Chapter 1

Preliminaries

1.1 Introduction

Historically, the origin of fixed point theorem is attributed to the work on differential equations by the French mathematicians H. Poincare and Emile Picard. Proving results like Poincare-Bendixson theorem requires the use of topological methods. This work at the end of the 19th century opened into several successive versions of the theorem. The general case was first proved in 1910 by Jacques Hadamard, and then in 1912 by Luitzen Egbertus Jan Brouwer.

The Brouwer fixed point theorem was one of the early achievements of algebraic topology, and is the basis of more general fixed point theorems which are important in functional analysis. The case $n = 3$ was first proved by Piers Bohl in 1904 (published in Journal für die reine und angewandte Mathematik). It was later rediscovered by Brouwer in 1909. Hadamard proved the general case in 1910, and Brouwer [25] found a different proof in 1912. Since these early proofs were all non-constructive indirect proofs, they ran contrary to Brouwer’s intuitionist ideals. Brouwer fixed point theorem states that for any continuous function $f$ (with certain properties), there is a point $x_0$ such that $f(x_0) = x_0$. The simplest form of Brouwer’s theorem is for continuous functions $f$ from a disk $D$ to itself. A more general form is for continuous functions from a convex compact subset $C$ of Euclidean space $\mathbb{R}^n$ to itself.

Among hundreds of fixed point theorems, Brouwer’s is particularly well known mainly due to its use across numerous fields of mathematics. In its original field, this result is one of the key theorems characterizing the topology of Euclidean spaces along with the Jordan curve theorem, the hairy ball theorem and the Borsuk-Ulam theorem. This earns Brouwer’s fixed point theorem a place among the fundamental theorems of topology. The theorem is also used for proving deep results about differential equations which is presently covered in most introductory courses on differential geometry. It appears in unlikely fields such as game theory. In economics, Brouwer’s fixed point theorem and its extension, the Kakutani fixed point
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Theorem, play a central role in the proof of existence of general equilibrium in market economies as developed in 1950s by economists and Nobel prize winners Gerard Debreu and Kenneth Arrow. For a relatively up to date account on Brouwer fixed point theorem, one is referred to the survey article of Sehie Park [128].

There exists a vast literature on fixed point theory and this still continues to be a very active field of research. Fixed point theory is broadly divided into three major areas:

(i) Topological Fixed Point Theory.

(ii) Metric Fixed Point Theory.

(iii) Discrete Fixed Point Theory.

Historically speaking, these three areas were originated by the discovery of following three major theorems:

(i) Brouwer’s Fixed Point Theorem.

(ii) Banach’s Fixed Point Theorem.

(iii) Tarski’s Fixed Point Theorem.

Theorems concerning the existence and properties of fixed points are known as fixed point theorems. Such theorems are very important tools for proving the existence and uniqueness of the solutions to various mathematical models representing phenomena arising in different fields such as: steady state temperature distribution, chemical equations, neutron transport theory, economic theories, epidemics and flow of fluids besides facilitating existence and uniqueness theories of differential, integral and partial differential equations, variational inequalities etc. They are also used to study the optimal control problems of suitable systems. Fixed point theorems on ordered Banach spaces provide us exact or approximate solutions of boundary value problems.

1.2 Banach contraction principle

The term metric fixed point theory refers to those fixed point theoretic results in which geometric conditions on the underlying spaces and/or mappings play a crucial role. Although a substantial number of definitive results have already been discovered, a few lying at the heart of the theory remain open and there are many questions waiting answers regarding the limits to which the theory may be extended. Some of these questions are merely tantalizing while others suggest substantial new avenues of research. The first contractive definition is due to Banach [17] which came in 1922. The conclusion of the theorem is that $T$ has a unique fixed point, which can be explicitly reached from any starting value $z_0 \in X$. Also, this definition implies that $T$ is continuous.
In 1977, in his survey article Rhoades [140] investigated all the available contractive definitions till date and analyzed the interrelationships among them. Some of these contractive conditions which are relevant to our presentation are noted as follows:

(i) Banach [17]: There exists a number \( k, 0 \leq k < 1 \), such that, for each \( x, y \in X \),
\[
d(Tx, Ty) \leq kd(x, y).
\]

(ii) Rakotch [137]: There exists a monotone decreasing function \( k : (0, \infty) \to [0, 1) \) such that, for each \( x, y \in X, x \neq y \),
\[
d(Tx, Ty) \leq kd(x, y).
\]

(iii) Kannan [95]:
\[
d(Tx, Ty) \leq k \left[ d(x, Tx) + d(y, Ty) \right]
\]
with \( k < \frac{1}{2} \).

