Chapter 2

Unified common fixed point theorems in symmetric spaces

2.1 Introduction

In 1986, G. Jungck [86] generalized the notion of weakly commuting pairs of mappings (due to Sessa [152]) by introducing compatible pair and also showed that compatible mappings commute at their coincidence points. Since then many interesting common fixed point theorems for compatible mappings satisfying contractive type conditions have been established by various researchers.

However, the study of common fixed points of non-compatible pairs of mappings is also equally interesting. Pant [124] initiated the study of non-compatible mappings and the notion of pointwise $R$-weakly commuting pairs. Using these concepts Pant [125] proved some interesting fixed point theorems for mappings satisfying Lipschitz type or non-contractive type conditions. Further, the results of Pant [125] were generalized and improved by Sastry et al. [147] (see also [155]) employing the notions of tangential mappings (or the property (E.A)) and $g$-continuity. In [53], Gopal et al. have made an attempt to generalize Pant’s (cf. [125]) results by introducing a new notion of absorbing pairs.

For the sake of completeness in our presentation, we recall the following definitions wherein a pair of self-mappings $(f, g)$ defined on a symmetric (semi-metric) space $(X, d)$ is said to be

(i) compatible (cf. [86]) if $\lim_{n \to \infty} d(fgx_n, gfx_n) = 0$ whenever $\{x_n\}$ is a sequence such that $\lim_{n \to \infty} fx_n = \lim_{n \to \infty} gx_n = t$ for some $t$ in $X$,

(ii) tangential (or the property (E.A) (cf. [1, 147]) if there exists a sequence $\{x_n\}$ in $X$ and some $t \in X$ such that $\lim_{n \to \infty} fx_n = \lim_{n \to \infty} gx_n = t$,

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(iii) partially commuting (or weakly compatible or coincidentally commuting (cf. [87])) if pair commutes on the set of coincidence points,

(iv) $g$-absorbing if there exists some real number $R > 0$ such that $d(gx, gf x) \leq Rd(fx, gx)$ for all $x$ in $X$. Analogously, $g$ will be called $f$-absorbing (cf. [53]) if there exists some real number $R > 0$ such that $d(fx, fg x) \leq Rd(fx, gx)$ for all $x$ in $X$. The pair of self mappings $(f, g)$ will be called absorbing if it both $g$-absorbing as well as $f$-absorbing and

(v) pointwise $g$-absorbing if for given $x$ in $X$, there exists some $R > 0$ such that $d(gx, gf x) \leq Rd(fx, gx)$.

On similar lines, we can define pointwise $f$-absorbing mapping. If we take $g = I$, the identity mapping on $X$, then $f$ is trivially $I$-absorbing. Similarly $I$ is $f$-absorbing in respect of any $f$. It has been shown in [53] that a pair of compatible or $R$-weakly commuting mappings need not be $g$-absorbing or $f$-absorbing. Also absorbing mappings are neither a subclass of compatible mappings nor a subclass of non-compatible mappings as the absorbing mappings need not commute at their coincidence points. For other properties and related results of absorbing mappings, one can consult [53].

Notice that pointwise $R$-weakly commutativity is equivalent to commutativity at coincidence points whereas compatible mappings are pointwise $R$-commutating as they commute at their coincidence points. Interestingly, the classes of compatible as well as non-compatible mappings are often proper subsets of the class of tangential pairs and this is the motivation to use the tangential property (or the property (E.A)) as opposed to compatibility or non-compatibility.

**Definition 2.1.1.** Let the pair of self-mappings $(f, g)$ defined on a symmetric (semi-metric) space $(X, d)$. Then $f$ is said to be $g$-continuous (cf. [147]) if $gx n \to gx \Rightarrow fx n \to fx$ whenever $\{x n\}$ is a sequence in $X$ with $x \in X$.

For the sake of completeness, we state the following theorems from Pant [125], Sastry and Murthy [147] and Cho et al. [31] respectively.

**Theorem 2.1.1.** (cf. [125]) Let $(f, g)$ be a pair of non-compatible pointwise $R$-weakly commuting self-mappings of a metric space $(X, d)$ satisfying:

(i) $\overline{fX} \subset gX,$

(ii) $d(fx, fy) \leq kd(gx, gy)$, for all $x, y \in X, k \geq 0$, and

(iii) $d(fx, f^2 x) \neq \max\{d(fx, gfx), d(f^2 x, gfx)\}$

whenever the right hand side is non-zero. Then $f$ and $g$ have a common fixed point.

A similar theorem also appears in Pant [126]. The following theorem due to Sastry and Murthy [147] generalizes Theorem 2.1.1.
Theorem 2.1.2. (cf. [147]) If (in the setting of Theorem 2.1.1) \( d(fx, fy) \leq kd(gx, gy) \), for all \( x, y \in X, k \geq 0 \) holds and further

(i) the pair \((f, g)\) is weakly commuting,

(ii) the pair \((f, g)\) is tangential,

(iii) \( f \) is \( g \)-continuous,

(iv) either \( \overline{f(X)} \subset g(X) \) or \( g(X) \) is closed,

then \( f \) and \( g \) have a common fixed point.

In an attempt to offer consolidation to certain results proved for contractive type mappings due to Imdad et al. [70], recently Cho et al. [31] proved two interesting fixed point theorems for nonexpasive type of mappings in symmetric spaces which run as follows:

Theorem 2.1.3. (cf. [31]) Let \( f, g, S \), and \( T \) be self-mappings of a symmetric (semi-metric) space \((X, d)\) where \( d \) satisfies \((W_3)\) and \((HE)\). Suppose that

(i) \( fX \subset TX \) and \( gX \subset SX \),

(ii) the pair \((g, T)\) satisfies the property \((E.A)\) (resp. \((f, S)\) satisfies the property \((E.A)\)),

(iii) \( SX \) is a \( d \)-closed \((\tau(d)\)-closed) subset of \( X \) (resp. \( TX \) is a \( d \)-closed\((\tau(d)\)-closed) subset of \( X \)) and

(iv) for any \( x, y \in X \), \( d(fx, gy) \leq m(x, y) \), where

\[
  m(x, y) = \max \left\{ d(Sx, Ty), \min \{d(fx, Sx), d(gy, Ty)\}, \min \{d(fx, Ty), d(gy, Sx)\} \right\}.
\]

Then, there exist \( u, w \in X \) such that \( fu = Su = gw = Tw \).

Theorem 2.1.4. (cf. [31]) Let \( f, g, S \), and \( T \) be self-mappings of a symmetric (semi-metric) space \((X, d)\) whereas \( d \) enjoys \((1C)\) and \((HE)\). Suppose that

(i) \( fX \subset TX \) and \( gX \subset SX \),

(ii) the pair \((g, T)\) enjoys the property \((E.A)\) (resp. \((f, S)\) enjoys the property \((E.A)\)),

(iii) \( SX \) is a \( d \)-closed \((\tau(d)\)-closed) subset of \( X \) (resp. \( TX \) is a \( d \)-closed\((\tau(d)\)-closed) subset of \( X \)) and

(iv) for any \( x, y \in X \), \( d(fx, gy) \leq m_1(x, y) \), where

\[
  m_1(x, y) = \max \left\{ d(Sx, Ty), \alpha[d(fx, Sx) + d(gy, Ty)], \alpha[d(fx, Ty) + d(gy, Sx)] \right\}
\]

\[ 0 < \alpha < 1. \]
Then, there exist \( u, w \in X \) such that \( fu = Su = gw = Tw \).

The main objective of this chapter is to obtain some results on coincidence and common fixed points without continuity requirements satisfying a slightly more general contractive condition which also admits a non-metric distance function \( d \) with the property that sequence \( \{x_n\} \) converges to \( x \) if and only if \( d(x_n, x) \to 0 \).

We choose symmetric spaces as well as semi-metric spaces as our underlying spaces. In process, some recent results due to Pant [125], Pant [126], Sastry and Murthy [147], Imdad et al. [70], Cho et al. [31] and some others are extended to symmetric (semi-metric) spaces. We further refined these results (Section 2.4) using the idea of absorbing pairs. We conclude this chapter by deriving some related results besides furnishing illustrative examples which exhibit the utility of our results proved in this chapter.

### 2.2 Results via common property (E.A)

In what follows, we utilize the common property (E.A) instead of the property (E.A) to prove our results. Firstly, on the lines of Liu et al. [109], we adopt the following:

**Definition 2.2.1.** Let \( Y \) be an arbitrary set and \( X \) be a non-empty set equipped with a symmetric (semi-metric) \( d \). Then two pairs \( (f, S) \) and \( (g, T) \) of mappings from \( Y \) into \( X \) share the common property (E.A) if there exist two sequences \( \{x_n\} \) and \( \{y_n\} \) in \( Y \) such that

\[
\lim_{n \to \infty} fx_n = \lim_{n \to \infty} Sx_n = \lim_{n \to \infty} gy_n = \lim_{n \to \infty} Ty_n = t
\]

for some \( t \in X \).

