modular form is called an eigenform if it is an eigenform for all Hecke operators $T_n$, where $n \in \mathbb{N}$.

### 4.3 Overview of Earlier Works

For $k \in \{12, 16, 18, 20, 22, 26\}$, let $\Delta_k$ denote the unique normalized cusp form of weight $k$ on $SL_2(\mathbb{Z})$.

Theorem 4.3.1. (Duke [8], Ghate [10]) The product of two eigenforms on $SL_2(\mathbb{Z})$ is not an eigenform except for the following cases.

$$E_4^2 = E_8, \quad E_4E_6 = E_{10}, \quad E_6E_8 = E_4E_{10} = E_{14},$$

$$E_4\Delta_{12} = \Delta_{16}, \quad E_6\Delta_{12} = \Delta_{18}, \quad E_4\Delta_{16} = E_8\Delta_{12} = \Delta_{20},$$

$$E_4\Delta_{18} = E_6\Delta_{16} = E_{10}\Delta_{12} = \Delta_{22},$$

$$E_4\Delta_{22} = E_6\Delta_{20} = E_8\Delta_{18} = E_{10}\Delta_{16} = E_{14}\Delta_{12} = \Delta_{26}.$$

Definition 4.3.1. (Rankin-Cohen bracket) Let $f \in M_{k_1}(N, \chi)$ and $g \in M_{k_2}(N, \psi)$. Then the $m^{th}$ Rankin-Cohen bracket of $f$ and $g$ is given by,

$$[f, g]_m(z) = \sum_{r+s=m} (-1)^r \binom{m + k_1 - 1}{s} \binom{m + k_2 - 1}{r} f^{(r)}(z)g^{(s)}(z),$$

where $f^{(r)}(z) = D^r f(z)$ and $g^{(s)}(z) = D^s g(z)$.

It is known that $[f, g]_m \in M_{k_1+k_2+2m}(N, \chi\psi)$ and $[f, g]_m$ is a cusp form if $m \geq 1$.

Theorem 4.3.1 was generalized to Rankin-Cohen brackets of eigenforms by Lanphier and Takloo-Bighash [25]. They proved the following theorem.

Theorem 4.3.2. (Lanphier–Takloo-Bighash, [25]) There are only a finite number of triples $(f, g, m)$ with the property that $f$ and $g$ are normalized eigenforms on $SL_2(\mathbb{Z})$ and $[f, g]_m$ is again an eigenform. The following describes all the possibilities.
1. \([E_4, E_6]_0 = E_{10}\) and \([E_4, E_{10}]_0 = [E_6, E_8]_0 = E_{14}\).

2. If \(k, l \in \{4, 6, 8, 10, 14\}\) and \(m \geq 1\) with \(k + l + 2m \in \{12, 16, 18, 20, 22, 26\}\), then
   \[ [E_k, E_l]_m = c_m(k, l)\Delta_{k+l+2m}, \]
   where
   \[ c_m(k, l) = -\frac{2l}{B_l} \left( \frac{m + l - 1}{m} \right) + (-1)^{m+1} \frac{2k}{B_k} \left( \frac{m + k - 1}{m} \right). \]

3. If \(k \in \{4, 6, 8, 10, 14\}\) and \(m \geq 0\) with \(l, k + l + 2m \in \{12, 16, 18, 20, 22, 26\}\), then
   \[ [E_k, \Delta_l]_m = c_m(l)\Delta_{k+l+2m}, \]
   where
   \[ c_m(l) = \left( \frac{m + l - 1}{m} \right). \]

In [11], Ghate considered the product of two a.e. eigenforms of square-free level. More precisely, he proved the following theorems.

**Theorem 4.3.3.** (Ghate, [11]) Let \(k_1, k_2 \geq 1\) be integers and \(N \geq 1\) be a square-free integer. If \(g \in S_{k_1}(\Gamma_1(N))\) and \(h \in S_{k_2}(\Gamma_1(N))\) are a.e. eigenforms, then \(gh\) is not an a.e. eigenform.

**Theorem 4.3.4.** (Ghate, [11]) Let \(k_1, k_2 \geq 3\) be integers, \(k = k_1 + k_2\), \(g \in S_{k_2}(N, \psi)\) and \(h \in \mathcal{E}_{k_1}(N, \chi)\) be a.e. eigenforms. Assume that \(N\) is square-free and \(g\) is a newform. If \(\dim(S_{k_1}^{\text{new}}(N, \psi\chi)) \geq 2\) then \(gh\) is not an a.e. eigenform.

**Theorem 4.3.5.** (Ghate, [11]) Let \(k_1, k_2 \geq 3\) be integers and \(k = k_1 + k_2\). Let \(g = f_{k_2}(z, \psi_1, \psi_2) \in \mathcal{E}_{k_2}(N, \psi)\) and \(h = f_{k_1}(z, \chi_1, \chi_2) \in \mathcal{E}_{k_1}(N, \chi)\) be eigenforms as in Theorem 4.2.1 with \(\psi\) and \(\chi\) primitive. Assume also that \(N\) is square-free and \(k_2 \neq k/2\). If
   \[ \dim(S_{k_1}^{\text{new}}(N, \psi\chi)) \geq \begin{cases} 1 & \text{when } gh \in \mathcal{E}_{k}(N, \psi\chi), \\ 2 & \text{when } gh \in S_{k}(N, \psi\chi), \end{cases} \]
   then \(gh\) is not an a.e. eigenform.

Recently, Beyerl et al. [4] have proved that in the case of a certain class of nearly holomorphic modular forms, the product of any two nearly holomorphic eigenforms is not an eigenform, except for a finitely many explicitly given cases. For \(r \geq 1\), let \(\delta_k^{(r)} := \delta_{k+2r-2} \circ \cdots \circ \delta_{k+2} \circ \delta_k\) and assume that \(\delta_k^{(0)}\) is the identity map.

**Theorem 4.3.6.** (Beyerl et al., [4]) Let \(f \in M_k\) and \(g \in M_l\) be such that \(\delta_k^{(r)}(f)\) and \(\delta_l^{(s)}(g)\) are eigenforms. Then \(\delta_k^{(r)}(f)\delta_l^{(s)}(g)\) is an eigenform only in the following cases:
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(1) \[ \begin{align*}
E_4^2 &= E_8, & E_4 E_6 &= E_{10}, & E_6 E_8 &= E_4 E_{10} = E_{14}, & E_4 \Delta_{12} &= \Delta_{16}, \\
E_6 \Delta_{12} &= \Delta_{18}, & E_4 \Delta_{16} &= E_8 \Delta_{12} = \Delta_{20}, & E_4 \Delta_{18} &= E_6 \Delta_{16} = E_{10} \Delta_{12} = \Delta_{22}, \\
E_4 \Delta_{22} &= E_6 \Delta_{20} = E_8 \Delta_{18} = E_{10} \Delta_{16} = E_{14} \Delta_{12} = \Delta_{26}. 
\end{align*} \]

(2) \[ \delta_4(E_4) E_4 = \frac{1}{2} \delta_8(E_8). \]

**Note:** We remark that the cases mentioned in case (1) above are the same modular cases given in [8] and [10].

### 4.4 Main Results

(a) Quasimodular forms:

**Theorem 4.4.1.** Let \( f \in M_k, g \in M_l \). For \( r, s \geq 0 \), assume that \( D^r f \in \hat{M}_{k+2r}, \)
\( D^s g \in \hat{M}_{l+2s} \) are eigenforms. Then \((D^r f)(D^s g)\) is an eigenform only in the following cases.

\[ \begin{align*}
E_4^2 &= E_8, & E_4 E_6 &= E_{10}, & E_6 E_8 &= E_4 E_{10} = E_{14}, \\
E_4 \Delta_{12} &= \Delta_{16}, & E_6 \Delta_{12} &= \Delta_{18}, & E_4 \Delta_{16} &= E_8 \Delta_{12} = \Delta_{20}, & E_4 \Delta_{18} &= E_6 \Delta_{16} = E_{10} \Delta_{12} = \Delta_{22}, \\
E_4 \Delta_{22} &= E_6 \Delta_{20} = E_8 \Delta_{18} = E_{10} \Delta_{16} = E_{14} \Delta_{12} = \Delta_{26}, \\
(D E_4) E_4 &= \frac{1}{2} D E_8.
\end{align*} \]

**Theorem 4.4.2.** Let \( f \in \hat{M}^{\leq p}_k \) and \( g \in \hat{M}^{\leq q}_l \) be eigenforms. Assume that \( p < k/2 \) and \( q < l/2 \), then except for the cases mentioned in Theorem 4.4.1, \( f g \) is not an eigenform.