(iv) Boyd and Wong [22]:
\[
d(Tx, Ty) \leq \phi(d(x, y)), \quad \text{where } \phi : \mathbb{R} \to \mathbb{R}, \phi(t) < t
\]
for \( t > 0 \) together with a semicontinuity condition of \( \phi \).

(v) Bianchini [19]:
\[
d(Tx, Ty) \leq k \max\{d(x, Tx), d(y, Ty)\}
\]
with \( k < 1 \).

(vi) Chatterjea [30]:
\[
d(Tx, Ty) \leq k \left[ d(x, Ty) + d(y, Tx) \right]
\]
with \( k < \frac{1}{2} \).

(vii) Hardy and Rogers [60]:
\[
d(Tx, Ty) \leq a \left[ d(x, Tx) + d(y, Ty) \right] + b \left[ d(x, Ty) + d(y, Tx) \right] + c \ d(x, y)
\]
for all \( x, y \in X \) with \( x \neq y \), \( a, b, c \geq 0 \) such that \( 2a + 2b + c < 1 \).

(viii) Khan et al. [101]:
\[
\phi(d(Tx, Ty)) \leq a \ d(x, y) \phi(d(x, y)) + b \ d(x, y) \left[ \phi(d(x, Tx)) + \phi(d(y, Ty)) \right] \\
+ c \ d(x, y) \min\{\phi(d(x, Ty)) + \phi(d(y, Tx))\}
\]
for all \( x, y \in X \) with \( x \neq y \), \( a, b, c \) are three decreasing functions from \( \mathbb{R}^+ - \{0\} \to [0, 1) \) such that \( a(t) + 2b(t) + c(t) < 1 \) for every \( t > 0 \), where \( \phi : \mathbb{R}^+ \to \mathbb{R}^+ \) is an increasing and continuous function satisfying \( \phi(t) = 0 \) if and only if \( t = 0 \).

During the past few years a number of contractive definitions have appeared in which the parameter \( k \) is 1, and the inequality is strict for \( x, y \) distinct. For example, Edelstein’s condition \( d(Tx, Ty) < d(x, y) \), Sehgal’s condition \( d(Tx, Ty) < \max\{d(x, y), d(x, Tx), d(y, Ty)\} \), Rhoades’s condition \( x, y \in X \),
\[
d(Tx, Ty) < m(x, y)
\]
where \( m(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), [d(x, Tx) + d(y, Ty)]/2\} \) are worth mentioning.
A few ideas in 1974 were given by Ciric [34] using the following contractive
definition: there exists a number $k$, $0 \leq k < 1$ such that, for each $x, y \in X$,
\[
d(Tx, Ty) \leq k \, m(x, y)
\]
where $m(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}$. Ciric showed
that any mapping $T$ satisfying the above inequality has a bounded orbit, and from
this fact he proved that $T$ has a unique fixed point.

The generalization of existing idea is a very important tool for the advancement
in the field of science and technology. The present thesis has been written on the
same basis. In this thesis, fixed point theorems and related results in metric spaces,
symmetric (semi-metric) spaces, Menger PM-spaces, fuzzy metric spaces, intuition-
istic fuzzy metric spaces, non-Archimedean fuzzy metric spaces and complex valued
metric spaces have been discussed and many results are generalized.

### 1.3 Implicit functions

In 1999, Popa [131] initiated the idea of implicit functions rather than a single
contraction condition to prove fixed point theorems in metric spaces whose strength
lies in its unifying power as an implicit function can cover several contraction condi-
tions in one go which include known as well as unknown contraction conditions. This
fact is evident from examples furnished in Popa [131]. In [131], a general fixed point
theorem for compatible mappings satisfying an implicit relation was proved and in
[80] the results from [131] were improved by lightning the compatibility conditions
to weak compatibility. Quite recently, Ali and Imdad [3] proved some common fixed
point theorems for two pairs of weakly compatible mappings under a new implicit
function.

