Chapter 2

Generalized Modular Functions

2.1 Introduction

In this chapter we discuss some properties of exponents of $q$-product expansions of certain class of generalized modular functions (GMF) on the Hecke congruence subgroup $\Gamma_0(N)$. The results of this chapter are contained in a joint work with W. Kohnen [22].

Let $f$ be a non-zero GMF. Then by a theorem of Eholzer and Skoruppa [9], each GMF $f$ has a product expansion

$$f(z) = cq^h \prod_{n\geq 1} (1 - q^n)^{c(n)} \quad (0 < |q| < \epsilon), \quad q = e^{2\pi iz}, \quad z \in \mathcal{H},$$

where $h \in \mathbb{Z}$ and $c, c(n)(n \geq 1)$ are uniquely determined complex numbers.

It was proved in [21] that for each square-free integer $N \geq 11$, one can find a GMF $f$ on $\Gamma_0(N)$ such that $f$ has no zeros on $\mathcal{H}$ and the $q$-exponents $c(n) (n \geq 1)$ take infinitely many different values. This result and the proof given are actually easily seen to be valid for arbitrary integer $N \geq 11$. Kohnen proved in [19] that for any non-constant GMF $f$ with empty divisor, the $q$-exponents $c(n) (n \geq 1)$ take infinitely many different values. The first result of the chapter sharpens the above statement under certain conditions on $f$. Let $\text{div}(f)$ denote the divisor of $f$, i.e., the set of zeros and poles of $f$ in $\mathcal{H}$ and at all cusps. Our second result shows that under the hypothesis that the divisor of $f$ empty, $c(n) (n \geq 1)$ change signs infinitely often, provided that $c(n)$ are real numbers.
2.2 Preliminaries

Definition 2.2.1. (Generalized modular function) A generalized modular function $f$ on $\Gamma_0(N)$ is a holomorphic function on $\mathcal{H}$, meromorphic at cusps such that

$$f\left(\frac{az+b}{cz+d}\right) = \chi(\gamma)f(z),$$

for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$, where $\chi : \Gamma_0(N) \to \mathbb{C}^\times$ is a homomorphism with $\chi(\gamma) = 1$ for $\gamma$ parabolic with trace = 2.

M. I. Knopp and G. Mason [15] introduced the notion of a generalized modular function. We abbreviate GMF for a generalized modular function. For further details on GMF, we refer to [15].

Let $g = \sum_{n \geq 1} b(n)q^n$ be a cusp form on $\Gamma_0(N)$. Then the $L$-series associated to this cusp form is given by

$$L(g, s) = \sum_{n \geq 1} \frac{b(n)}{n^s}.$$

It is known that $L(g, s)$ can be analytically continued over the whole $s$-plane to an entire function.

2.3 Overview of Earlier Works

Theorem 2.3.1. (Knopp-Mason, [15]) If $f$ is a GMF on $\Gamma_0(N)$, then $g = \frac{1}{2\pi i} \frac{f'}{f}$ is a modular function of weight 2 with trivial character. If the GMF $f$ doesn’t have any zero or pole in $\mathcal{H}$ as well as at cusps, then $g$ is a cusp form. Conversely, if $g$ is a cusp form of weight 2 on $\Gamma_0(N)$, then there exists a GMF $f$ on $\Gamma_0(N)$ such that $g = \frac{1}{2\pi i} \frac{f'}{f}$ and $f$ is uniquely determined up to multiplication with non-zero scalars.

Remark 2.3.1. Since $g = \frac{1}{2\pi i} \frac{f'}{f}$, the exponents of $q$-expansion of $f$ and the Fourier coefficients of $g$ are related. In fact, if $g(z) = \sum_{n \geq 1} b(n)q^n$ is a cusp form, then

$$b(n) = -\sum_{d|n} dc(d), \quad (n \geq 1). \quad (2.3.1)$$

Regarding the properties of $c(n)$, we have a theorem of Kohnen and Martin, which is the following.
Theorem 2.3.2. (Kohnen-Martin, [21]) For each square-free integer $N \geq 11$, there exists a GMF $f$ on $\Gamma_0(N)$ with $\text{div}(f) = \emptyset$ such that the exponents $c(n)$ $(n \geq 1)$ take infinitely many different values.

Remark 2.3.2. The result and the proof given in [21] are actually easily seen to be valid for any integer $N \geq 11$ and to hold for any non-constant $f$, if one exploits the fact proved in [15] that GMFs $f$ with $\text{div}(f) = \emptyset$ correspond to cusp forms of weight 2 by taking logarithmic derivatives.

More generally, Kohnen [17] has proved the following result for any GMF with empty divisor.

Theorem 2.3.3. (Kohnen, [17]) For any non-constant GMF $f$ on $\Gamma_0(N)$ with $\text{div}(f) = \emptyset$, $c(n)$ take infinitely many different values.