Since \( \hat{M}_2 \) is generated by \( E_2, E_2 \) is an eigenform.

**Remark 4.4.1.** Following the same proof as in the case of \( M_k \), one can prove that a quasimodular form in \( \hat{M}_k \) with non-zero constant Fourier coefficient is an eigenform iff \( f \in \mathbb{C} E_k \). Also \( f \) is an eigenform iff \( D^r f \) is an eigenform (See Proposition 4.5.3).

In the following we consider product of derivatives of \( E_k \).

**Theorem 4.4.3.** For \( k \geq 2 \) and \( r, s \geq 0 \), \((D^r E_2)(D^s E_k)\) is not an eigenform.

As a consequence of the above theorem, we get the following corollary.

**Corollary 4.4.4.** Let \( f \in M_k \) be an eigenform. Then \((D^r E_2)f\) is an eigenform iff \( r = 0 \) and \( f \in \mathbb{C} \Delta_{12} \).
(b) Almost everywhere (a.e.) eigenforms:
We now state the results which generalize the results of Ghate (Theorem 4.3.3, 4.3.4, 4.3.5) for the Rankin-Cohen brackets.

Theorem 4.4.5. Let \( k_1, k_2 \geq 1 \) be integers and \( N \geq 1 \) be a square-free integer and \( m \geq 1 \). If \( g \in S_{k_1}(\Gamma_1(N)) \) and \( h \in S_{k_2}(\Gamma_1(N)) \) are a.e. eigenforms, then \([g, h]_m\) is not an a.e. eigenform.

Theorem 4.4.6. Let \( k_1, k_2, k, m \) be positive integers such that \( k = k_1 + k_2 + 2m \) and \( N \) be a square-free integer. Then we have the following.

1. For \( k_1 \geq 3 \) and \( k_2 \geq k_1 + 2 > 2 \), suppose that \( g \in S_{k_1}(N, \chi) \) is an a.e. eigenform which is a newform and \( h \in \mathcal{E}_{k_2}(N, \psi) \). If \( \dim(S^\text{new}_{k}(N, \chi \psi)) \geq 2 \), then \([g, h]_m\) is not an a.e. eigenform.

2. For \( k_1, k_2 \geq 3, |k_1-k_2| \geq 2 \), let \( g = f_{k_1}(z, \chi_1, \chi_2) \in \mathcal{E}_{k_1}(N, \chi) \) and \( h = f_{k_2}(z, \psi_1, \psi_2) \in \mathcal{E}_{k_2}(N, \psi) \) be a.e. eigenforms as described earlier with \( \chi \) and \( \psi \) being primitive. If \( \dim(S^\text{new}_{k}(N, \chi \psi)) \geq 2 \), then \([g, h]_m\) is not an a.e. eigenform.

(c) Nearly holomorphic modular forms:
In the case of nearly holomorphic modular forms, we prove the following theorem.

Theorem 4.4.7. Let \( f \in M_k \) be an eigenform. Then \( E^*_zf \) is an eigenform iff \( f = \mathbb{C}\Delta_{12} \).

4.5 Proofs

4.5.1 Proof of Theorem 4.4.1
Before we proceed to the proof of the theorem, we present some preliminary results which will be needed in the proof. Recall the following proposition from [4].

**Proposition 4.5.1.** (Beyerl et al., [4]) Let \( f \in M_k \) and \( g \in M_l \). Then

\[
\delta_k^{(r)}(f)\delta_l^{(s)}(g) = \sum_{j=0}^{r+s} \frac{1}{k+l+2j-2} \left( \sum_{m=\max(j-r,0)}^{s} (-1)^{j+m} \binom{s}{m} \binom{r+m}{j} \binom{k+r+m-1}{r+m-j} \right) \delta_{k+l+2j}^{(r+s-j)}([f, g]_j(z)).
\]

Both sides of the equality in the above proposition are polynomials in \( \frac{1}{Im(z)} \), where the coefficients are holomorphic functions on the upper half plane \( \mathcal{H} \). Comparing the constant coefficients on both the sides of the equality, we get the following corollary.
Corollary 4.5.2. Let \( f \in M_k \) and \( g \in M_l \). Then
\[
D^r(f)D^s(g) = \sum_{j=0}^{r+s} \frac{1}{(k+l+2j-2)} \left( \sum_{m=\text{max}(j-r,0)}^{s} (-1)^{j+m} \binom{s}{m} \binom{k+r+m-1}{r+m-j} \right) D^{r+s-j}([f,g]_j)(z).
\]

Next, we prove the following proposition which shows the commuting relation between the operators \( D \) and \( T_n \).

Proposition 4.5.3. If \( f \in \tilde{M}_k \), then \((D^r(T_n f))(z) = \frac{1}{n^r}(T_n(D^r f))(z)\), for \( r \geq 0 \). Moreover, we have \( D^r f \) is an eigenform for \( T_n \) iff \( f \) is. In this case, if \( \lambda_n \) is the eigenvalue of \( T_n \) associated to \( f \), then \( n^r \lambda_n \) is the eigenvalue of \( T_n \) associated to \( D^r f \).

Proof. We have
\[
(T_n f)(z) = n^{k-1} \sum_{d|n} d^{-k} \sum_{b=0}^{d-1} \left( \frac{nz + bd}{d^2} \right).
\]
So
\[
D(T_n f)(z) = n^{k-1} \sum_{d|n} d^{-k} \sum_{b=0}^{d-1} \frac{1}{2\pi i} \frac{d^2}{dz} \left( \frac{nz + bd}{d^2} \right).
\]
Again one computes that
\[
T_n(D f)(z) = n \left[ n^{k-1} \sum_{d|n} d^{-k} \sum_{b=0}^{d-1} \frac{1}{2\pi i} \frac{d^2}{dz} \left( \frac{nz + bd}{d^2} \right) \right],
\]
from which we obtain
\[
D(T_n f)(z) = \frac{1}{n}(T_n(D f)).
\]
By using induction on \( r \), we get the first assertion of the proposition.
Now assume that \( f \) is an eigenform. So, \((T_n f)(z) = \lambda_n f(z)\). Then applying \( D^r \) on both sides and using the first assertion, we obtain the following:
\[
T_n(D^r f)(z) = n^r \lambda_n(D^r f)(z).
\]
Hence, \( D^r f \) is an eigenform. Conversely, let us assume that \( D^r f \) is an eigenform for \( T_n \) with eigenvalue \( \beta_n \). Then \( T_n(D^r f)(z) = \beta_n(D^r f)(z) \). Again using the first assertion, we obtain
\[
D^r(T_n f)(z) = \frac{1}{n^r} T_n(D^r f)(z) = \frac{\beta_n}{n^r} D^r f(z),
\]
\[
\Rightarrow D^r(T_n f - \frac{\beta_n}{n^r} f) = 0.
\]
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⇒ $T_n f - \frac{\beta_n}{n^r} f$ is a constant.