We prove our first result using the \( S \)-continuity of \( f \) (or \( T \)-continuity of \( g \)) instead of utilizing Lipschitzian or contractive type condition which runs as follows:

**Theorem 2.2.1.** Let \( Y \) be an arbitrary nonempty set whereas \( X \) be another nonempty set equipped with a symmetric (semi-metric) \( d \) which enjoys \( (W_3) \) (Hausdorffness of \( \tau(d) \)) and (HE). Let \( f, g, S, T : Y \to X \) be four mappings which satisfy the following conditions:

(i) \( f \) is \( S \)-continuous and \( g \) is \( T \)-continuous,

(ii) the pairs \( (f, S) \) and \( (g, T) \) share the common property (E.A),

(iii) \( SY \) and \( TY \) are \( d \)-closed (\( \tau(d) \)-closed) subsets of \( X \) (resp. \( fY \subset TY \) and \( gY \subset SY \)),

then there exist \( u, w \in Y \) such that \( fu = Su = Tw = gw \).

Moreover, if \( Y = X \) along with
(iv) the pairs \((f, S)\) and \((g, T)\) are weakly compatible and

\[ d(fx, gfx) \neq \max \left\{ d(Sx, Tf x), d(gfx, Tf x), d(fx, Tf x), d(fx, Sx), d(gfx, Sx) \right\}, \]

whenever the right hand side is non-zero,

then \(f, g, S,\) and \(T\) have a common fixed point in \(X.\)

**Proof.** Notice that \(Y\) is an arbitrary set but \(fY\) lies in \(X,\) therefore a sequence \(\{fx_n\}\) in a semi-metric space \((X, d)\) converges to a point \(fx\) with respect to \(\tau(d)\) iff \(d(fx_n, fx) \to 0.\) To substantiate this, suppose \(fx_n \to fx\) and let \(\epsilon > 0.\) Since \(B(fx, \epsilon)\) is a neighborhood of \(fx,\) there exists \(U \in \tau(d)\) such that \(fx \in U \subset B(fx, \epsilon).\)

As \(fx_n \to fx,\) one can find a \(m \in \mathbb{N}\) (where \(\mathbb{N}\) stands for the set of natural numbers) such that \(fx_n \in U \subset B(fx, \epsilon)\) for \(n \geq m\) implying thereby \(d(fx_n, fx) < \epsilon\) for \(n \geq m\) i.e. \(d(fx_n, fx) \to 0.\) The converse part is obvious in view of the definition of \(\tau(d).\)

Since the pairs \((f, S)\) and \((g, T)\) share the common property \((E.A),\) there exist two sequence \(\{x_n\}\) and \(\{y_n\}\) in \(Y\) and some \(t \in X\) such that

\[ \lim_{n \to \infty} d(fx_n, t) = \lim_{n \to \infty} d(Sx_n, t) = \lim_{n \to \infty} d(gy_n, t) = \lim_{n \to \infty} d(Ty_n, t) = 0. \]

Since \(SY\) is a \(d\)-closed (or \(\tau(d)\)-closed) subspace of \(X,\) one can find a \(u \in Y\) such that \(Su = t\) which in turn yields that \(\lim_{n \to \infty} d(Sx_n, Su) = 0.\) Now using \(S\)-continuity of \(f\) along with \((W_3)\) and \((HE),\) one finds \(d(fu, Su) = 0\) implying thereby \(fu = Su.\)

Similarly using the \(d\)-closedness of the subspace \(TY\) and \(T\)-continuity of \(g\) along with condition \((W_3)\) and \((HE),\) one can also show that \(d(gw, Tw) = 0\) implying thereby \(gw = Tw\) which in turn yields \(fu = Su = gw = Tw = t.\) Thus both the pairs have a coincidence point.

Now using weak compatibility of the pairs \((f, S)\) and \((g, T),\) we have \(fSu = Sfu,\)

\[ ffu = fSu = Sfu = SSu \quad \text{and} \quad TTw = gT w = T gw = ggw. \]

Now, we assert that \(fu = w.\) Otherwise employing \((v),\) we have

\[ d(fu, ffu) = d(ffu, gw) \]

\[ \neq \max \left\{ d(Sfu, Tw), d(gw, Tw), d(ffu, Tw), d(ffu, Sfu), d(gw, Sfu) \right\} \]

\[ = \max \left\{ d(ffu, fu), 0, d(ffu, fu), 0, d(ffu, fu) \right\} = d(ffu, fu) \]

or

\[ d(fu, ffu) \neq d(ffu, fu) \]

which is a contradiction yielding thereby \(fu = w.\) Similarly, in case \(u \neq gw,\) we again arrive at a contradiction. Thus, \(fu = w = Su = Tw = gw = u,\) and \(w\) is a common fixed point of \(f, g, S,\) and \(T.\)

By restricting \(f, g, S,\) and \(T\) suitably, one can derive corollaries involving two as well as three mappings. Here, it may be pointed out that any result involving three mappings is itself a new result. For the sake of brevity, we opt to mention just one such corollary by restricting Theorem 2.2.1 to a triode of mappings \(f, S,\) and \(T\) which...
is still new and presents yet another sharpened form of Theorem 2.1.2 to symmetric (semi-metric) spaces besides admitting a non-self setting up to coincidence points.

**Corollary 2.2.1.** Let $Y$ be an arbitrary set whereas $(X, d)$ be a symmetric (semi-metric) space equipped with a symmetric (semi-metric) $d$ which enjoys $(W_3)$ (Hausdorffness of $\tau(d)$) and (HE). Let $f, S, T : Y \to X$ be a triode of mappings which satisfy the following conditions:

(i) $f$ is $S$-continuous and $f$ is $T$-continuous,

(ii) the pair $(f, S)$ is tangential and $(f, T)$ is tangential,

(iii) $SY$ and $TY$ are $d$-closed ($\tau(d)$-closed) subsets of $X$ (resp. $fY \subset TY \cap SY$),
then there exist $u, w \in Y$ such that $fu = Su = Tw$.
Moreover, if $Y = X$ along with,

(iv) the pairs $(f, S)$ and $(f, T)$ are weakly compatible and

(v) $d(fx, f^2x) \neq \max \{d(Sx, Tf x), d(f^2x, Tf x), d(fx, Sx), d(fx, Sx)\}$, whenever the right hand side is non-zero.

Then $f, S,$ and $T$ have a common fixed point in $X$.

Our next theorem is essentially inspired by the condition (iv) of Theorem 2.1.3 wherein a nonexpansive type condition is utilized. Here, we employ a corresponding Lipschitzian type generalized condition.

**Theorem 2.2.2.** Let $Y$ be an arbitrary set whereas $(X, d)$ be a symmetric (semi-metric) space equipped with a symmetric (semi-metric) $d$ which enjoys $(W_3)$ (Hausdorffness of $\tau(d)$) and (HE). Let $f, g, S, T : Y \to X$ be four mappings which satisfy the following conditions:

(i) the pair $(g, T)$ satisfies the property (E.A) (resp. $(f, S)$ satisfies the property (E.A)),

(ii) $TY$ is a $d$-closed ($\tau(d)$-closed) subset of $X$ and $gY \subset SY$ (resp. $SY$ is a $d$-closed ($\tau(d)$-closed) subset of $X$ and $fY \subset TY$) and

(iii) $d(fx, gy) \leq km(x, y)$, for any $x, y \in Y$, where $k \geq 0$ and $m(x, y)$ is the same as in Theorem 2.1.3, then there exist $u, w \in Y$ such that $fu = Su = Tw = gw$.
Moreover, if $Y = X$ along with

(iv) the pairs $(f, S)$ and $(g, T)$ are weakly compatible and

(v) $d(fx, gf x) \neq \max \{d(Sx, Tf x), d(gf x, Tf x), d(fx, Tfx), d(fx, Sx), d(gfx, Sx)\}$, whenever the right hand side is non-zero.
Then, $f, g, T,$ and $S$ have a common fixed point.

Proof. Since the pair $(g, T)$ satisfies the property (E.A), there exists a sequence $\{x_n\}$ in $Y$ and a point $t \in X$ such that $\lim_{n \to \infty} d(Tx_n, t) = \lim_{n \to \infty} d(gx_n, t) = 0$. As $SY$ is a $d$-closed ($\tau(d)$-closed) subset of $X$ and $fY \subset TY$, one can always find a sequence $\{y_n\}$ in $Y$ such that $gx_n = Sy_n$ so that $\lim_{n \to \infty} d(Sy_n, t) = 0$. By property (HE), $\lim_{n \to \infty} d(gx_n, Tx_n) = \lim_{n \to \infty} d(Sy_n, Tx_n) = 0$. Since $SY$ is a $d$-closed ($\tau(d)$-closed) subset of $X$ and $fY \subset TY$, there exists a point $u \in Y$ such that $Su = t$. Now using condition (iii), we have

\[ d(fu, gx_n) \leq k \max \left\{ d(Su, Tx_n), \min \{d(fu, Su), d(gx_n, Tx_n)\}, \min \{d(fu, Tx_n), d(gx_n, Su)\} \right\}, \]

which on letting $n \to \infty$, gives rise $\lim_{n \to \infty} d(fu, gx_n) = 0$. Now appealing to ($W_3$), we get $fu = Su$. Since $fY \subset TY$, there exists a point $w \in Y$ such that $fu = Tw$. Now, we show that $Tw = gw$. To accomplish this, using (iii), we have

\[ d(fu, gw) \leq k \max \left\{ d(Su, Tw), \min \{d(fu, Su), d(gw, Tw)\}, \min \{d(fu, Tw), d(gw, Su)\} \right\} = 0, \]

implying thereby $fu = gw$ and hence in all $fu = Su = gw = Tw$ which shows that both the pairs have a coincidence point each.