2.4 Main Results

To state our results, we define certain operators on a GMF. Let $f$ be a GMF on $\Gamma_0(N)$ and $M | N$. Assume that $\mathcal{R} = \{\gamma_1, \ldots, \gamma_r\}$ is a set of representatives for $\Gamma_0(N)$ modulo $\Gamma_0(M)$, then we define a “norm” of $f$ w.r.t. $\mathcal{R}$ by

$$\mathcal{N}_{\mathcal{R},N}^M(f) := \prod_{\nu=1}^r f|_0 \gamma_\nu.$$  \hfill (2.4.1)

Lemma 2.4.1. Let $f$ be a GMF on $\Gamma_0(N)$ and $M | N$. If we have two sets of representatives $\mathcal{R}$ and $\mathcal{R}'$ for $\Gamma_0(N) \setminus \Gamma_0(M)$, then $\mathcal{N}_{\mathcal{R},N}^M(f)$ and $\mathcal{N}_{\mathcal{R}',N}^M(f)$ differ by a non-zero scalar. Further, for any set of representatives $\mathcal{R}$, $\mathcal{N}_{\mathcal{R},N}^M(f)$ is a GMF.

Proof. Let $\mathcal{R} = \{\gamma_1, \ldots, \gamma_r\}$ and $\mathcal{R}' = \{\alpha_1, \ldots, \alpha_r\}$ be two sets of representatives for $\Gamma_0(N) \setminus \Gamma_0(M)$. Then for each $\gamma_i$ there exists a unique $\alpha_j$ such that $\gamma_i \alpha_j^{-1} \in \Gamma_0(N)$. Since $f$ is a GMF on $\Gamma_0(N)$, we get

$$f|_0 (\gamma_i \alpha_j^{-1}) = \chi(\gamma_i \alpha_j^{-1}) f,$$

where $\chi$ is the character associated to the GMF $f$. Thus,

$$f|_0 \gamma_i = \chi(\gamma_i \alpha_j^{-1}) f|_0 \alpha_j,$$

this implies

$$\prod_{i=1}^r f|_0 \gamma_i = \prod_{j=1}^r \chi(\gamma_i \alpha_j^{-1}) f|_0 \alpha_j.$$

\[ N_{R,N}^M(f) = \prod_{j=1}^r \chi(\gamma_j \alpha_j^{-1}) N_{R',N}^M(f). \]

This proves the first assertion. The second assertion follows from the first, since if \( \{\gamma_1, \ldots, \gamma_r\} \) is a set of representatives for \( \Gamma_0(N) \backslash \Gamma_0(M) \), then for any \( \gamma \in \Gamma_0(M) \), \( \{\gamma \gamma_1, \ldots, \gamma \gamma_r\} \) is another set of representatives for \( \Gamma_0(N) \backslash \Gamma_0(M) \).

The “trace” operator on the space of cusp forms \( S_k(N) \) is defined as

\[ F|Tr^M_N = \sum_{\nu=1}^r F|k\gamma_\nu. \tag{2.4.2} \]

The trace operator maps \( S_k(N) \) to \( S_k(M) \).

Remark 2.4.1. The norm and trace operators have a relation. Let \( g \in S_2(N) \) be the cusp form corresponding to a GMF \( f \), i.e., \( g = \frac{1}{2\pi i} f' \). Then, we have

\[ \frac{1}{2\pi i} N_{R,N}^M(f) = \frac{1}{2\pi i} \sum_{\nu=1}^r \left( \frac{f|_0 \gamma_\nu}{f|0} \right)' = \sum_{\nu=1}^r \frac{1}{2\pi i} \left( \frac{f'}{f} \right)|2\gamma_\nu = \sum_{\nu=1}^r g|2\gamma_\nu = g|Tr^M_N. \tag{2.4.3} \]

With respect to the trace operator, one has a nice characterization of the space of newforms \( S^\text{new}_k(N) \) obtained by A. P. Ogg. Below we give a version from [26].

**Theorem 2.4.2.** (Li, [26]) If \( F \in S_k(N) \), then \( F \in S^\text{new}_k(N) \) iff for all primes \( p|N \), we have \( F|Tr^N_N/f = 0 = (F|kH_N)|Tr^N_N/f \), where \( H_N \) is the Fricke involution defined by, \( z \mapsto \frac{-1}{Nz} \).

**Remark 2.4.2.** Let \( f \) be a GMF on \( \Gamma_0(N) \). Since \( H_N \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & -c/N \\ bN & a \end{pmatrix} H_N \), for \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) \), it follows that \( f|_0 H_N \) is also a GMF on \( \Gamma_0(N) \).

We now state the main results of this chapter.

**Theorem 2.4.3.** Let \( f \) be a non-constant GMF on \( \Gamma_0(N) \) with \( \text{div}(f) = \emptyset \) and assume that \( f \) has algebraic Fourier coefficients. Suppose further that \( N_{R,N}^M(f) \) and \( N_{R,N}^M(f|_0 H_N) \) are constants for every divisor \( M|N, M \neq N \). Then the exponents \( c(p) \) (\( p \) prime) take infinitely many different values.