Since $T_n f - \frac{\beta_n}{n^r} f$ is a quasimodular form and any non-zero constant function is not a quasimodular form of positive weight, we get

$$T_n f(z) = \frac{\beta_n}{n^r} f(z).$$

Hence $f$ is an eigenform, proving the proposition.

Now we prove the result on a sum of eigenforms with distinct weights.

**Proposition 4.5.4.** Suppose that $\{f_i\}_i$ is a collection of modular forms of distinct weights $k_i$. Then $\sum_{i=1}^t a_i D^{(r - \frac{k_i}{2})}(f_i), a_i \in \mathbb{C}^*$ is an eigenform iff each $D^{(r - \frac{k_i}{2})}(f_i)$ is an eigenform and each function has the same set of eigenvalues.

**Proof.** We prove the proposition for $t = 2$ and the proof follows by induction. Let $f \in M_k$ and $g \in M_l$ be such that $D^r f$ and $D^{(r + \frac{k}{2} - \frac{l}{2})}(g)$ are eigenforms with same eigenvalues. If $T_n(D^r(f)) = \mu_n D^r(f)$ and $T_n(D^{(r + \frac{k}{2} - \frac{l}{2})}(g)) = \mu_n D^{(r + \frac{k}{2} - \frac{l}{2})}(g)$, then by linearity of $T_n$,

$$T_n(D^r(f)) + D^{(r + \frac{k}{2} - \frac{l}{2})}(g)) = \mu_n(D^r(f) + D^{(r + \frac{k}{2} - \frac{l}{2})}(g)).$$

Hence, $D^r(f) + D^{(r + \frac{k}{2} - \frac{l}{2})}(g)$ is also an eigenform.

Conversely, suppose that $D^r(f) + D^{(r + \frac{k}{2} - \frac{l}{2})}(g)$ is an eigenform. Then by Proposition 4.5.3 and linearity of $D^r$, $f + D^{(\frac{k-l}{2})}(g)$ is also an eigenform. Write

$$T_n(f + D^{(\frac{k-l}{2})}(g)) = \lambda_n(f + D^{(\frac{k-l}{2})}(g)).$$

Using the linearity of $T_n$ and Proposition 4.5.3, we get

$$T_n(f) + n \frac{k-l}{2} D^{(\frac{k-l}{2})}(T_n(g)) = \lambda_n f + \lambda_n D^{(\frac{k-l}{2})}(g).$$

Rearranging this, we get

$$T_n(f) - \lambda_n f = D^{(\frac{k-l}{2})}(\lambda_n g - n \frac{k-l}{2} T_n(g)).$$

Now, we see that the left hand side is a modular form and the right hand side is not a modular form since $k \neq l$. Hence both sides must be equal to zero. Thus we have

$$T_n(f) = \lambda_n f \quad \text{and} \quad T_n(g) = \lambda_n n \frac{k-l}{2} g.$$

Therefore, $f$ is an eigenvector for $T_n$ with eigenvalue $\lambda_n$ and $g$ is an eigenvector for $T_n$ with eigenvalue $\lambda_n n \frac{k-l}{2}$. Then by Proposition 4.5.3, we see that $D^{(\frac{k-l}{2})}(g)$ is an
eigenvector for $T_n$ with eigenvalue $\lambda_n$. Therefore, $f$ and $D^{(k-l)}(g)$ are eigenvectors for $T_n$ with eigenvalues $\lambda_n$. So $D^r(f)$ and $D^{(k-l+r)}(g)$ must have the same eigenvalue with respect to $T_n$. Hence, for all $n \in \mathbb{N}$, $D^r(f)$ and $D^{(k-l+r)}(g)$ must be eigenforms with the same eigenvalues.

Using the above proposition, we prove the following lemma.

**Lemma 4.5.5.** Let $l < k$ and $f \in M_k$, $g \in M_l$ be eigenforms. Then $D^{(k-l)}(g)$ and $f$ do not have the same eigenvalues.

**Proof.** Suppose on the contrary that they have the same eigenvalues. Let $g$ have eigenvalues $\lambda_n(g)$, then by Proposition 4.5.3, $f$ has eigenvalues $n^{k-l} \lambda_n(g)$. Thus the Fourier expansion of $f$ is of the following form

$$f(z) = \sum_{n \geq 1} c n^{k-l} \lambda_n(g) q^n + c_0,$$

for some constants $c$ and $c_0$. Thus, we have

$$f(z) = \frac{1}{(2\pi i)^{(k-l)/2}} \frac{d^{(k-l)/2}}{dz^{(k-l)/2}} g(z) + c_0.$$

This says that $f$ equals some derivative of $g$ plus some constant. However, from direct computation, we see that this function is not modular, which contradicts that $f$ is modular.

We shall need a special case of the above lemma.

**Corollary 4.5.6.** Let $k > l$ and $f \in M_k$, $g \in M_l$ be eigenforms. Then $D^{(k-l+r)}(g)$ and $D^r(f)$ do not have the same eigenvalues.

Next, we need the following lemma.

**Lemma 4.5.7.** Let $D^r(f) \in \tilde{M}_{k+2r}$, $D^s(g) \in \tilde{M}_{l+2s}$. In the following cases $[f, g]_m \neq 0$.

**Case 1:** $f$ a cusp form, $g$ not a cusp form.

**Case 2:** $f = g = E_k$, $m$ even.

**Case 3:** $f = E_k$, $g = E_l$, $k \neq l$.

**Proof.** Case 1: Let $f = \sum_{n \geq 1} a_n q^n$, $g = \sum_{n \geq 0} b_n q^n$ and let $n_0$ be the smallest positive integer for which $a_{n_0} \neq 0$. Then a direct computation of the coefficient of $q^{n_0}$ in $[f, g]_m$ yields

$$a_{n_0} b_0 (-1)^m \binom{m + k - 1}{m} \neq 0.$$
Case 2: In this case, the $q$-coefficient of $[E_k, E_k]_m$ is
\[ 2 \cdot \frac{-2k}{B_k} \binom{m + k - 1}{m}, \]
which is non-zero.

Case 3: Without loss of generality, let us assume that $k > l$. Then the coefficient of $q$ in the Fourier expansion of $[E_k, E_l]$ is
\[ -\frac{2l}{B_l} \binom{m + k - 1}{m} + (-1)^m \frac{-2k}{B_k} \binom{m + l - 1}{m}. \]
If $m$ is even, then both of these terms are non-zero and of same sign, and so the sum is non-zero. If $m$ is odd, then using the fact that $|B_k| > |B_l|$ for $l > 4$ and $l$ even, we get
\[ \left| \frac{B_k}{k} \binom{m + k - 1}{m} \right| > \left| \frac{B_l}{l} \binom{m + l - 1}{m} \right|. \] (4.5.1)

Thus, for $l > 4$ and $m$ odd, the coefficient of $q$ is non-zero. If $l = 4$, then also (4.5.1) holds true for $m > 1$. For $m = 1$, (4.5.1) simplifies to $|B_k| > |B_l|$, which is true for $(k, l) \neq (8, 4)$. Now if $m = 1$, $k = 8$ and $l = 4$, then $B_k = B_l$ and hence $\frac{-2l}{B_l} \binom{m + k - 1}{m} + (-1)^m \frac{-2k}{B_k} \binom{m + l - 1}{m} \neq 0$. This proves the lemma.