Popa [133] utilized the idea of implicit functions to prove hybrid fixed point
theorems which can be described as follows:

Let $\Psi$ be the set of all real lower semi-continuous functions $F(t_1, t_2, \ldots, t_6) : \mathbb{R}_+^6 \to \mathbb{R}$ which satisfy the following conditions $G_1, G_2$ and $G_3$:

1. $(G_1)$: $F$ is non-increasing in variables $t_2, \ldots, t_6$ and non-decreasing in $t_1$,
2. $(G_2)$: There exists $h \in (0, 1)$ and $k > 1$ with $hk < 1$ such that
   \[
   u \leq kt \text{ and } F(t, v, v, u, u + v, 0) \leq 0 \text{ implies } t \leq hv,
   \]
3. $(G_3)$: $F(t, t, t, 0, t, t) > 0, \forall t > 0$.

Popa [133] also contains the following examples of implicit functions.

**Example 1.3.1.** $F(t_1, \ldots, t_6) = t_1 - a \max\{t_2, t_3, t_4, \frac{1}{2}(t_5 + t_6)\}$, where $a \in (0, 1)$. 
Example 1.3.2. \( F(t_1, \ldots, t_6) = t_1 - h[a \max\{t_2, t_3, t_4, \frac{1}{2}(t_5 + t_6)\}]
\]
\[ + (1 - a)[\max\{t_2^2, t_3t_4, t_5t_6, \frac{1}{2}t_3t_6, \frac{1}{2}t_4t_5\}]^{\frac{1}{2}}, \]
where \( h \in (0, 1) \) and \( 0 \leq a \leq 1 \).

Example 1.3.3. \( F(t_1, \ldots, t_6) = t_1^2 + t_2^2 + \frac{h}{1 + t_5t_6} - [at_2^2 + bt_3^2 + ct_4^2], \) where \( 0 < a + b + c < 1 \).

Most recently Imdad and Ali [67], defined a suitable implicit function to prove common fixed point theorems in fuzzy metric spaces which can be described as follows:

Let \( \Psi \) be the family of all continuous functions \( F : [0, 1]^4 \to \mathbb{R} \) satisfying the following conditions ([67]):

\((F_1)\) : For every \( u > 0, v \geq 0 \) with \( F(u, v, u, v) \geq 0 \) or \( F(u, v, v, u) \geq 0 \), we have \( u > v \).

\((F_2)\) : \( F(u, u, 1, 1) < 0, \forall u > 0 \).

Imdad and Ali [67] also contains the following examples of implicit functions.

Example 1.3.4. Define \( F : [0, 1]^4 \to \mathbb{R} \) as \( F(t_1, t_2, t_3, t_4) = t_1 - \phi(\min\{t_2, t_3, t_4\}), \) where \( \phi : [0, 1] \to [0, 1] \) is continuous and \( \phi(s) > s \) for \( 0 < s < 1 \).

Example 1.3.5. Define \( F : [0, 1]^4 \to \mathbb{R} \) as
\[
F(t_1, t_2, t_3, t_4) = t_1 - a \min\{t_2, t_3, t_4\}, \text{ where } a > 1.
\]

Example 1.3.6. Define \( F : [0, 1]^4 \to \mathbb{R} \) as
\[
F(t_1, t_2, t_3, t_4) = t_1 - at_2 - \min\{t_3, t_4\}, \text{ where } a > 0.
\]

Example 1.3.7. Define \( F : [0, 1]^4 \to \mathbb{R} \) as
\[
F(t_1, t_2, t_3, t_4) = t_1 - at_2 - b(t_3 + t_4),
\]
where \( a > 1, \ b \geq 0(\neq 1) \).