Now employing weak compatibility of the pairs $(f, S)$ and $(g, T)$, we have $fSu = Sfu$, $ffu = fSu = Sfu = SSu$ and $TTw = gTw = Tgw = ggw$.

If $fu \neq w$, then from (v), we have either

\[ d(fu, ffu) = d(fu, gw) \]

\[ > \max \left\{ d(Sfu, Tw), d(gw, Tw), d(ffu, Tw), d(fsu, Tw), d(gw, Sfu) \right\} = \max \left\{ d(ffu, fu), 0, d(ffu, fu), 0, d(ffu, fu) \right\} = d(ffu, fu) \]

or

\[ d(ffu, fu) = d(ffu, gw) \]

\[ < \max \left\{ d(Sfu, Tw), d(gw, Tw), d(ffu, Tw), d(fsu, Sgu), d(gw, Sfu) \right\} = \max \left\{ d(ffu, fu), 0, d(ffu, fu), 0, d(ffu, fu) \right\} = d(ffu, fu) \]

which gives a contradiction (in both the cases). Similarly, if $u \neq gw$, we again arrive at a contradiction. Thus, $fu = w = Su = Tw = gw = u$, and $w$ is a common fixed point of $f, g, S,$ and $T$. 

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Remark 2.2.1. Choosing $k = 1$ in Theorem 2.2.2, we derive a slightly sharpened form of Theorem 2.1.3 as conditions on the ranges of involved mappings are relatively lightened.

By restricting $f$, $g$, $S$ and $T$ suitably, one can derive corollaries for two as well as three mappings. For the sake of brevity, we derive just one corollary by restricting Theorem 2.2.2 to a triode of mappings which is yet another sharpened and unified form of Theorem 2.1.1 due to Pant [125] (also relevant result in Pant [126]) in symmetric spaces.

Corollary 2.2.2. Let (in the setting of Theorem 2.2.2) $d$ satisfies $(W_3)$ and $(HE)$. If $f, S, T : Y \to X$ are three mappings which satisfy the following conditions:

(i) the pair $(f, S)$ satisfies the property (E.A) (resp. $(f, T)$ satisfies the property (E.A)),

(ii) $SY$ is a $d$-closed ($\tau(d)$-closed) subset of $X$ and $fY \subset TY$ (resp. $TY$ is a $d$-closed ($\tau(d)$-closed) subset of $X$ and $fY \subset SY$) and

(iii) $d(fx, fy) \leq km_2(x, y)$, for any $x, y \in Y$, where $k \geq 0$ and

$$m_2(x, y) = \max \left\{d(Sx, Ty), \min\{d(fx, Sx), d(fy, Ty)\}, \min\{d(fx, Ty), d(fy, Sx)\}\right\},$$

then there exist $u, w \in Y$ such that $fu = Su = Tw$.

Moreover, if $Y = X$ along with

(iv) the pairs $(f, S)$ and $(f, T)$ are weakly compatible and

(v) $d(fx, f^2x) \neq \max \left\{d(Sx, Tf x), d(f^2x, Tf x), d(fx, Tf x), d(fx, Sx), d(f^2x, Sx)\right\}$, whenever the right hand side is non-zero,

then $f$, $g$, $S$ and $T$ have a common fixed point.

Corollary 2.2.3. Let $(X, d)$ be a symmetric (semi metric) space wherein $d$ satisfies $(W_3)$ (Hausdorffness of $\tau(d)$) and $(HE)$. If $f, g, S, T : X \to X$ are four self mappings of $X$ which satisfy the following conditions:

(i) the pair $(f, S)$ satisfies the property (E.A) (resp. $(g, T)$ satisfies the property (E.A)),

(ii) $SX$ is a $d$-closed ($\tau(d)$-closed) subset of $X$ and $fX \subset TX$ (resp. $TX$ is a $d$-closed $\tau(d)$-closed) subset of $X$ and $gX \subset SX$),

(iii) $d(fx, gy) < m(x, y)$, where $m(x, y)$ is nonzero and carries the same meaning as in Theorem 2.1.3, then there exist $u, w \in X$ such that $fu = Su = Tw = gw$.

Also, if

(iv) the pairs $(f, S)$ and $(g, T)$ are weakly compatible, then $f, g, S$ and $T$ have a unique common fixed point.
Proof. Notice that all the conditions of Theorem 2.2.2 are satisfied except (v) besides being $Y = X$. Therefore there exist $u, w \in X$ such that $fu = Su = gw = Tw$ which on using weak compactability of the pairs yields that $ffu = fSu = Sfu = SSu$ and $gTw = Tgw = TTw = ggw$. If $fu \neq w$, then employing (iii), we have $d(fu, ffu) = d(ffu, gw)$

$$< \max \left\{ d(Sfu, Tw), \min \{ (d(gw, Tw), d(ffu, Sfu)) \} \right\}$$

$$= \max \{ d(ffu, fu), 0, d(ffu, fu) \} = d(ffu, fu),$$

which is a contradiction. Thus $fu = ffu = Sfu$ which shows that $fu$ is a common fixed point of $f$ and $S$. Similarly using $u \neq gw$, one can show that $gw$ is a common fixed point of $g$ and $T$. This concludes the proof.

Our next theorem is essentially inspired by the condition (iv) of Theorem 2.1.4.

**Theorem 2.2.3.** Theorem 2.2.2 remains true if $(W_3)$ is replaced by $(1C)$ whereas condition (iii) of Corollary 2.2.3 is replaced by the following condition besides retaining rest of the hypotheses:

$$d(fx, gy) \leq km(x, y),$$

for any $x, y \in X$, where $k \geq 0$ with $k \alpha < 1$ and $m(x, y)$ is the same as in Theorem 2.1.4.

Proof. The proof can be completed on the lines of proof of Theorem 2.2.2, hence details are not included.

By restricting $f, g, S$ and $T$ suitably, one can derive corollaries for two as well as three mappings. For the sake of brevity, we derive just one corollary by restricting Theorem 2.2.3 to a triode of mappings which is yet another sharpened form of Theorem 2.1.1 due to Pant [125] (also relevant result in Pant [126]) in symmetric spaces.

**Corollary 2.2.4.** If (in the setting of Theorem 2.2.3) $d$ satisfies $(IC)$ and $(HE)$. Suppose that the triode of mappings $f, S, T : Y \to X$ satisfy the following conditions:

(i) the pair $(f, S)$ satisfies the property (E.A) (resp. $(f, T)$ satisfies the property (E.A)),

(ii) $SY$ is a $d$-closed $(\tau(d)$-closed) subset of $X$ and $fY \subset TY$(resp. $TY$ is a $d$-closed $(\tau(d)$-closed) subset of $X$ and $fY \subset SY$) and

(iii) $d(fx, fy) \leq km(x, y)$, for any $x, y \in X$, where $k \geq 0$, $0 < \alpha < 1$ together with $k \alpha < 1$

$$m(x, y) = \max \left\{ d(Sx, Ty), \alpha[d(fx, Sx) + d(fy, Ty)], \alpha[d(fy, Ty) + d(fy, Sx)] \right\}$$

Then there exist $u, w \in Y$ such that $fu = Su = Tw$.

Moreover, if $Y = X$ together with

(iv) the pairs $(f, S)$ and $(f, T)$ are weakly compatible and
(v) \( d(fx, f^2x) \neq \max \left\{ d(Sx,Tfx), d(f^2x,Tfx), d(fx,Sx), d(f^2x,Sx) \right\} \),

whenever the right hand side is non-zero.

Then \( f, T, \) and \( S \) have a common fixed point.

**Corollary 2.2.5.** Let \((X,d)\) be symmetric (semi-metric) space wherein \( d \) satisfies (1C) (Hausdorffness of \( \tau(d) \)) and (HE). If \( f, g, S \) and \( T \) are four self mappings of \( X \) which satisfy the following conditions:

(i) the pair \((f,S)\) satisfies the property (E.A) (resp. \((g,T)\) satisfies the property (E.A)),

(ii) \( SX \) is a \( d \)-closed \((\tau(d))-closed\) subset of \( X \) and \( fX \subset TX \) (resp. \( TX \) is a \( d \)-closed \((\tau(d))-closed\) subset of \( X \) and \( gX \subset SX \)).

(iii) \( d(fx,gy) < m_1(x,y) \) where \( m_1(x,y) \) carries the same meaning as in Theorem 2.1.4. Then there exist \( u,w \in X \) such that \( fu = Su = Tw = gw \). Also, if

(iv) the pairs \((f,S)\) and \((g,T)\) are weakly compatible, then \( f, g, S \) and \( T \) have a unique common fixed point.