We shall need a criterion for an eigenform, given in the following lemma.

**Lemma 4.5.8.** If $f(z) = \sum_{n \geq 0} a_n q^n \in \widetilde{M}_k$ is a non-zero eigenform, then $a_1 \neq 0$.

**Proof.** If $T_m$ is the $m$th Hecke operator, then the Fourier expansion of $T_m f(z)$ is given by
\[ T_m f(z) = \sum_{n \geq 0} \left( \sum_{d | (m, n), d > 0} d^{k-1} a_{mn/d^2} \right) q^n. \] (4.5.2)
If $f$ is an eigenform, then for all $m \geq 1$, there exist $\lambda_m \in \mathbb{C}$ such that $T_m f = \lambda_m f$.

Substituting in (4.5.2), we get
\[ \sum_{n \geq 0} \left( \sum_{d | (m, n), d > 0} d^{k-1} a_{mn/d^2} \right) q^n = \lambda_m \sum_{n \geq 0} a_n q^n. \] (4.5.3)

Now, comparing the coefficients of $q$ both the sides in the above equation, we get
\[ \lambda_m a_1 = a_m, \]
for all \( m \geq 1 \).
If \( a_1 = 0 \), then \( a_m = 0 \) for all \( m \geq 1 \), and so \( f = a_0 \). Since any non-zero constant is not a quasimodular form of positive weight, this implies that \( a_0 = 0 = f \). This proves the lemma.

Finally, we prove a lemma, which is a driving force to prove the theorem.

**Lemma 4.5.9.** Let \( D^r(f) \in \tilde{M}_{k+2r} \) and \( D^s(g) \in \tilde{M}_{l+2s} \) be eigenforms. Let us assume that not both \( f \) and \( g \) are cusp forms. Then in the expansion given in Corollary 4.5.2, either the term involving \([f,g]_r \) is non-zero, or the term involving \([f,g]_{r+s-1} \) is non-zero.

**Proof.** There are three cases.
Case 1: \( f = g = E_k \). If \( r + s \) is even, then by Lemma 4.5.7, \([f,g]_{r+s} \neq 0 \) and from Corollary 4.5.2, we see that the coefficient of \([f,g]_{r+s} \) is non-zero. If \( r + s \) is odd, then by Lemma 4.5.7, \([f,g]_{r+s-1} \) is non-zero. Now, the coefficient of \([f,g]_{r+s-1} \) is also non-zero, because if it were zero, after simplification we would have \( k = -(r + s) + 1 \leq 0 \), a contradiction.
Case 2: If \( f \) is a cusp form and \( g \) is not a cusp form, then by Lemma 4.5.7, \([f,g]_{r+s} \) is non-zero and by computing the coefficient, we see that the term involving \([f,g]_{r+s} \) is non-zero.
Case 3: If \( f = E_k \), \( g = E_l \), \( k \neq l \), then again by Lemma 4.5.7, \([f,g]_{r+s} \neq 0 \) and thus the term involving \([f,g]_{r+s} \) is non-zero.

We are now ready to prove Theorem 4.4.1. By Corollary 4.5.2, we have

\[
D^r(f)D^s(g) = \sum_{j=0}^{r+s} \alpha_j D^{(r+s-j)}([f,g]_j), \quad \text{for} \quad \alpha_j \in \mathbb{C}.
\]

By Proposition 4.5.4, the above sum is an eigenform iff every summand is either an eigenform with a single common eigenvalue or is zero. Again by Corollary 4.5.6, \( \alpha_j D^{(r+s-j)}([f,g]_j) \) are always of different eigenvalues for different \( j \). Hence for \( D^r(f)D^s(g) \) to be an eigenform, all but one term in the summation must be zero and the remaining non-zero term must be an eigenform.

If both \( f \) and \( g \) are cusp forms, then by Lemma 4.5.8, \( D^r(f)D^s(g) \) is not an eigenform. Otherwise, from Lemma 4.5.9 either the term involving \([f,g]_{r+s} \) or the term involving \([f,g]_{r+s-1} \) is non-zero. By Theorem 4.3.2, this is an eigenform for finitely many cases only. Hence, there are only finitely many \( f, g, r, s \) such that \( D^r(f)D^s(g) \) becomes an eigenform. So, we have to rule out the remaining cases not mentioned in the theorem.
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In particular, consider the cases for which \([f, g]_{r+s}\) is an eigenform. From Theorem 4.3.2, we find that there are 29 cases with \(g\) a cusp form and 81 cases with \(f\) and \(g\) both Eisenstein series. We also must consider the infinite class with \(f = g = E_k\) and \(r + s\) odd, where \([f, g]_{r+s} = 0\).

For the last case when \(f = g = E_k\) and \(r + s\) is odd, we have \([f, g]_0\) is not an eigenform. Thus there are two non-zero terms. Then applying Proposition 4.5.4 and Corollary 4.5.6, we conclude that \(D^r(f)D^s(g)\) is not an eigenform.

Now, consider the rest. In the last finitely many cases, we find computationally that there are two non-zero terms involving \([f, g]_0\) and \([f, g]_{r+s}\). Therefore, in these cases also the fact that \(D^r(f)D^s(g)\) is not an eigenform follows from Proposition 4.5.4 and Corollary 4.5.6. Below we consider a typical case

\[
D(E_4) \cdot D(E_4) = \frac{-1}{45} [E_4, E_4]_2 + 0 \cdot D([E_4, E_4]_1) + \frac{10}{45} D^2([E_4, E_4]_0),
\]

\[
D(E_6) \cdot E_8 = \frac{-1}{14} [E_6, E_8]_1 + \frac{3}{7} D([E_6, E_8]_0),
\]

which can not be eigenforms because of the fact that there are multiple terms of different weights.

4.5.2 Proof of Theorem 4.4.2

To prove the theorem, we recall a result given in ([6], Proposition 20 (iii)).

Lemma 4.5.10. If \(p < k/2\), then \(M_k \subseteq_p \bigoplus_{r=0}^p D^r(M_{k-2r})\).

Now, if \(f \in M_k \subseteq_p\) and \(p < k/2\), then by the above lemma, we have

\[
f = \sum_{r=0}^p D^r(f_r), \quad \text{for } f_r \in M_{k-2r}.
\]

By hypothesis, \(f\) is an eigenform. Therefore, applying Proposition 4.5.4 and Corollary 4.5.6, we conclude that \(f = D^r(f_r)\) for some \(r\). Similarly, we get \(g = D^s(g_s)\) for some \(g_s \in M_{l-2s}\). Now applying Theorem 4.4.1, we get the desired result.

4.5.3 Proof of Theorem 4.4.3

We first prove the following proposition.
Proposition 4.5.11. Let \( f \in M_k \) be an eigenform. Then \( E_2 f \) is an eigenform if and only if \( f \in \mathbb{C} \Delta_{12} \).

Proof. Since \( D \Delta_{12} = E_2 \Delta_{12} \), by Proposition 4.5.3, \( E_2 \Delta_{12} \) is an eigenform.