Example 1.3.8. Define \( F : [0, 1]^4 \to \mathbb{R} \) as
\[
F(t_1, t_2, t_3, t_4) = t_1^3 - a t_2t_3t_4, \text{ where } a > 1.
\]

1.4 Symmetric spaces

A symmetric (semi-metric) \( d \) in respect of a non-empty set \( X \) is a function \( d : X \times X \to [0, \infty) \) which satisfies \( d(x, y) = d(y, x) \) and \( d(x, y) = 0 \iff x = y \) (for
all \( x, y \in X \). If \( d \) is a symmetric (semi-metric) on a set \( X \), then for \( x \in X \) and \( \epsilon > 0 \), we write \( B(x, \epsilon) = \{ y \in X : d(x, y) < \epsilon \} \). A topology \( \tau(d) \) on \( X \) is given by the sets \( U \) (along with empty set) in which for each \( x \in U \), one can find some \( \epsilon > 0 \) such that \( B(x, \epsilon) \subset U \). A set \( S \subset X \) is a neighbourhood of \( x \in X \) if and only if there is a \( U \) containing \( x \) such that \( x \in U \subset S \). Thus a symmetric (semi-metric) space \((X, d)\) is a topological space whose topology \( \tau(d) \) on \( X \) is induced by a symmetric (semi-metric) \( d \). A symmetric \( d \) is said to be a potent semi-metric (cf. [4]) if for each \( x \in X \) and for each \( \epsilon > 0 \), \( B(x, \epsilon) \) is a neighbourhood of \( x \) in the topology \( \tau(d) \). Notice that \( \lim_{n \to \infty} d(x_n, x) = 0 \) if and only if \( x_n \to x \) in the topology \( \tau(d) \). The distinction between a symmetric and a potent semi-metric is apparent as one can easily construct a symmetric \( d \) such that \( B(x, \epsilon) \) need not be a neighbourhood of \( x \) in \( \tau(d) \). As symmetric (or semi-metric) spaces are not essentially Hausdorff, therefore in order to prove fixed point theorems, some additional axioms are required. The following axioms are relevant to this thesis which are available in Aliouche [5], Cho et al. [31], Galvin and Shore [48], Hicks and Rhoades [62] and Wilson [172]. From now on symmetric as well as potent semi-metric spaces will be denoted by \((X, d)\).

\( (W_3) \) : [172] Given \( \{x_n\} \), \( x \) and \( y \) in \( X \) with \( d(x_n, x) \to 0 \) and \( d(x_n, y) \to 0 \) imply \( x = y \).

\( (W_4) \) : [172] Given \( \{x_n\}, \{y_n\} \) and an \( x \) in \( X \) with \( d(x_n, x) \to 0 \) and \( d(x_n, y_n) \to 0 \) imply \( d(y_n, x) \to 0 \).

\( (HE) \) : [5, 48] Given \( \{x_n\}, \{y_n\} \) and an \( x \) in \( X \) with \( d(x_n, x) \to 0 \) and \( d(y_n, x) \to 0 \) imply \( d(x_n, y_n) \to 0 \).

\( (1C) \) : [31, 48] A symmetric \( d \) is said to be 1-continuous if \( \lim_{n \to \infty} d(x_n, x) = 0 \) implies \( \lim_{n \to \infty} d(x_n, y) = d(x, y) \).

\( (CC) \) : [31, 48] A symmetric \( d \) is said to be continuous if \( \lim_{n \to \infty} d(x_n, x) = 0 \) and \( \lim_{n \to \infty} d(y_n, y) = 0 \) imply \( \lim_{n \to \infty} d(x_n, y_n) = d(x, y) \) where \( x_n, y_n \) are sequences in \( X \) and \( x, y \in X \).

Clearly, the continuity (i.e. \( (CC) \)) of a symmetric is a stronger property than \( (1C) \) (or 1-continuity) i.e. \( (CC) \) implies \( (1C) \) but not conversely. Also \( (W_4) \) implies \( (W_3) \) and \( (1C) \) implies \( (W_3) \) but converse implications are not true. All other possible implications amongst \( (W_3) \), \( (1C) \) and \( (HE) \) are not true in general whose nice illustrations via demonstrative examples are available in Cho et al. [31]. But \( (CC) \) implies all the remaining four conditions namely: \( (W_3), (W_4), (HE) \) and \( (1C) \).

The following examples due to Cho et al. [31], show the relationships amongst \( (W_3), (W_4), (HE), (1C) \) and \( (CC) \).