**Proof.** The proof can be completed on the lines of Corollary 2.2.3, hence details are not included.

The following lemma enunciates a set of conditions which interrelates property (E.A) with common property (E.A).

**Lemma 2.2.1.** Let \( Y \) be an arbitrary set whereas \((X,d)\) be a symmetric (semi-metric) space wherein \( d \) satisfies (\( W_3 \)) (Hausdorffness of \( \tau(d) \)) and (HE). If \( f, g, S, T : Y \to X \) are four mappings which satisfy the following conditions:

(i) the pair \((f,S)\) (or \((g,T)\)) satisfies the property (E.A),

(ii) \( fY \subset TY \) (or \( gY \subset SY \)),

(iii) \( d(fx,gy) \leq km(x,y) \), for any \( x, y \in Y \), where \( k \geq 0 \) and \( m(x,y) \) is the same as earlier, then the pairs \((f,S)\) and \((g,T)\) share the common property (E.A).

**Proof.** Since the pair \((f,S)\) enjoys the property (E.A), one can find a sequence \( \{x_n\} \subset Y \) such that \( \lim_{n \to \infty} fx_n = \lim_{n \to \infty} Sx_n = t \) for some \( t \in X \). Since \( fY \subset TY \), therefore for each sequence \( \{x_n\} \) one can find a sequence \( \{y_n\} \subset Y \) such that \( fx_n = Ty_n \) which in turn yields that \( \lim_{n \to \infty} fx_n = \lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Ty_n = t \). Now we assert that \( \lim_{n \to \infty} gy_n = t \). If not, then using (iii), we have

\[
\begin{align*}
d(fx_n, gy_n) \\ \leq k \max \left\{ d(Sx_n,Ty_n), \min \{d(fx_n,Sx_n),d(gy_n,Ty_n)\}, \min \{d(fx_n,Ty_n),d(gy_n,Sx_n)\} \right\}
\end{align*}
\]
which on letting \( n \to \infty \) and making use of \((W_3)\) and \((HE)\), one gets
\[
\lim_{n \to \infty} d(t, gy_n) \leq k \max\{0, 0, 0\} \leq 0
\]
yielding thereby \( \lim_{n \to \infty} gy_n = t \) which shows that the pairs \((f, S)\) and \((g, T)\) share the common property \((E.A)\).

**Lemma 2.2.2.** Lemma 2.2.1 remains true if \((W_3)\) is replaced by \((1C)\) whereas condition (iii) of Lemma 2.2.1 is replaced by
\[
d(fx, gy) \leq km(x, y),
\]
for any \( x, y \in Y \), where \( k \geq 0 \) with \( k \alpha < 1 \) and \( m(x, y) \) is the same as earlier besides retaining rest of the hypotheses.

**Proof.** The proof can be completed on the lines of Lemma 2.2.1, hence details are not included.

**Theorem 2.2.4.** Let \( f, g, S, T : Y \to X \) be four mappings where \( Y \) is an arbitrary non-empty set and \( X \) is a non-empty set equipped with a symmetric (semi-metric) \( d \) wherein \( d \) satisfies \((W_3)\) (Hausdorffness of \( \tau(d) \)) and \((HE)\). Suppose that

(i) the pairs \((f, S)\) and \((g, T)\) share the common property \((E.A)\),

(ii) \( TY \) and \( SY \) are \( d \)-closed \((\tau(d)\) closed\) subsets of \( X \) and

(iii) \( d(fx, gy) \leq km(x, y) \),

for any \( x, y \in Y \), where \( k \geq 0 \) and \( m(x, y) \) is the same as in Theorem 2.1.3.

Then the pairs \((f, S)\) and \((g, T)\) have a coincidence point each.

Moreover if \( Y = X \) along with,

(iv) the pairs \((f, S)\) and \((g, T)\) are weakly compatible and

(v) \( d(fx, gfx) \neq \max \{d(Sx, Tfx), d(gfx, Tfx), d(fx, Tfx), d(fx, Sx), d(gfx, Sx)\} \),

whenever the right hand side is non-zero.

Then \( f, g, S \) and \( T \) have a common fixed point.

**Proof.** Since the pairs \((f, S)\) and \((g, T)\) share the common property \((E.A)\), therefore there exists two sequences \( \{x_n\} \) and \( \{y_n\} \) in \( Y \) such that
\[
\lim_{n \to \infty} d(fx_n, t) = \lim_{n \to \infty} d(Sx_n, t) = \lim_{n \to \infty} d(gy_n, t) = \lim_{n \to \infty} d(Ty_n, t) = 0
\]
for some \( t \in X \), which due to the property \((HE)\), gives rise \( \lim_{n \to \infty} d(gy_n, Ty_n) = 0 \) and \( \lim_{n \to \infty} d(fx_n, Sx_n) = 0 \). Since \( SY \) is a closed subset of \( X \), hence \( \lim_{n \to \infty} Sx_n = t \in SY \) and hence there exists a point \( u \in Y \) such that \( Su = t \) which in turn yields that
\[
\lim_{n \to \infty} d(gy_n, Ty_n) = 0, \quad \lim_{n \to \infty} d(Su, Ty_n) = 0 \quad \text{and} \quad \lim_{n \to \infty} d(Su, gy_n) = 0. \]

Now we assert that \(fu = Su\). If not, then using (iii), we have
\[
d(fu, gy_n) \leq k \max \left\{ d(Su, Ty_n), \min\{d(fu, Su), d(gy_n, Ty_n)\}, \min\{d(fu, Ty_n), d(gy_n, Su)\} \right\}
\]
which on letting \(n \to \infty\), and making use \((W_3)\) and \((HE)\) gives rise
\[
d(gu, t) \leq 0
\]
a contradiction, implying thereby \(fu = t\). Hence \(fu = Su\). Therefore, \(u\) is a coincidence point of the pair \((f, S)\).

Since \(TY\) is also a closed subset of \(X\), hence \(\lim_{n \to \infty} Ty_n = t \in TY\) so that there exists a point \(w \in Y\) such that \(Tw = t\) which in turn yields \(\lim_{n \to \infty} d(Sx_n, Tw) = 0\), \(\lim_{n \to \infty} d(fx_n, Tw) = 0\) and \(\lim_{n \to \infty} d(fx_n, Sx_n) = 0\). Now we assert that \(gw = Tw\). If not, then again using (iii), we have
\[
d(fx_n, gw) \leq k \max \left\{ d(Sx_n, Tw), \min\{d(fx_n, Sx_n), d(gw, Tw)\}, \min\{d(fx_n, Tw), d(gw, Sx_n)\} \right\}
\]
which on letting \(n \to \infty\), and making use of \((W_3)\) and \((HE)\) gives rise
\[
\lim_{n \to \infty} d(fx_n, gw) = 0 \quad \text{i.e.} \quad \lim_{n \to \infty} fx_n = gw,
\]
a contradiction implying thereby \(gw = t\). Hence \(gw = Tw\), which shows that \(w\) is a coincidence point of the pair \((g, T)\) and in all \(fu = Su = gw = Tw = t\). The rest of the proof is similar to that of Theorem 2.2.1, hence it is omitted.

**Theorem 2.2.5.** Theorem 2.2.4 remains true if condition (iii) (of Theorem 2.2.4) is replaced by
\[
d(fx, gy) \leq km_1(x, y)
\]
for all \(x, y \in Y\) with \(k\alpha < 1\) wherein \(m_1(x, y)\) is the same as earlier whereas \((W_3)\) is replaced by \((1C)\) besides retaining rest of the hypotheses.

**Proof.** The proof can be completed on the lines of proof of Theorem 2.2.4, hence details are not included.

**Theorem 2.2.6.** Let \(Y\) be an arbitrary set whereas \((X, d)\) be a symmetric (semi-metric) space equipped with a symmetric (semi-metric) \(d\) which enjoys \((W_3)\) and \((HE)\). Let \(f, g, S, T : Y \to X\) be four mappings which satisfy the following conditions:

(i) the pair \((f, S)\) (or \((g, T))\) satisfies the property \((E.A)\),

(ii) \(fY \subset TY\) (or \(gY \subset SY\)),

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(iii) $TY$ (or $SY$) is $d$-closed ($\tau(d)$ closed) subset of $X$ and

(iv) $d(fx, gy) \leq km(x, y)$, for any $x, y \in Y$, where $k \geq 0$ and $m(x, y)$ is the same as earlier, then the pairs $(f, S)$ and $(g, T)$ have a coincidence point.

Moreover, if $Y = X$ along with,

(v) the pairs $(f, S)$ and $(f, T)$ are weakly compatible and

(vi) $d(fx, gx) \neq \max \left\{ d(Sx, Tx), d(gfx, Tfx), d(fx, Sx), d(gfx, Sx) \right\}$, whenever the right hand side is non-zero.

Then $f, g, S$ and $T$ have a common fixed point.