Conversely, suppose that \( E_2 f \) is an eigenform with eigenvalues \( \beta_n \) and

\[
f = \sum_{m \geq 0} a_m q^m \in M_k
\]

is an eigenform with eigenvalues \( \lambda_n \). Since \( g = Df - \frac{k}{12} E_2 f \in M_{k+2} \), we have

\[
T_n(g) = T_n(Df) - \frac{k}{12} T_n(E_2 f) = n\lambda_n Df - \frac{k}{12} n\lambda_n E_2 f + \frac{k}{12} (n\lambda_n - \beta_n) E_2 f \quad \text{for all} \quad n \geq 1.
\]

(4.5.4)

As \( E_2 f \) is not a modular form and \( n\lambda_n g = n\lambda_n (Df - \frac{k}{12} E_2 f) \) is a modular form, we have \( n\lambda_n = \beta_n \) for all \( n \geq 1 \). Thus \( g = Df - \frac{k}{12} E_2 f \in M_{k+2} \) is an eigenform with eigenvalues \( n\lambda_n \). Now suppose that \( f \) is a non cusp eigenform, so we may assume that \( f = E_k \). This implies that \( g \) has to be a non cusp eigenform of weight \( k+2 \) and hence \( g = \alpha E_{k+2} \) for some \( \alpha \in \mathbb{C} \). Then applying \( T_n \) to \( \alpha E_{k+2} = DE_k - \frac{k}{12} E_2 E_k \), we get

\[
T_n(\alpha E_{k+2}) = T_n(DE_k - \frac{k}{12} E_2 E_k)
\]

\[
\Rightarrow \sigma_{k+1}(n) = n\sigma_{k-1}(n).
\]

Thus, we get for all \( n \geq 1 \), \( \sigma_{k+1}(n) = n\sigma_{k-1}(n) \), which is not true. Hence \( f \) can not be a non cusp eigenform.

If \( f \) is a cusp form, without loss of generality we may assume that \( f \) is normalized. Let \( g = \sum_{m \geq 1} b_m q^m \). Since \( b_1 = 1 - \frac{k}{12} \), we have for all \( n \geq 1 \),

\[
b_n = na_n \left( 1 - \frac{k}{12} \right). \quad (4.5.5)
\]

Now computing the values of \( b_n \) from \( Df - \frac{k}{12} E_2 f \) in terms of \( a_n \) and then substituting in the previous equation, we see that \( a_2 = -24 \), \( a_3 = 252 \) and \( a_4 = -1472 \). These are nothing but the second, third and fourth Fourier coefficients of \( \Delta_{12} \) respectively. But Theorem 1 of [12] says that if \( f_1 = \sum_{n \geq 1} a_n(f_1) q^n \) and \( f_2 = \sum_{n \geq 1} a_n(f_2) q^n \) are two cuspidal eigenforms on \( \Gamma_0(N) \) of different weights, then there exists \( n \leq 4(\log(N) + 1)^2 \) such that \( a_n(f_1) \neq a_n(f_2) \). Applying this theorem to \( f_1 = f, f_2 = \Delta_{12} \) and \( N = 1 \), we conclude that \( k = 12 \). Thus we have \( f = \Delta_{12} \).

Remark 4.5.1. Since \( DE_2 = \frac{E_2^3 - E_4}{12} \) and \( DE_2, E_4 \) are eigenforms with different eigenvalues, \( E_2^3 \) is not an eigenform.
Now, we are ready to prove Theorem 4.4.3. By Lemma 4.5.8, Proposition 4.5.11 and Remark 4.5.1, we only have to prove in the following cases that it is not an eigenform.

1. \( r = 0 \) and \( s \geq 1 \),
2. \( r \geq 1 \) and \( s = 0 \).

Let us assume on contrary that \((D^s E_k)E_2\) is an eigenform for \( s \geq 1 \). Let
\[
\frac{-B_k}{2k} (D^s E_k)E_2 = \sum_{n \geq 1} a_n q^n
\]
be the normalized form. The first few coefficients in the Fourier expansion of the normalized form are as follows:

\[
a_1 = 1, \quad a_2 = 2^s \sigma_{k-1}(2) - 24, \\
a_3 = 3^s \sigma_{k-1}(3) - 24\{2^s \sigma_{k-1}(2) + 3\}, \\
a_4 = 4^s \sigma_{k-1}(4) - 24\{3^s \sigma_{k-1}(3) + 3 \cdot 2^s \sigma_{k-1}(2) + 4\}.
\]

Since \(\frac{-B_k}{2k} (D^s E_k)E_2\) is a normalized eigenform, we have
\[
a_4 = a_2^2 - 2^{k+2s+1}.
\]

(4.5.6)

Substituting the values of \(a_2\) and \(a_4\) in the above equation, we get
\[
4^s \sigma_{k-1}(4) - 24\{3^s \sigma_{k-1}(3) + 3 \cdot 2^s \sigma_{k-1}(2) + 4\} = 2^2 \sigma_{k-1}(2)^2 - 48 \cdot 2^s \sigma_{k-1}(2) + 576 - 2^{k+2s+1}
\]

\[
\Rightarrow 24\{3^s(1 + 3^{k-1}) + 3 \cdot 2^s(1 + 2^{k-1}) + 4\} - 48 \cdot 2^s(1 + 2^{k-1}) + 576 - 3 \cdot 2^{k+2s-1} = 0
\]

\[
\Rightarrow 3^s(1 + 3^{k-1}) + 3 \cdot 2^s(1 + 2^{k-1}) + 4 - 2^{s+1}(1 + 2^{k-1}) + 24 - 2^{k+2s-4} = 0
\]

\[
\Rightarrow 3^s(1 + 3^{k-1}) + 2^s(1 + 2^{k-1}) - 2^{k+2s-4} + 28 = 0
\]

(4.5.7)

\[
\Rightarrow 3^s(1 + 3^{k-1}) + 2^s + 28 = 2^{k+s-4}(2^s - 2^3).
\]

(4.5.8)

We also have \(a_6 = a_2 a_3\).

\[
\Rightarrow 6^s \sigma_{k-1}(6) - 24\{5^s \sigma_{k-1}(5) + 3 \cdot 4^s \sigma_{k-1}(4) + 4 \cdot 3^s \sigma_{k-1}(3) + 7 \cdot 2^s \sigma_{k-1}(2) + 6\}
\]

\[
= (2^s \sigma_{k-1}(2) - 24)\{3^s \sigma_{k-1}(3) - 24(2^s \sigma_{k-1}(2) + 3)\}
\]

\[
\Rightarrow 5^s \sigma_{k-1}(5) + 3 \cdot 4^s \sigma_{k-1}(4) + 4 \cdot 3^s \sigma_{k-1}(3) + 7 \cdot 2^s \sigma_{k-1}(2) + 6 - 3^s \sigma_{k-1}(3)
\]

\[
-2^s \sigma_{k-1}(2)(2^s \sigma_{k-1}(2) + 3) + 24(2^s \sigma_{k-1}(2) + 3) = 0
\]
\[ 5s \sigma_{k-1}(5) + 3^{s+1} \sigma_{k-1}(3) + 2^{2s+1} \sigma_{k-1}(2)^2 + 7 \cdot 2^{s+2} \sigma_{k-1}(2) - 3 \cdot 2^{k+2s-1} + 78 = 0. \]  

(4.5.9)

In the above equalities, we have used the fact that \( \sigma_{k-1}(6) = \sigma_{k-1}(2) \sigma_{k-1}(3) \) and \( \sigma_{k-1}(4) = \sigma_{k-1}(2)^2 - 2^{k-1} \).