**Theorem 2.4.4.** Let \( f \) be a non-constant GMF on \( \Gamma_0(N) \) with \( \text{div}(f) = \emptyset \) and suppose that the \( c(n) \) \((n \geq 1)\) are real. Then the \( c(n) \) \((n \geq 1)\) change signs infinitely often.
2.5 Proofs

2.5.1 Proof of Theorem 2.4.3

By assumption \( f \) has algebraic Fourier coefficients. Therefore, the same is true for \( g = \frac{1}{2\pi i} f' f \). Then, by bounded denominator argument, there exists an integer \( A \in \mathbb{N} \) such that the Fourier coefficients of \( Ag \) are algebraic integers. Hence replacing \( f \) by \( f^A \) we may assume without loss of generality that \( g \) has integral algebraic Fourier coefficients. By hypothesis, \( N^M_{K,N}(f) \) and \( N^M_{K,N}(f|_0 H_N) \) are constants for every divisor \( M|N, M \neq N \). This implies that \( g|Tr^M_N \) and \( (g|_2 H_N)|Tr^M_N \) are zero for each \( M|N, M \neq N \). Hence, by Theorem 2.4.2, we conclude that \( g \in S^\text{new}_2(N) \). Let us write the Fourier expansion of \( g \) as

\[
g(z) = \sum_{n \geq 1} b(n) q^n.
\]

Since \( g = \frac{1}{2\pi i} f' f \), we get the following relation.

\[
b(n) = -\sum_{d|n} dc(d), \quad (n \geq 1).
\]

In particular, for each prime \( p \), we have the relation

\[
b(p) = -c(1) - pc(p).
\]

Now assume on contrary that \( c(p) \) take finitely many different values. Then using the Deligne’s bound

\[
b(p) \ll_g \sqrt{p}
\]

for the Fourier coefficients \( b(p) \) of \( g \), where the constant depends only on \( g \), we get

\[
-c(1) - pc(p) \ll_g \sqrt{p}.
\]

Thus, for sufficiently large prime \( p \), the above relation implies that \( c(p) = 0 \). Hence, for sufficiently large prime \( p \), we get the following relation.

\[
b(p) = -c(1) = b(1). \tag{2.5.1}
\]

On the other hand, since \( 0 \neq g \in S^\text{new}_2(N) \) has integral algebraic Fourier coefficients, by a result of Ono and Skinner (Lemma on p. 459, [39]), there exists a positive proportion of primes \( p \) with \( b(p) \neq \alpha \), for any fixed algebraic integer \( \alpha \). This gives a contradiction to (2.5.1). This proves our theorem.
2.5.2 Proof of Theorem 2.4.4

To prove this theorem, we first recall Landau’s theorem on Dirichlet series with non-negative coefficients. See (Theorem 11.13, [1]) for more details.

**Theorem 2.5.1. (Landau)** Let \( h(s) \) be represented in the half-plane \( \sigma = \text{Re}(s) > c \) by the Dirichlet series

\[
h(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s},
\]

where \( c \) is finite, and assume that \( a(n) \geq 0 \) for all \( n \geq n_0 \), for some \( n_0 \). If the Dirichlet series has a finite abscissa of convergence \( \sigma_c \), then \( h(s) \) has a singularity on the real axis at the point \( s = \sigma_c \).

Now we recall a Lemma of Kohnen.

**Lemma 2.5.2. (Kohnen, [20])** Suppose that \( 0 \neq F \in S_k(N) \). Then the abscissa of absolute convergence of the \( L \)-series \( L(F, s) \) associated to \( F \) is \( \frac{k+1}{2} \).

We are now ready to prove Theorem 2.4.4. Since \( f \) is not constant, \( g \neq 0 \). The identity between the exponents \( c(n) \) of the product expansion of \( f \) and the Fourier coefficients \( b(n) \) of \( g \) can be rewritten as an identity between Dirichlet series

\[
L(g, s) = -\zeta(s) \sum_{n \geq 1} \frac{c(n)}{n^{s-1}}, \quad \sigma = \text{Re}(s) > \frac{3}{2}.
\]  

Let us assume on the contrary that \( c(n) \geq 0 \) for almost all \( n \). Since \( L(g, s) \) extends to an entire function and \( \zeta(s) \) is holomorphic for \( \sigma > 1 \), by Theorem 2.5.1, the series

\[
\sum_{n \geq 1} \frac{c(n)}{n^{s-1}}
\]

converges in the range \( \sigma > 1 \). Further, almost all of its coefficients are non-negative by hypothesis. Therefore the convergence in this range must be absolute. From (2.5.2) we therefore conclude that \( L(g, s) \) is absolutely convergent for \( \sigma > 1 \). This is a contradiction, since the abscissa of absolute convergence of \( L(g, s) \) is exactly \( 3/2 \) by Lemma 2.5.2. This completes the proof.