**Proof.** Notice that all the conditions of Lemma 2.2.1 are satisfied, therefore the pairs $(f, S)$ and $(g, T)$ share the common property (E.A), i.e. there exist two sequences $\{x_n\}, \{y_n\} \subset Y$ such that

$$
\lim_{n \to \infty} fx_n = \lim_{n \to \infty} Sx_n = \lim_{n \to \infty} gy_n = \lim_{n \to \infty} Ty_n = t \in X.
$$

If $SY$ is closed subset of $X$, then in view of Theorem 2.2.2, the pair $(f, S)$ has a coincidence point say $u$ i.e. $fu = Su$. Since $fY \subset TY$ and $fu \in fY$, there exists $w \in Y$ such that $fu = Tw$. Now we assert that $gw = Tw$. If not, then using (iv) we have

$$
d(fx_n, gw) \leq k \max \left\{ d(Sx_n, Tw), \min\{d(fx_n, Sx_n), d(gw, Tw)\}, \min\{d(Sx_n, gw), d(Tw, fx_n)\} \right\}
$$

which on making $n \to \infty$, gives rise

$$
d(t, gw) \leq 0
$$

which is a contradiction yielding thereby $gw = Tw$. The rest of the proof is similar to that of Theorem 2.2.4, hence it is omitted.

**Theorem 2.2.7.** The conditions of Theorem 2.2.6 remains true if $(W_3)$ is replaced by (1C) whereas condition (iv) (of Theorem 2.2.6) is replaced by

$$
d(fx, gy) \leq km_1(x, y)
$$

for all $x, y \in Y$ where $m_1(x, y)$ is the same as earlier with $k\alpha < 1$ besides retaining rest of the hypotheses.

**Proof.** Proceeding on the lines of the proof of Theorem 2.2.4, one can complete the proof of this theorem, hence details are not included.
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2.3 Illustrative examples

Now, we furnish examples demonstrating the validity of the hypotheses and degree of generality of our results over some recently established results due to Cho et al. [31] and others. Our first example demonstrates Theorem 2.2.1.

Example 2.3.1. Consider $Y = X = [2, 20]$ equipped with the symmetric $d(x, y) = (x - y)^2$. In order to illustrate Theorem 2.2.1, we set $f = g$ and $S = T$. Define $f, S : X \to X$ as

$$
f(2) = 2, \quad f(x) = 7 \text{ if } 2 < x \leq 5, \quad f(x) = 2 \text{ if } x > 5,
$$

$$
S(2) = 2, \quad S(x) = 7 \text{ if } 2 < x \leq 5, \quad S(x) = \frac{x + 1}{3} \text{ if } x > 5.
$$

Then the pair $(f, S)$ satisfies all the conditions of Theorem 2.2.1 and has a coincidence point at $x = 2$ which also remains a common fixed point of the pair. Notice that $f$ is not $S$–Lipschitzian whenever $x \in (2, 5]$ and $y = 20$ as (with $k \geq 0$)

$$
d(f5, f20) \leq k \ d(S5, S20) \Rightarrow 25 \leq 0
$$

which is a contradiction. Here it is interesting to notice that the condition

$$
d(fx, fx) \neq \max \left\{ d(Sx, Sfx), d(fx, Sx), d(ffx, Sfx), d(fx, Sfx), d(Sx, Sfx) \right\}
$$

is not satisfied e.g. for $x = 5$ we have

$$
d(5, f5) \neq \max \left\{ d(S5, Sf5), d(5, S5), d(f5, Sf5), d(f5, Sf5), d(S5, Sf5) \right\}
$$

or

$$
d(7, 2) \neq \max \left\{ d(7, \frac{8}{3}), d(7, 7), d(2, \frac{8}{3}), d(7, \frac{8}{3}), d(7, 2) \right\}
$$

or

$$
25 \neq 25
$$

which is a contradiction in case right hand side of above inequality is nonzero. This confirms that condition (v) of Theorem 2.2.1 is only a necessary condition (but not sufficient).

The following example exhibits that the axioms (HE) and (1C) are necessary in Theorem 2.2.3. The idea of this example essentially appears in Cho et al. [31].

Example 2.3.2. Consider $Y = X = [0, \infty)$ and define a symmetric $d$ on $X$ as

$$
d(x, y) = \begin{cases} 
|x - y|, & \text{if } x \neq 0, \ y \neq 0 \\
\frac{1}{x}, & \text{if } x \neq 0, \ y = 0 \\
\frac{1}{y}, & \text{if } y \neq 0, \ x = 0.
\end{cases}
$$
The following example demonstrates Theorem 2.2.4.

In order to verify \( d(fx, fy) \leq kn_1(x, y) \) where \( n_1(x, y) \) denotes the restriction of \( m_1(x, y) \) to mappings \( f \) and \( S \), we distinguish two cases:

Case (i). If \( x > 0, y > 0 \), then

\[
d(fx, fy) = |x - y| = 3|\frac{x-y}{3}| = 3d(Sx, Sy) \leq 3n_1(x, y)
\]

where \( m_1(x, y) \) is the same as earlier.

Case (ii). If \( x = 0 \) and \( y > 0 \), then

\[
d(fx, fy) = d(0, y) = \frac{1}{y} = \frac{3}{y} = \frac{1}{3}d(fx, Sy)
\]

\[
< \frac{1}{3}[d(fx, Sy) + d(Sx, fy)]
\]

\[
< 3m_1(x, y)
\]

which shows that the condition (v) of Theorem 2.2.3 is satisfied for all \( x, y \in X \) with \( k = 3 \), \( \alpha = \frac{1}{3} \) and \( m_1(x, y) \) is the same as earlier. Also the pair \((f, S)\) enjoys the property (E.A) (e.g. \( x_n = n \)) whereas \( f(X) \) is \( d \)-closed (or \( \tau(d) \)) closed subset of \( X \). Thus all the conditions of Theorem 2.2.3 are satisfied. Notice that the pair \((f, S)\) has no coincidence or common fixed point.

The following example demonstrates Theorem 2.2.4.

**Example 2.3.3.** Consider \( Y = X = [-1, 1] \) equipped with the symmetric \( d(x, y) = (x - y)^2 \) which satisfies \( (W_3) \) and \( (HE) \). Define self mappings \( f, g, S \) and \( T \) on \( X \) as

\[
f(-1) = f1 = 3/5, \quad fx = x/4, \quad -1 < x < 1,
\]

\[
g(-1) = g1 = 3/5, \quad gx = -x/4, \quad -1 < x < 1,
\]

\[S(-1) = -1/8, \quad Sx = x/8, \quad -1 < x < 1, \text{ and } S1 = -1/8, \text{ and}
\]

\[T(-1) = -1/8, \quad Tx = -x/8, \quad -1 < x < 1, \text{ and } T1 = 1/8.
\]

Consider sequences \( \{x_n = \frac{1}{n}\} \) and \( \{y_n = \frac{-1}{n}\} \) in \( X \). Clearly,

\[
\lim_{n \to \infty} fx_n = \lim_{n \to \infty} Sx_n = \lim_{n \to \infty} gy_n = \lim_{n \to \infty} Ty_n = 0
\]

which shows that pairs \((f, S)\) and \((g, T)\) share the common property (E.A). Notice that \( f(X) = g(X) = \{\frac{3}{5}\} \cup (\frac{-1}{4}, \frac{1}{4}) \not\subset S(X) = T(X) = [\frac{-1}{8}, \frac{1}{8}] \). In order to verify condition (iii) of Theorem 2.2.4, notice that

\[
d(fx, gy) = (x/4 + y/4)^2 = ((x + y)/4)^2 = 4((x + y)/8)^2 \leq 4 d(Sx, Ty) \leq 4 m(x, y)
\]

where \( m(x, y) \) is the same as earlier. Thus, all the conditions of Theorem 2.2.4 are satisfied and 0 is a common fixed point of the pairs \((f, S)\) and \((g, T)\) which is also their coincidence point as well.
Here it is worth noting that Theorems 2.1.3. and 2.1.4. due to Cho et al. [31] can be used only when \( k \) is at most 1 whereas our results are valid for any \( k \geq 0 \). Notice that in the present example \( k \) is 4 and hence Theorems 2.1.3 and 2.1.4 due to Cho et al. [31] can not work in the context of this example which substantiate the utility of our results over the cited ones.

Our last example highlights the non-uniqueness of common fixed points in the present context.

Example 2.3.4. In order to highlight the non-uniqueness of common fixed point in Theorems 2.2.1, consider \( Y = X = \{0, 1, 1/2, 1/3, \ldots, 1/n, \ldots\} \) under the symmetric

\[ d(x, y) = e^{|x-y|} - 1. \]

Set \( f = S, g = T \) and define \( f \) and \( g \) on \( X \) by

\[ f(1/x) = 1/x^2, \quad g(1/x) = 1/x^3, \quad f(0) = 0 = g(0). \]

Clearly \( f(X) \not\subset g(X) \) but \( g(X) \) is a closed subset of \( X \). Also, rest of the conditions of Theorem 2.2.1 are trivially satisfied. Notice that \( f \) and \( g \) have two common fixed points namely: 0 and 1.