Now if \( s \leq 3 \), then the left hand side of (4.5.8) is positive, but the right hand side of the equation is non-positive. Thus \( s \geq 4 \). Now from (4.5.7), we have

\[ 3^s \left( \frac{1+3^{k-1}}{4} \right) + 2^{s-2}(1 + 2^{k-1}) + 7 = 2^{k+2s-6}. \]  

(4.5.10)

If \( k \equiv 2 \pmod{4} \) and \( s \) is odd, then \( 7 + 3^s \left( \frac{1+3^{k-1}}{4} \right) \equiv 2 \pmod{4} \), but the remaining terms of (4.5.10) are divisible by 4, giving a contradiction. If \( k \equiv 2 \pmod{4} \) and \( s \equiv 0 \pmod{4} \), then \( 3^s \left( \frac{1+3^{k-1}}{4} \right) + 2^{s-2}(1 + 2^{k-1}) + 7 \equiv 0 \pmod{5} \), but 5 does not divide \( 2^{k+2s-6} \). This gives a contradiction. If \( k \equiv 2 \pmod{4} \) and \( s \equiv 2 \pmod{4} \), then \( 3^{s+1} \sigma_{k-1}(3) + 2^{2s+1} \sigma_{k-1}(2)^2 + 7 \cdot 2^{s+2} \sigma_{k-1}(2) - 3 \cdot 2^{k+2s-1} + 78 \equiv 4 \pmod{5} \), but the remaining term of left hand side of (4.5.9) is divisible by 5, giving a contradiction. If \( k \equiv 0 \pmod{4} \) and \( s \) is even or \( s \equiv 1 \pmod{4} \), then we get a contradiction from (4.5.10) and if \( k \equiv 0 \pmod{4} \) and \( s \equiv 3 \pmod{4} \), we get a contradiction from (4.5.9). This proves the theorem in the first case.

Now, we prove the theorem in the second case. Let us assume on contrary that \((D^rE_2)E_k\) is an eigenform for \( r \geq 1 \). Let \( \frac{1}{24}(D^rE_2)E_k = \sum_{n \geq 1} b_nq^n \) be the normalized eigenform. We have the first few coefficients of the expansion as follows:

\[ b_1 = 1, \ b_2 = 3 \cdot 2^r - \frac{2k}{B_k}, \]

\[ b_3 = 4 \cdot 3^r - \frac{2k}{B_k}(3 \cdot 2^r + \sigma_{k-1}(2)), \]

\[ b_4 = 7 \cdot 4^r - \frac{2k}{B_k}(4 \cdot 3^r + 3 \cdot 2^r \sigma_{k-1}(2) + \sigma_{k-1}(3)). \]

Since \( \frac{1}{24}(D^rE_2)E_k \) is a normalized eigenform, we have

\[ b_4 = b_2^2 - 2^{k+2r+1}. \]  

(4.5.11)

Substituting the values of \( b_2 \) and \( b_4 \) in the above equation, we get

\[ 7 \cdot 4^r - \frac{2k}{B_k}(4 \cdot 3^r + 3 \cdot 2^r \sigma_{k-1}(2) + \sigma_{k-1}(3)) = \left( 3 \cdot 2^r - \frac{2k}{B_k} \right)^2 - 2^{k+2r+1} \]

\[ \Rightarrow \left( \frac{2k}{B_k} \right)^2 + \frac{2k}{B_k}(4 \cdot 3^r + 3 \cdot 2^r \sigma_{k-1}(2) + \sigma_{k-1}(3) - 3 \cdot 2^{r+1}) + 3^2 \cdot 2^{2r} - 7 \cdot 2^{2r} - 2^{k+2r+1} = 0. \]
\[
\Rightarrow \left( \frac{2k}{B_k} \right)^2 + \frac{2k}{B_k} \left( 4 \cdot 3^r + 3 \cdot 2^r (1 + 2^{k-1}) + 1 + 3^{k-1} - 3 \cdot 2^{r+1} \right) + 2^{2r+1}(1 - 2^k) = 0
\]

\[
\Rightarrow \left( \frac{2k}{B_k} \right)^2 + \frac{2k}{B_k} \left( 4 \cdot 3^r + 3 \cdot 2^r (2^{k-1} - 1) + 1 + 3^{k-1} \right) + 2^{2r+1}(1 - 2^k) = 0
\]

\[
\Rightarrow \frac{2k}{B_k} = \frac{-b \pm \sqrt{b^2 + 2^{2r+3}(2^k - 1)}}{2}, \quad (4.5.12)
\]

where

\[
b = 4 \cdot 3^r + 3 \cdot 2^r (2^{k-1} - 1) + 1 + 3^{k-1}. \quad (4.5.13)
\]

Since \( \frac{2k}{B_k} \) is a rational number, \( b^2 + 2^{2r+3}(2^k - 1) \) is a perfect square. Again, since 2 divides \( b \), \( \frac{2k}{B_k} \) is an integer. This implies that \( k \in \{2, 4, 6, 8, 10, 14\} \). Since the case when \( k = 2 \) is considered in the earlier case, we have to consider the cases \( k \in \{4, 6, 8, 10, 14\} \).

\( k = 4 \)

In this case, \( \frac{2k}{B_k} = -240 \). Since \( \frac{2k}{B_k} \) is negative, from (4.5.12), we get

\[
-b - \sqrt{b^2 + 2^{2r+3}(2^4 - 1)} = -480
\]

\[
\Rightarrow b^2 + 15 \cdot 2^{2r+3} = (b - 480)^2
\]

\[
\Rightarrow b = 240 - 2^{2r-3}.
\]

Substituting this value of \( b \) in (4.5.13), we get

\[
2^{2r-3} + 4 \cdot 3^r + 21 \cdot 2^r - 212 = 0. \quad (4.5.14)
\]

Now, we see that (4.5.14) is not satisfied for any positive integer \( r \), giving a contradiction.

\( k = 6 \)

Here \( \frac{2k}{B_k} = 504 \). Since \( \frac{2k}{B_k} \) is positive in this case, from (4.5.12), we get

\[
-b + \sqrt{b^2 + 2^{2r+3}(2^6 - 1)} = 1008
\]

\[
\Rightarrow b = 2^{2r-2} - 504.
\]

Substituting this value of \( b \) in (4.5.13), we get

\[
4^{r-1} - 4 \cdot 3^r - 93 \cdot 2^r - 748 = 0. \quad (4.5.15)
\]

Since the left hand side of the above equation is positive for \( r \geq 11 \) and is non-zero for any positive integer \( r < 11 \), we get a contradiction.

\( k = 8 \)

Since \( \frac{2k}{B_k} = -480 \) is negative in this case, we have from (4.5.12)

\[
-b - \sqrt{b^2 + 2^{2r+3}(2^8 - 1)} = -960
\]

\[
\Rightarrow b = 480 - 17 \cdot 2^{2r-4}.
\]

Then from (4.5.13), we get

\[
17 \cdot 2^{2r-4} + 4 \cdot 3^r + 3 \cdot 2^r (2^7 - 1) + 3^7 + 1 = 480. \quad (4.5.16)
\]
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Now, the left hand side of the above equation is greater than 480 for any integer \( r \geq 1 \).
Thus, this case is excluded.

\( k = 10 \)

Here we have \( \frac{2k}{B_k} = 264 \) is positive. Then from (4.5.12), we have
\[
-b + \sqrt{b^2 + 2^{2r+3}(2^{10} - 1)} = 528
\]
\[
\Rightarrow b = 31 \cdot 2^{2r-2} - 264.
\]
Then from (4.5.13), we get
\[
31 \cdot 4^{r-1} - 4 \cdot 3^r - 3 \cdot 2^r (2^9 - 1) - (3^9 + 265) = 0. \tag{4.5.17}
\]
Here the left hand side of the above equation is positive for any integer \( r \geq 8 \) and is non-zero for any positive integer \( r < 8 \). Thus, this case is also excluded.