Example 1.4.1. ([31]) \( (W_4) \not\Rightarrow (HE) \) and \( (W_4) \not\Rightarrow (CC) \) and so \( (W_3) \not\Rightarrow (HE) \) and \( (W_3) \not\Rightarrow (CC) \).
Let $X = [0, \infty)$ and let

$$d(x, y) = \begin{cases} 
|x - y| & \text{if } x \neq 0, \ y \neq 0 \\
\frac{1}{x} & \text{if } x \neq 0 \\
\frac{1}{y} & \text{if } y \neq 0.
\end{cases}$$

Then, $(X, d)$ is a symmetric space which satisfies $(W_4)$ but does not satisfy $(HE)$ for \{x_n = n\}, \{y_n = n + 1\}. Also $(X, d)$ does not satisfy $(CC)$.

**Example 1.4.2.** ([31]) $(HE) \not\Rightarrow (W_4)$ and so $(HE) \not\Rightarrow (W_4)$ and $(HE) \not\Rightarrow (CC)$. Let $X = [0, 1] \cup \{2\}$ and let

$$d(x, y) = \begin{cases} 
|x - y| & \text{if } 0 \leq x \leq 1, \ 0 \leq y \leq 1 \\
|x| & \text{if } 0 < x \leq 1, \ y = 2 \\
|y| & \text{if } 0 < y \leq 1, \ x = 2.
\end{cases}$$

and $d(0, 2) = 1$ Then, $(X, d)$ is a symmetric space which satisfies $(HE)$. Let \{x_n = 1/n\}. Then, \(\lim_{n \to \infty} d(x_n, 0) = \lim_{n \to \infty} d(x_n, 2) = 0\). But $d(0, 2) \neq 0$ and hence the symmetric space $(X, d)$ does not satisfy $(W_3)$.

**Example 1.4.3.** ([31]) $(CC) \not\Rightarrow (W_4)$ and so $(W_3) \not\Rightarrow (W_4)$.

Let $X = \{\frac{1}{n} : n = 1, 2, \ldots\} \cup \{0\}$ and let $d(0, \frac{1}{n}) = \frac{1}{n}$ (n is odd), $d(0, \frac{1}{n}) = 1$ (n is even) and

$$d(\frac{1}{n}, \frac{1}{m}) = \begin{cases} 
|\frac{1}{n} - \frac{1}{m}| & \text{if } m + n \text{ is even}, \\
|\frac{1}{n} - \frac{1}{m}| & \text{if } m + n \text{ is odd and } |m + n| = 1, \\
1 & \text{if } m + n \text{ is odd and } |m + n| > 2.
\end{cases}$$

Then, the symmetric space $(X, d)$ satisfies $(CC)$ but does not satisfy $(W_4)$ for \(\{x_n = \frac{1}{2n+1}\}\) and \(\{y_n = \frac{1}{2n}\}\).

**Example 1.4.4.** ([31]) $(CC) \not\Rightarrow (HE)$. Let $X = \{\frac{1}{n} : n = 1, 2, \ldots\} \cup \{0\}$ and

$$d(\frac{1}{n}, \frac{1}{m}) = \begin{cases} 
|\frac{1}{n} - \frac{1}{m}| & \text{if } |m + n| \geq 2, \\
1 & \text{if } |m + n| = 1.
\end{cases}$$

and $d(\frac{1}{n}, 0) = \frac{1}{n}$. Then, $(X, d)$ is a symmetric space which satisfies $(CC)$. If \(\{x_n = \frac{1}{n}\}\) and \(\{y_n = \frac{1}{(n+1)}\}\), then, $\lim_{n \to \infty} d(x_n, 0) = \lim_{n \to \infty} d(y_n, 0) = 0$. But $\lim_{n \to \infty} = d(x_n, y_n) \neq 0$. Hence, the symmetric space $(X, d)$ does not satisfy $(HE)$. 


Recall that a sequence \( \{x_n\} \) in a symmetric (semi-metric) space \((X, d)\) is said to be \(d\)-Cauchy sequence if it satisfies the usual metric condition. Here, one needs to notice that in a symmetric (semi-metric) space, Cauchy convergence criterion is not a necessary condition for the convergence of a sequence but this criterion becomes a necessary condition if symmetric (semi-metric) is suitably restricted (cf. Wilson [172]). In [28], Burke furnished an illustrative example to show that a convergent sequence in symmetric (semi-metric) spaces need not admit Cauchy subsequence. But, he was able to formulate an equivalent condition under which every convergent sequence in symmetric (semi-metric) space admits a Cauchy subsequence. There are several concepts of completeness in symmetric (or semi-metric) space, e.g. \(S\)-completeness, \(d\)-Cauchy completeness, strong and weak completeness whose details are available in Galvin and Shore [48] but we omit the details as such notions are not relevant to our presentation.