2.4 Analogous results via absorbing pairs

We begin with the following proposition which enunciates a set of conditions under which pointwise \( S \)-absorbing (also pointwise \( T \)-absorbing property) is equivalent to pointwise absorbing property.

Proposition 2.4.1. Let \((X, d)\) be a symmetric space (semi-metric) equipped with a symmetric \( d \) whereas \( f, S \) and \( T \) be three self mappings defined on \( X \) which satisfy the Lipschitz type condition

\[ d(f(x), f(y)) \leq km(x, y), \quad k \geq 0, \]

where

\[ m(x, y) = \max \left\{ d(Sx, Ty), \min\{d(fx, Sx), dfy, Ty\}, \min\{d(fx, Ty), dfy, Sx\} \right\}. \]

Then, the pair \((f, S)\) as well as \((f, T)\) are pointwise absorbing iff the pair \((f, S)\) is pointwise \( S \)-absorbing while the pair \((f, T)\) is pointwise \( T \)-absorbing.

Proof. To prove the if part, suppose that the pair \((f, S)\) is pointwise \( S \)-absorbing whereas the pair \((f, T)\) is pointwise \( T \)-absorbing. Then, we distinguish two cases.

Case I. If \( x \in X \) such that \( fx \neq Sx \), then choosing \( R = \frac{d(fx, Sx)}{d(fx, Sx)} \), we can write

\[ d(fx, Sx) \leq Rd(fx, Sx) \]

i.e. the pair \((f, S)\) is pointwise absorbing. In case \( x \in X \) such that \( fx \neq Tx \), then choosing \( R = \frac{d(fx, Tx)}{d(fx, Tx)} \), we can have

\[ d(fx, Tx) \leq Rd(fx, Tx) \]

i.e. the pair \((f, T)\) is pointwise absorbing.

Case II. If for \( x \in X \) such that \( fx = Sx \), then employing pointwise \( S \)-absorbing property of the pair \((f, S)\), we have \( fx = Sx = Sfx \), which in turn yields \( Sfx = SSx = fx = Sx \) while by dint of pointwise \( T \)-absorbing property of the pair \((f, T)\), we have \( fy = Ty = Tfy \), which in turn yields \( Tfy = TTy = fy = Ty \). Now using condition (i) with \( x = Sy \) and keeping \( y \) as its stands, we get

\[ d(fSy, fy) \leq \max \left\{ d(SSy, Ty), \min\{d(fSy, SSy), dfy, Ty\} \right\}, \]
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\[
\min \{d(fSy, Ty), d(fy, SSy)\}
\]

or

\[
d(fSy, fy) \leq \max \left\{ d(fy, fy), \min \{d(fSy, fy), d(fy, fy)\}, \min \{d(fSy, fy), d(fy, fy)\}\right\}
\]

or

\[
d(fSy, fy) \leq \max \left\{ 0, \min \{d(fSy, fy), 0\}, \min \{d(fSy, fy), 0\}\right\}
\]

or

\[
d(fSy, fy) \leq 0,
\]

implying thereby \( fy = fSy \), i.e. the pair \((f, S)\) is pointwise absorbing. Similarly, it can be shown that the pair of mappings \((f, T)\) is also pointwise absorbing. Only if part is obvious. This concludes the proof.

In our next result by making use of \(S\)-continuity of \(f\) and \(T\)-continuity of \(g\) (instead of utilizing some Lipschitzian type conditions (e.g. Pant [125])) along with absorbing properties of the involved mappings, we prove the following:

**Theorem 2.4.1.** Let \( Y \) be an arbitrary nonempty set whereas \( X \) be another nonempty set equipped with a symmetric (semi-metric) \( d \) which enjoys \((W_3)\) (Hausdorffness of \( \tau(d) \)) and \((HE)\). Let \( f, g, S, T : Y \to X \) be four mappings which satisfy the following conditions:

(i) \( f \) is \( S\)-continuous and \( g \) is \( T\)-continuous,

(ii) the pairs \((f, S)\) and \((g, T)\) share the common property \((E.A)\),

(iii) \( SY \) and \( TY \) are \( d\)-closed \((\tau(d)\)-closed\) subset of \( X \) (resp. \( fY \subset TY \) and \( gY \subset SY \)), then, there exist \( u, w \in Y \) such that \( fu = Su = Tw = gw \).

Moreover, if \( Y = X \), then \( f, g, S \) and \( T \) have a common fixed point provided the pairs \((f, S)\) and \((g, T)\) are pointwise absorbing.

**Proof.** Following the lines of the proof of Theorem 2.2.1, one can show that \( fu = Su = gw = Tw = t \) i.e. both the pairs have a coincidence point.

As the pairs \((f, S)\) and \((g, T)\) are pointwise absorbing, one can write

\[
Su = Sfu, \quad fu = fSu, \quad Tw = Tgw, \quad gw = gTw,
\]

\[
\Rightarrow fu = Sfu, \quad fu = ffu \quad \text{and} \quad gw = Tgw, \quad gw = ggw,
\]

which show that \( fu (fu = gw) \) is a common fixed point of \( f, g, S \) and \( T \).

With a view to demonstrate the utility of Theorem 2.4.1 over Theorem 2.2.1, we adopt the following example.
Example 2.4.1. Consider $X = Y = (-1, 1] \cup \{2, 3, 4\}$ equipped with the symmetric defined by $d(x, y) = (x-y)^2$ for all $x, y \in X$ which naturally satisfies $(W_3)$ and $(HE)$. Define self mappings $f, g, S$ and $T$ on $X$ as

$$
f(x) = \begin{cases} 
\frac{3}{5}, & \text{if } -1 < x < -1/2 \\
\frac{x}{4}, & \text{if } -1/2 \leq x \leq 1/2 \\
\frac{3}{5}, & \text{if } 1/2 < x < 1 \\
3, & \text{if } x = 1, 4 \\
2, & \text{if } x = 2, 3, 
\end{cases}
$$

$$
g(x) = \begin{cases} 
\frac{-x}{4}, & \text{if } -1 < x < -1/2 \\
\frac{3}{5}, & \text{if } 1/2 < x < 1 \\
3, & \text{if } x = 1, 4 \\
2, & \text{if } x = 2, 3, 
\end{cases}
$$

$$
S(x) = \begin{cases} 
\frac{3}{4}, & \text{if } -1 < x < -1/2 \\
\frac{x}{2}, & \text{if } -1/2 \leq x \leq 1/2 \\
\frac{-3}{4}, & \text{if } 1/2 < x < 1 \\
2, & \text{if } x = 1, 2, 3, 4, 
\end{cases}
$$

$$
T(x) = \begin{cases} 
\frac{-x}{2}, & \text{if } -1 < x < -1/2 \\
\frac{3}{4}, & \text{if } 1/2 < x < 1 \\
2, & \text{if } x = 1, 2, 3, 4. 
\end{cases}
$$

Consider sequences $\{x_n\} = \{\frac{1}{n+1}\}$ and $\{y_n\} = \{-\frac{1}{n+1}\}$ in $X$. Clearly,

$$
\lim_{n \to \infty} f(x_n) = \lim_{n \to \infty} S(x_n) = \lim_{n \to \infty} g(y_n) = \lim_{n \to \infty} T(y_n) = 0
$$

and

$$
\lim_{n \to \infty} S(x_n) = 0 = S(0) \Rightarrow \lim_{n \to \infty} f(x_n) = 0 = f(0), \\
\lim_{n \to \infty} T(y_n) = 0 = T(0) \Rightarrow \lim_{n \to \infty} g(y_n) = 0 = g(0)
$$

which shows that $(f, S)$ and $(g, T)$ share the common property $(E.A)$ whereas the map $f$ is $S$-continuous and the mapping $g$ is $T$-continuous. Further $f(X) = \{\frac{3}{5}, 2, 3\} \cup [\frac{-3}{4}, \frac{3}{4}] \not\subseteq S(X) = \{\frac{-3}{4}, \frac{3}{4}\} \cup [\frac{-1}{4}, \frac{1}{4}]$ and $g(X) = \{\frac{3}{5}, 2, 3\} \cup [\frac{-1}{4}, \frac{1}{4}] \not\subseteq T(X) = \{\frac{-3}{4}, \frac{3}{4}\} \cup [\frac{-1}{4}, \frac{1}{4}]$ which show that $S(X)$ and $T(X)$ are closed subsets of $X$. Also by a routine calculation, one can easily verify that the pairs $(f, S)$ and $(g, T)$ are pointwise absorbing. Thus, the pairs of mappings $(f, S)$ and $(g, T)$ satisfy all the conditions of Theorem 2.4.1 and have two common fixed points namely: 0 and 2.

Notice that at $x = 1$, the involved mappings do not satisfy the condition (v) of Theorem 2.2.1 namely:

$$
d(f(x), g(x)) \neq \max\{d(S(x), f(x)), d(g(f(x), T(f(x))), d(f(x), T(f(x))), d(f(x)), S(x)), d(g(f(x), S(x))\}
$$
whenever the right hand side is non-zero. Moreover, it can also be verified that at points \( x = 1 \) and \( y = 2 \), the involved mappings do not satisfy the Lipschitz type condition for any \( k \). Thus this example substantiates the fact that Theorem 2.4.1 is a genuine extension of Theorem 2.2.1.