\( k = 14 \)

Here \( \frac{2k}{B_k} = 24 \) is positive. Then from (4.5.12), we have
\[
-b + \sqrt{b^2 + 2^{2r+3}(2^{14} - 1)} = 48
\]
\[
\Rightarrow b = 5161 \cdot 2^{2r-2} - 24.
\]
Substituting this value of \( b \) in (4.5.13), we get
\[
5161 \cdot 4^{r-1} - 4 \cdot 3^r - 3 \cdot 2^r (2^{14} - 1) - (3^{13} + 25) = 0. \tag{4.5.18}
\]
Here the left hand side of the above equation is positive for any integer \( r \geq 6 \) and is non-zero for any positive integer \( r < 6 \). Thus, this case is also excluded. This concludes the proof of the theorem.

4.5.4 Proof of Corollary 4.4.4

Applying Theorem 4.4.3, we get \( D^r(E_2)E_k \) is not an eigenform. If \( f \) is a cusp form, then for \( r \geq 1 \), \( D^r(E_2)f \) has Fourier coefficients starting from \( q^2 \). So in this case also it is not an eigenform by Lemma 4.5.8. Now applying Proposition 4.5.11, we get the required result.

4.5.5 Proof of Theorem 4.4.5

We recall the following result given in [24] (Proposition 1(ii)).

Lemma 4.5.12. \((\text{Lanphier, [24]})\) If \( g \in M_{k_1}(N, \chi) \) and \( h \in M_{k_2}(N, \psi) \) and \( k = k_1 + k_2 + 2m \), then \( [g, h]_m|_k W_Q = [g|_{k_1} W_Q, h|_{k_2} W_Q] \).
Now, assume on the contrary that
\[ [g, h]_m = f \] (4.5.19)
is an a.e. eigenform. Then \( f(z) = \alpha f_0(Qz) \), where \( M|N, Q|(N/M) \), \( \alpha \in \mathbb{C}^* \) and \( f_0 \in S_k(M, \chi) \) is a normalized newform. Since \( N \) is square-free, for any divisor \( Q \) of \( N \), \( (Q, N/Q) = 1 \). We also compute that
\[
\begin{align*}
f|_k W_Q &= Q^{k/2}(Nvz + Qw)^{-k} f \left( \frac{Qxz + y}{Nvz + Qw} \right) \\
&= \alpha Q^{-k/2}(Nvz/Q + Qw)^{-k} f_0 \left( \frac{Qxz + y}{Nvz/Q + w} \right) \\
&= \alpha \chi(w) Q^{-k/2} f_0(z),
\end{align*}
\]
since \( \begin{pmatrix} Qx \\ Nv/Q \\ y \\ w \end{pmatrix} \in \Gamma_0(M) \).

Applying the operator \( W_Q \) to (4.5.19) and by Lemma 4.5.12, we get \([g|_k W_Q, h|_k W_Q]_m = f|_k W_Q = \text{const. } f_0\). This gives a contradiction since the Fourier expansion of \( f_0 \) (being primitive) starts with \( q \), whereas the Fourier expansion of \([g|_k W_Q, h|_k W_Q]_m \) starts with at least \( q^2 \), a contradiction. This proves the theorem.

### 4.5.6 Proof of Theorem 4.4.6

We recall Proposition 6 of [51].

**Proposition 4.5.13.** (Zagier, [51]) Let \( k_1, k_2, m \) be integers satisfying \( k_2 \geq k_1 + 2 > 2 \) and let \( k = k_1 + k_2 + 2m \). If \( f(z) = \sum_{n=1}^{\infty} a_n q^n \in S_k(N, \chi') \) and \( g(z) = \sum_{n=0}^{\infty} b_n q^n \in M_{k_1}(N, \chi) \), then
\[
\langle f, [g, E_{k_2}^{(N, \psi)}]_m \rangle = \frac{\Gamma(k - 1) \Gamma(k_2 + m)}{(4\pi)^{k-1} m \Gamma(k_2)} \sum_{n=1}^{\infty} \frac{a_n \overline{b_n}}{n^{k_1+k_2+m-1}},
\]
where \( \langle \cdot, \cdot \rangle \) is the Petersson inner product and \( E_{k_2}^{(N, \psi)} \) is the Eisenstein series defined by (4.2.5).

Next, we recall Proposition 1 of [11], which gives the action of the \( W_Q \) operator on a.e. eigenforms belonging to \( \mathcal{E}_k(N, \psi) \).
Proposition 4.5.14. (Ghate, [11]) Let $k \geq 3$. Let $f_k(z, \psi_1, \psi_2) \in \mathcal{E}_k(N, \psi)$ be an a.e. eigenform as described in Theorem 4.2.1. Let $Q_1 = (Q, M_1)$, $Q_2 = (Q, M_2)$. Let $\psi_1 = \psi_{Q_1} \psi_{M_1/Q_1}$ and $\psi_2 = \psi_{Q_2} \psi_{M_2/Q_2}$ be the decompositions of $\psi_1$ and $\psi_2$ into their $Q$ and prime to $Q$ parts respectively. Set
\[
\alpha = \frac{\psi_{Q_2}(-M_2/Q_2)\psi_{M_2/Q_2}(Q_2)}{\psi_{Q_1}(M_2/Q_2)\psi_{M_1/Q_1}(Q_2)},
\]
then
\[
f_k(z, \psi_1, \psi_2)|_k W_Q = \alpha Q^{k/2} Q_2^{-k} f_k\left(\frac{Qz}{Q_1 Q_2}, \psi_{Q_2} \psi_{M_1/Q_1}; \psi_{Q_1} \psi_{M_2/Q_2}\right).
\]

The Eisenstein series $E_{k_1}^{(N, \psi)}$ are related to the a.e. eigenforms given in Theorem 4.2.1 in the following way. (See Lemma 1 of [11]).

Lemma 4.5.15. (Ghate, [11]) For $k \geq 3$, we have
\[
E_{k_1}^{(N, \psi)}(z) = \frac{-2k}{B_{k, \psi}} f_k(Qz, \psi_0, \psi_2),
\]
where $\psi_0$ is the principal character of level $M_1 = 1$, $\psi_2$ is the primitive character of conductor $M_2$ associated to $\psi$ and $Q = N/M_2$.

We now prove the following propositions.

Proposition 4.5.16. Let $k$, $k_1$, $k_2$, $m$ be positive integers satisfying $k_2 \geq k_1 + 2 > 2$ and $k = k_1 + k_2 + 2m$. Let $g \in S_{k_1}(N, \chi)$ be an a.e. eigenform which is a newform and $h = E_{k_1}^{(N, \psi)} \in \mathcal{E}_{k_1}(N, \psi)$. If $\dim(S_{k_1}^{\text{new}}(N, \chi)) \geq 2$, then $[g, h]_m$ is not an a.e. eigenform.