**Definition 1.4.1.** ([28]) If \((X, d)\) is a symmetric (semi-metric) space, then a sequence \(\{x_n\}\) in \(X\) is said to be Cauchy if for any \(\epsilon > 0\) there is some \(k \in \mathbb{N}\) such that \(d(x_n, x_m) < \epsilon\) for all \(n, m \geq k\).

**Example 1.4.5.** ([28]) Let \(X = A \cup B\) where \(A = \{(0, y) \in \mathbb{R}^2: -1 \leq y \leq 1\}\) and \(B\) is the graph of the equation \(y = \sin(1/x)\) for \(0 < x \leq 1\). If \(u = (x_1, y_1)\) and \(v = (x_2, y_2)\) are elements of \(X\), then define \(d(u, r) = [(x_1 - x_2)^2 + (y_1 - y_2)^2]^{1/2}\) if at least one of \(u\) or \(v\) is in \(A\). Also, define \(d(u, v)\) to be the arc length between \(u\) and \(v\) if both \(u\) and \(v\) are in \(B\). Then \(d\) is a symmetric (semi-metric) for the usual topology on \(X\) and there are no Cauchy sequences in \(B\) which converge to points in \(A\).

**Theorem 1.4.1.** ([28]) Let \((X, d)\) be any symmetric space (semi-metric space). The following conditions are equivalent:

(i) Every convergent sequence in \(X\) has a Cauchy subsequence.

(ii) If \(\{x_n\}\) is a convergent sequence in \(X\) and \(\epsilon\) is a positive number then, there is a subsequence \(\{z_n\}\) of \(\{x_n\}\) such that \(d(z_n, z_m) < \epsilon\) for all \(n, m \in \mathbb{N}\).

(iii) If \(F \subseteq X\) and there is a positive \(\epsilon\) such that \(d(x, y) \geq \epsilon\) (for all distinct \(x, y \in F\)), then \(F\) is closed.

Theorem 1.4.1 is useful in the sense that it reflects the impact that condition (i) has on the topology of \(X\) (as in condition (iii)). In the succeeding Theorem 1.4.2, it turns out that condition (iii) is much easier to use than condition (i). Condition (iii) has been discussed previously in several papers (e.g. [12, 102, 122]), and is known as the weak condition of Cauchy. In [102] there is an example of a symmetric space which has no symmetric satisfying the weak condition of Cauchy. Hence, the analogue of Theorem 1.4.2 is not true in symmetric spaces.
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Theorem 1.4.2. ([28]) Every symmetric (semi-metric) space \((X, d)\) is semi-metrizable by a compatible symmetric \(d\) where every convergent sequence in \(X\) has a Cauchy subsequence.

1.5 Probabilistic metric spaces

The concept of an abstract metric space was introduced by M. Frechet in 1906, presents a natural frame of a large number of mathematical, physical and other scientific constructs in which the notion of “distance”, appears. The objects under consideration may be of most varied type. They may be points, functions, sets and even the subjective experiences of sensations. What turns out to be very sweet is the possibility of associating a non-negative real number with each ordered pair of elements of a certain set, and that the numbers associated with pairs and triples of such elements satisfy certain natural axioms patterned after natural distance.

However, in numerous instances in which the theory of metric spaces is applied, this very association of a single number with a pair of elements is, realistically speaking, an over-idealization. This is so even in the measurement of an ordinary length, where the number given as the distance between two points is often not the result of a single measurement, but the average of a series of measurement. Indeed, in this and many similar situations, it is appropriate to look upon the distance concept as a statistical rather than a determinate one. More precisely, Menger [110] replaced the metric function \(d : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}^+\) with a distribution function \(F_{p,q} : \mathbb{R} \rightarrow [0, 1]\) and then for any positive number \(x\) the value \(F_{p,q}(x)\) was interpreted as the probability that the distance between \(p\) and \(q\) is less than \(x\). When this is done, one obtains a generalization of the concept of a metric space which was first introduced by K. Menger in 1942 and, following him, is called a statistical metric space.