By restricting \( f, \ g, \ S \) and \( T \) suitably, one can derive corollaries involving two as well as three mappings. Here, it may be pointed out that any result involving three mappings is itself a new result. For the sake of brevity, we opt to mention just one such corollary by restricting Theorem 2.4.1 to a triode of mappings \( f, \ S \) and \( T \) which is still new and presents yet another sharpened form of a theorem contained in [147] to symmetric (semi-metric) spaces besides admitting a non-self setting upto coincidence points.

**Corollary 2.4.1.** Let \( Y \) be an arbitrary set whereas \((X, d)\) be a symmetric (semi-metric) space equipped with a symmetric (semi-metric) \( d \) which enjoys \((W_3)\) (Hausdorffness of \( \tau(d) \)) and \((HE)\). Let \( f, S, T : Y \to X \) be a three mappings which satisfy the following conditions:

1. \( f \) is \( S \)-continuous and \( f \) is \( T \)-continuous,
2. the pairs \((f, S)\) as well as \((f, T)\) are tangential,
3. \( SY \) and \( TY \) are \( d \)-closed \((\tau(d)\)-closed) subsets of \( X \) (resp. \( fY \subset TY \cap SY \)), then, there exist \( u, w \in Y \) such that \( fu = Su = Tw \).

Moreover, if \( Y = X \), then \( f, S \) and \( T \) have a common fixed point provided the pairs \((f, S)\) and \((f, T)\) are pointwise absorbing.

Our next theorem is essentially inspired by Theorem 2.2.2.

**Theorem 2.4.2.** Let \( Y \) be an arbitrary set whereas \((X, d)\) be a symmetric (semi-metric) space equipped with a symmetric (semi-metric) \( d \) which enjoys \((W_3)\) (Hausdorffness of \( \tau(d) \)) and \((HE)\). Let \( f, g, S, T : Y \to X \) be four mappings which satisfy the following conditions:

1. the pair \((g, T)\) satisfies the property \((E.A)\) (resp. \((f, S)\) satisfies the property \((E.A))\),
2. \( TY \) is a \( d \)-closed \((\tau(d)\)-closed) subset of \( X \) and \( gY \subset SY \) (resp. \( SY \) is a \( d \)-closed \((\tau(d)\)-closed) subset of \( X \) and \( fY \subset TY \)) and
3. \( d(fx, gy) \leq km(x, y) \), for any \( x, y \in Y \), where \( k \geq 0 \) and
   \[
   m(x, y) = \max \left\{ \dfrac{d(Sx, Ty)}{d(Sx, Ty) + d(gy, Ty)}, \min \left\{ \dfrac{d(fx, Sx)}{d(fx, Sx) + d(gy, Ty)}, \min \left\{ \dfrac{d(fx, Sx)}{d(fx, Sx) + d(gy, Sx)} \right\} \right\},
   \]
   then, there exist \( u, w \in Y \) such that \( fu = Su = Tw = gw \).

Moreover, if \( Y = X \), then \( f, g, S \) and \( T \) have a common fixed point provided the pairs \((f, S)\) and \((g, T)\) are pointwise absorbing.
Proof. Following the lines of proof of Theorem 2.2.2, we can show that \( fu = Su = gw = Tw = t \). Thus both the pairs have a coincidence point. As \((f,S)\) and \((g,T)\) are pointwise absorbing pairs, one can write

\[
Su = Sfu, \quad fu = fSu \quad \text{and} \quad Tw = Tgw, \quad gw = gTw
\]

\[
\Rightarrow fu = Sfu, \quad fu = ffu \quad \text{and} \quad gw = Tgw, \quad gw = ggw,
\]

which show that \( fu (fu = gw) \) is a common fixed point of \( f, g, S \) and \( T \).

The following example demonstrates Theorem 2.4.2.

Example 2.4.2. Let \( Y = X = [0, \infty) \) equipped with the symmetric \( d(x,y) = (x - y)^2 \) which naturally satisfies \((W_3)\) and \((HE)\). Set \( f = g, \quad S = T \) and define \( f, S : Y \longrightarrow X \) as follows:

\[
f x = \begin{cases} 
2 , & \text{if} \quad 0 \leq x \leq 2, \quad \text{or} \quad x > 5, \quad x \neq 10, \\
10 , & \text{if} \quad x = 10, \\
6 , & \text{if} \quad 2 < x \leq 5
\end{cases}
\]

and

\[
S x = \begin{cases} 
2 , & \text{if} \quad 0 \leq x \leq 2, \quad \text{or} \quad x > \frac{11}{2}, \quad x \neq 10, \\
4 , & \text{if} \quad 2 < x \leq 5, \\
\frac{x + 1}{3} , & \text{if} \quad x \in (5, \frac{11}{2}), \\
10 , & \text{if} \quad x = 10.
\end{cases}
\]

Then, by a routine calculation, it can be easily verified that \( f \) and \( S \) satisfy condition (iii) (of Theorem 2.4.2) with constant \( k = 8 \). Also, \( S(X) = [2, \frac{13}{2}] \cup \{4, 10\} \) which is closed in \( \mathbb{R} \). Notice that the pair \((f,S)\) is non-compatible (e.g. \( x_n = 5 + \frac{1}{n} \)) and hence tangential. The verification of the pointwise absorbing property of the pair \((f,S)\) is straight forward. Thus, \( f \) and \( S \) satisfy all the conditions of the Theorem 2.4.2 and have two common fixed points namely: \( x = 2 \) and \( x = 10 \).

However, the closure of \( f(X) = \{2, 6, 10\} \) is not contained in \( S(X) \). Further, it is also worth noting that for all \( x \) with \( 2 < x \leq 5 \) with \( f = g \) and \( S = T \), the involved pair \((f,S)\) does not satisfy the condition

\[
d(fx, gfx) \neq \max \{d(Sx, Tf x), d(gfx, Tf x), d(fx, Tf x), d(fx, Sx), d(gfx, Sx)\},
\]

whenever the right hand side is non-zero. Thus this example also establishes the utility of Theorem 2.4.2 over Theorem 2.2.2.
Remark 2.4.1. Choosing $k = 1$ in Theorem 2.4.2, we derive a slightly sharpened form of a theorem due to Cho et al. [31] as conditions on the ranges of involved mappings are relatively lightened.

By restricting $f, g, S$ and $T$ suitably and making use of Proposition 2.4.1, one can derive corollaries for two as well as three mappings. For the sake of brevity, we derive just one corollary by restricting Theorem 2.4.2 to a triode of mappings which is yet another sharpened and unified form of a theorem due to Pant [125] (also relevant to some results in Pant [126]) extended to symmetric spaces.

Corollary 2.4.2. Suppose that (in the setting of Theorem 2.4.2) $d$ satisfies ($W_3$) and ($HE$). If $f, S, T : Y \to X$ are three mappings which satisfy the following conditions:

(i) the pair $(f, S)$ satisfies the property (E.A) (resp. $(f, T)$ satisfies the property (E.A)),

(ii) $SY$ is a $d$-closed ($\tau(d)$-closed) subset of $X$ and $fY \subset TY$ (resp. $TY$ is a $d$-closed ($\tau(d)$-closed) subset of $X$ and $fY \subset SY$) and

(iii) $d(fx, fy) \leq km_2(x, y)$,

for any $x, y \in Y$, where $k \geq 0$ and

$$m_2(x, y) = \max\{d(Sx, Ty), \min\{d(fx, Sx), d(fy, Ty)\}, \min\{d(fx, Ty), d(fy, Sx)\}\},$$

then there exist $u, w \in Y$ such that $fu = Su = Tw$.

Moreover, if $Y = X$, then $f, S$ and $T$ have a common fixed point provided the pair $(f, S)$ is pointwise $S$-absorbing whereas the pair $(f, T)$ is pointwise $T$-absorbing.

Corollary 2.4.3. Let $(X, d)$ be a symmetric (semi-metric) space wherein $d$ satisfies ($W_3$) (Hausdorffness of $\tau(d)$) and ($HE$). If $f, g, S, T : X \to X$ are four self mappings of $X$ which satisfy the following conditions:

(i) the pair $(f, S)$ satisfies the property (E.A) (resp. $(g, T)$ satisfies the property (E.A)),

(ii) $SX$ is a $d$-closed ($\tau(d)$-closed) subset of $X$ and $fX \subset TX$ (resp. $TX$ is a $d$-closed ($\tau(d)$-closed) subset of $X$ and $gX \subset SX$).

(iii) $d(fx, gy) < m(x, y)$ where $m(x, y)$

$$= \max\{d(Sx, Ty), \min\{d(fx, Sx), d(gy, Ty)\}, \min\{d(fx, Ty), d(gy, Sx)\}\},$$

then there exist $u, w \in X$ such that $fu = Su = Tw = gw$.

Also $f, g, S$ and $T$ have a unique common fixed point provided the pair $(f, S)$ is pointwise $S$-absorbing whereas the pair $(f, T)$ is pointwise $T$-absorbing.