Proof. Assume on the contrary that $[g, h]_m \in S_k(N, \chi)$ is an a.e. eigenform. Since $\dim(S_{k_1}^{\text{new}}(N, \chi)) \geq 2$, there exists a newform $f \in S_{k_1}^{\text{new}}(N, \chi)$ which is orthogonal to $[g, h]_m$ with respect to Petersson inner product. By Theorem 4.5.13, we also have
\[
\langle f, [g, E_{k_2}^{(N, \psi)}]_m \rangle = \frac{\Gamma(k-1) \Gamma(k_2 + m)}{(4\pi)^{k-1} m! \Gamma(k_2)} \sum_{n=1}^{\infty} \frac{a_n b_n}{n^{k_1 + k_2 + m - 1}}.
\]

Since $f$ is orthogonal to $[g, h]_m = [g, E_{k_2}^{(N, \psi)}]_m$, the left hand side of the above identity is zero. But the series $\sum_{n=1}^{\infty} \frac{a_n b_n}{n^s}$ has an Euler product which is absolutely convergent for $\Re(s) > k_1 + k_2 + m$. Hence, for $\Re(s) > k_1 + k_2 + m$, the series $\sum_{n=1}^{\infty} \frac{a_n b_n}{n^s}$ does not vanish. Since $k_1 > 2$, we have $k_1 + k_2 + m - 1 > k_1 + k_2 + m$. Therefore, $\sum_{n=1}^{\infty} \frac{a_n b_n}{n^{k_1 + k_2 + m - 1}} \neq 0$. This gives a contradiction, since the left hand side is zero, whereas the right hand side is not zero. \(\square\)
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Proposition 4.5.17. Let $k, k_1, k_2, m$ be positive integers satisfying the same conditions as in Proposition 4.5.16. Suppose that $g = f_{k_1}(z, \chi_1, \chi_2)$ is an a.e. eigenform as described in Theorem 4.2.1 with $\chi$ primitive and $h = E^{(N, \psi)}_{k_2}(N, \psi)$. If $\dim(S_{k}^\text{new}(N, \chi_1 \psi)) \geq 2$, then $[g, h]_m$ is not an a.e. eigenform.

Proof. Assume that $[g, h]_m$ is an a.e. eigenform. Since $\dim(S_k^\text{new}(N, \chi_1 \psi)) \geq 2$, there exists a newform $f \in S_k^\text{new}(N, \chi_1 \psi)$ which is orthogonal to $[g, h]_m$ with respect to Petersson inner product. Applying Theorem 4.5.13, we get

$$\langle f, [g, E^{(N, \psi)}_{k_2}]_m \rangle = \Gamma(k - 1)\Gamma(k_2 + m) \sum_{n=1}^{\infty} \frac{a_n \overline{b_n}}{n^{k_1+k_2+m-1}}.$$ 

According to the choice of $f$, the left hand side is zero. We shall show that the right hand side is not zero, i.e., the series on the right hand side does not vanish. Let

$$D(k - m - 1, f, g) = \sum_{n=1}^{\infty} \frac{a_n \overline{b_n}}{n^{k-m-1}}.$$ 

We have from Lemma 1 of [42],

$$D(k - m - 1, f, g) = L(k - m - 1, f, \chi_1)L(k - m - k_1, f, \chi_2)/L(k - k_1, \psi),$$

where $L(k - m - 1, f, \chi_1) = \sum_{n=1}^{\infty} \chi_1(n) \frac{a_n}{n^{k-m-1}}$, $L(k - m - k_1, f, \chi_2) = \sum_{n=1}^{\infty} \chi_2(n) a_n$ and $L(k - k_1, \psi) = \sum_{n=1}^{\infty} \psi(n) n^{-k_1-k}$. The denominator of (4.5.20) is a finite non-zero quantity and each term of the numerator is not zero since $k_2 \geq k_1 + 2$. Thus the right hand side is not zero. This completes the proof.

We are now ready to prove the theorem. In the notation of Theorem 4.2.1, we may write $h = f_{k_2}(Qz, \psi_1, \psi_2)$, where $Q|(N/M_1 M_2)$. Since $N$ is square-free, for any divisor $Q$ of $N$, we have $(Q, N/Q) = 1$. Now applying the $W$-operator $W_{N/Q M_2}$ on $h$ and using Proposition 4.5.14 and Lemma 4.5.15, we get

$$h|_{k_2} W_{N/Q M_2} = c_1 \cdot f_{k_2}(\frac{Nz}{M_1 M_2}, \psi_0, \overline{\psi}_1 \psi_2) = c_2 \cdot E^{(N, \overline{\psi}_1 \psi_2)}_{k_2},$$

where $\psi_0$ is the principal character and $c_1, c_2$ are some constants. Now assume on the contrary that $[g, h]_m$ is an a.e. eigenform. Applying $W_{N/Q M_2}$ to $[g, h]_m$ and using Lemma 4.5.12, we see that

$$[g|_{k_1} W_{N/Q M_2}, h|_{k_2} W_{N/Q M_2}]_m \in S_k(N, \chi Q M_2 \overline{\chi}_{N/Q M_2} \overline{\psi}_1 \psi_2)$$
is an a.e. eigenform. Since the $W$-operator is an isomorphism and it takes newform space to newform space, we have $\dim(S^\text{new}_k(N,\chi_{QM2}\chi_{N/QM2}\overline{\psi_1}\psi_2)) \geq 2$. Now applying Proposition 4.5.16 to $g|_{k_1}W_{N/QM_2} \in S_{k_1}(N,\chi_{QM2}\chi_{N/QM2})$ and $h|_{k_2}W_{N/QM_2} = c_2 \cdot E_{k_2}^{(N,\overline{\psi_1}\psi_2)}$, we conclude that $[g|_{k_1}W_{N/QM_2}, h|_{k_2}W_{N/QM_2}]_m$ is not an a.e. eigenform. This gives a contradiction, proving the first assertion.

If $k_2 - k_1 \geq 2$, then as in the proof of the previous assertion, applying the operator $W_{N/M_2}$ to $h$ and $g$, we get $h|_{k_2}W_{N/M_2} = c_3 \cdot E_{k_2}^{(N,\overline{\psi_1}\psi_2)}$ and $g|_{k_1}W_{N/M_2}$ continues to be a form with primitive character, where $c_3$ is a constant. Then applying Proposition 4.5.17, we get the result. If $k_1 - k_2 \geq 2$, then interchanging the role of $g$ and $h$ gives the required result. This proves the theorem.

### 4.5.7 Proof of Theorem 4.4.7

To prove the theorem, we first recall Proposition 2.5 from [4].

**Proposition 4.5.18.** (Beyerl et al., [4]) Let $f \in M_k$. Then $\delta_k^{(r)}(f)$ is an eigenform for $T_n$ iff $f$ is. In this case, if $\lambda_n$ denotes the eigenvalue of $T_n$ associated to $f$, then the eigenvalue of $T_n$ associated to $\delta_k^{(r)}(f)$ is $n^r \lambda_n$.

For any modular form $f \in M_k$, we have

$$\delta_k(f) - \frac{k}{12}E_2^s f = Df - \frac{k}{12}E_2 f \in M_{k+2}. \quad (4.5.21)$$

Moreover, $\delta_k(f) - \frac{k}{12}E_2^s f$ is a cusp form if $f$ is a cusp form. Then $\delta_{12}(\Delta_{12}) - E_2^s \Delta_{12}$ is a cusp form of weight 14, and since there is no non-zero cusp form of weight 14 on $SL_2(Z)$, we get

$$\delta_{12}(\Delta_{12}) = E_2^s \Delta_{12}.$$

Now applying Proposition 4.5.18, we see that $E_2^s \Delta_{12}$ is an eigenform.

Conversely, assume that $f \in M_k$ is an eigenform such that $E_2^s f$ is an eigenform. Then applying (4.5.21) and proceeding as in the proof of Proposition 4.5.11, we conclude that $f \in \mathbb{C} \Delta_{12}$. This completes the proof of the theorem.