Proof. Proof follows from Theorem 2.4.2 by setting $k = 1$. 

Proof.
Our next theorem is essentially inspired by the Lipschitzian condition utilized in Cho et al. [31].

**Theorem 2.4.3.** Theorem 2.4.2 remains true if \((W_3)\) is replaced by \((1C)\) whereas condition (iii) (of Theorem 2.4.2) is replaced by the following condition besides retaining rest of the hypotheses:

(i) \(d(fx, gy) \leq km_1(x, y)\), for any \(x, y \in Y\), where \(k \geq 0\) with \(k\alpha < 1\) and

\[
m_1(x, y) = \max \left\{ d(Sx, Ty), \alpha [d(fx, Sx) + d(gy, Ty)], \alpha [d(fx, Ty) + d(gy, Sx)] \right\}.
\]

**Proof.** The proof can be completed on the lines of proof of Theorem 2.4.2, hence details are not included.

By restricting \(f, g, S\) and \(T\) suitably, one can derive corollaries for two as well as three mappings. For the sake of brevity, we derive just one corollary by restricting Theorem 2.4.3 to a triode of mappings which is yet another sharpened form of a theorem contained in Pant [125] (also relevant to some results in Pant [126]) extended to symmetric spaces.

**Corollary 2.4.4.** Suppose that (in the setting of Theorem 2.4.3) \(d\) satisfies \((1C)\) and \((HE)\). Let us assume that a triode of mappings \(f, S, T : Y \to X\) satisfy the following conditions:

(i) the pair \((f, S)\) satisfies the property (E.A) (resp. \((f, T)\) satisfies the property (E.A)),

(ii) \(SY\) is a \(d\)-closed \((\tau(d)\)-closed) subset of \(X\) and \(fY \subset TY\) (resp. \(TX\) is a \(d\)-closed \((\tau(d)\)-closed) subset of \(X\) and \(fY \subset SY\)) and

(iii) \(d(fx, fy) \leq km_3(x, y)\), where

\[
m_3(x, y) = \max \left\{ d(Sx, Ty), \alpha [d(fx, Sx) + d(fy, Ty)], \alpha [d(fx, Ty) + d(fy, Sx)] \right\}
\]

for any \(x, y \in Y\), where \(k \geq 0, 0 < \alpha < 1\) together with \(k\alpha < 1\),

then there exist \(u, w \in Y\) such that \(fu = Su = Tw\).

Moreover, if \(Y = X\), then \(f, S\) and \(T\) have a common fixed point provided the pair \((f, S)\) is pointwise \(S\)-absorbing whereas the pair \((f, T)\) is pointwise \(T\)-absorbing.

**Corollary 2.4.5.** Let \((X, d)\) be a symmetric (semi-metric) space wherein \(d\) satisfies \((1C)\) (Hausdorffness of \(\tau(d)\)) and \((HE)\). If \(f, g, S\) and \(T\) are four self mappings of \(X\) which satisfy the following conditions:

(i) the pair \((f, S)\) satisfies the property (E.A) (resp. \((g, T)\) satisfies the property (E.A)),

(ii) \(SX\) is a \(d\)-closed \((\tau(d)\)-closed) subset of \(X\) and \(fX \subset TX\) (resp. \(TX\) is a \(d\)-closed \((\tau(d)\)-closed) subset of \(X\) and \(gX \subset SX\)).
(iii) \( d(fx, gy) < m_1(x, y) \), where

\[
m_1(x, y) = \max \left\{ d(Sx, Ty), \alpha [d(fx, Sx) + d(gy, Ty)], \alpha [d(fx, Ty) + d(gy, Sx)] \right\},
\]
then there exist \( u, w \in X \) such that \( fu = Su = Tw = gw \).

Also \( f, g, S \) and \( T \) have a unique common fixed point provided the pair \((f, S)\) is pointwise \( S \)-absorbing whereas the pair \((f, T)\) is pointwise \( T \)-absorbing.

**Proof.** The proof can be completed on the lines of Corollary 2.4.3, hence details are not included.

**Theorem 2.4.4.** Let \( f, g, S, T : Y \to X \) be four mappings where \( Y \) is an arbitrary non-empty set and \( X \) is a non-empty set equipped with a symmetric (semi-metric) distance \( d \) wherein \( d \) satisfies \((W_3)\) (Hausdorffness of \( \tau(d) \)) and \((HE)\). Suppose that

(i) the pairs \((f, S)\) and \((g, T)\) share the common property \((E.A)\),

(ii) \( TY \) and \( SY \) are \( d \)-closed \((\tau(d)\)-closed) subsets of \( X \) and

(iii) \( d(fx, gy) \leq km(x, y) \),

for any \( x, y \in Y \), where \( k \geq 0 \) and \( m(x, y) \) is the same as earlier,

then the pairs \((f, S)\) and \((g, T)\) have a coincidence point each.

Moreover, if \( Y = X \), then \( f, g, S \) and \( T \) have a common fixed point provided the pair \((f, S)\) is pointwise \( S \)-absorbing whereas the pair \((f, T)\) is pointwise \( T \)-absorbing.

**Proof.** The proof can be completed on the lines of Theorem 2.2.4 and Theorem 2.4.1, hence details are not included.

**Theorem 2.4.5.** Theorem 2.4.4 remains true if condition (iii) (of Theorem 2.4.4) is replaced by

(i) \( d(fx, gy) \leq km_1(x, y) \),

for all \( x, y \in Y \), \( k\alpha < 1 \) with \( m_1(x, y) \) is the same as earlier whereas \((W_3)\) is replaced by \((1C)\) besides retaining rest of the hypotheses.

**Proof.** The proof can be completed on the lines of proof of Theorem 2.4.4, hence details are not included.

**Theorem 2.4.6.** Let \( Y \) be an arbitrary set whereas \((X, d)\) be a symmetric (semi-metric) space equipped with a symmetric (semi-metric) \( d \) which enjoys \((W_3)\) and \((HE)\). Let \( f, g, S, T : Y \to X \) be four mappings which satisfy

(i) the pair \((f, S)\) (or \((g, T)\)) satisfies the property \((E.A)\),

(ii) \( TY \) and \( SY \) are \( d \)-closed \((\tau(d)\)-closed) subsets of \( X \),

(iii) \( fY \subset TY \) or \((gY \subset SY)\) and
(iv) \(d(fx, gy) \leq km(x, y)\),

for any \(x, y \in Y\), where \(k \geq 0\) and \(m(x, y)\) is the same as earlier, then the pairs \((f, S)\) and \((g, T)\) have a coincidence point.

Moreover, if \(Y = X\), then \(f, g, S\) and \(T\) have a common fixed point provided the pair \((f, S)\) is pointwise \(S\)-absorbing while the pair \((f, T)\) is pointwise \(T\)-absorbing.

**Proof.** The proof can be completed on the lines of Theorem 2.2.6 and Theorem 2.4.4, hence details are not included.

**Theorem 2.4.7.** Theorem 2.4.6 remains true if \((W_3)\) is replaced by \((1C)\) whereas condition (iv) (of Theorem 2.4.6) is replaced by

(i) \(d(fx, gy) \leq km_1(x, y)\),

for all \(x, y \in Y\) where \(m_1(x, y)\) is the same as earlier with \(k\alpha < 1\) besides retaining rest of the hypotheses.

**Proof.** Proceeding on the lines of the proof of Theorem 2.4.4, one can complete the proof of this theorem, hence details are not included.

Finally, we present an example which illustrates the applicability (and at the same time non-applicability of commuting type mappings e.g. \([83, 125, 147, 155]\)) of pointwise absorbing mappings for producing common fixed points for mappings satisfying Lipschitz type or non-contractive type conditions.

**Example 2.4.3.** Let \(Y = X = [2, 20]\) equipped with the symmetric \(d(x, y) = (x - y)^2\) which also satisfies \((W_3)\) and \((HE)\). Set \(f = g\) and \(S = T\) and define as follows:

\[f, S : X \rightarrow X\] as follows:

\[
fx = \begin{cases} 
6, & \text{if } 2 \leq x < 6, \text{ or } x > 6, \\
\frac{13}{2}, & \text{if } x = 6 
\end{cases}
\]

and

\[
Sx = \begin{cases} 
5, & \text{if } 2 \leq x \leq 5, \\
\frac{x + 7}{2}, & \text{if } 5 < x \leq 6, \\
10, & \text{if } 6 < x < \frac{13}{2}, \text{ or } x > \frac{13}{2}, \\
6, & \text{if } x = \frac{13}{2}.
\end{cases}
\]

Then, it can be verify that
(i) the closure of $fX$ is contained in $SX = TX$,

(ii) $f$ and $S$ satisfy the Lipschitz type condition for any $k > 1$,

(iii) also, the pair $(f, S)$ is non-compatible (hence tangential).

Notice that 6 is the coincidence point of the pair $(f, S)$, but $f$ and $S$ have no common fixed point in $X$. Here, it may be pointed out that pair $(f, S)$ is not pointwise absorbing at $x = 6$. 