The history of statistical metric spaces is brief. In the original paper, Menger [110] gave postulates for the distribution functions \(F_{p,q}\). These included a generalized triangle inequality. In addition, he constructed a theory of betweenness and indicated possible fields of application. In 1951 Menger continued his study of statistical metric spaces in a paper [110] devoted to a resume of the earlier work, the construction of several specific examples and further considerations of the possible applications of the theory.

While defining PM-Spaces, we use the notion of distribution function which runs as follows:

**Definition 1.5.1.** A mapping \(F : \mathbb{R} \rightarrow \mathbb{R}^+\) is said to be a distribution function if it is non decreasing and left continuous with \(\inf F = 0\) and \(\sup F = 1\).

A frequently used specific distribution function \(H\) defined as

\[
H(x) = \begin{cases} 
1, & \text{if } x > 0 \\
0, & \text{if } x \leq 0.
\end{cases}
\]
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Generally the set of all distribution function is denoted by \( D \).

**Definition 1.5.2.** A PM-Space in the sense of Schweizer and Sklar [149] is an ordered pair \((X, d)\), where \( X \) is a non-empty set and \( F \) is a map \( F : X \times X \to D \)
i.e. \( F \) associates a distribution function \( F(p, q) \) with every pair \((p, q) \in X\).

We denote distribution function \( F(p, q) \) by \( F_{p,q} \) for the real argument \( x \).

**Definition 1.5.3.** The function \( F_{p,q} \) satisfies the following properties:

(i) (PM-1): \( F_{p,q}(x) = 1 \) for all \( x > 0 \iff p = q \).

(ii) (PM-2): \( F_{p,q}(0) = 0 \),

(iii) (PM-3): \( F_{p,q} = F_{q,p} \) and

(iv) (PM-4): If \( F_{p,q}(x) = 1 \) and \( F_{q,r}(y) = 1 \), then \( F_{p,r}(x + y) = 1 \).

**Remark 1.5.1.** In view of the condition (PM-2) which obviously implies that \( F_{p,q}(x) = 0 \) for \( x \leq 0 \) and condition (PM-2) is equivalent to statement \( p = q \iff F_{p,q} = H \).

**Remark 1.5.2** Every metric space can be regarded as a PM-Space (of a special kind) if we set \( F_{p,q}(x) = H(x - d(p, q)) \).

The condition (PM-4) of above definition is always satisfied in a metric space wherein it reduces to the ordinary triangle inequality. However, in those PM-Spaces in which the equality \( F_{p,q}(x) = 1 \) does not hold for any finite \( x \), the condition (PM-4) will be satisfied vacuously.

**Definition 1.5.4.** A triangular inequality is said to hold universally in a PM-space iff it holds for all triples of points (distinct or not) in that space.

Let \( \Delta : [0, 1] \times [1, 0] \to [0, 1] \) be a 2-place function satisfying the following conditions:

(i) (\( \Delta \)-1): \( 0 \leq \Delta(a, b) \leq 1 \),

(ii) (\( \Delta \)-2): \( \Delta(c, d) \geq \Delta(a, b) \), for \( c \geq a, d \geq b \),

(iii) (\( \Delta \)-3): \( \Delta(a, b) = \Delta(b, a) \),

(iv) (\( \Delta \)-4): \( \Delta(1, 1) = 1 \),

(v) (\( \Delta \)-5): \( \Delta(a, 1) > 0 \) for all \( a > 0 \).

**Remark 1.5.3.** K. Menger introduced as generalized triangle inequality (also referred as Menger triangle inequality) as (PM-5): \( F_{p,r}(x + y) \geq \Delta(F_{p,q}(x), F_{q,r}(y)) \), where \( \Delta \) is a 2-place function satisfies above (\( \Delta \)-1) to (\( \Delta \)-5). From condition (\( \Delta \)-4), we see that (PM-5) contains the condition (PM-4) as a special case. There are numerous possible choice for \( \Delta \). The following six are most natural and simplest examples of 2-place functions:
(i) $\Delta_1(a, b) = \max(a + b - 1, 0)$,
(ii) $\Delta_2(a, b) = a.b,$
(iii) $\Delta_3(a, b) = \min(a, b)$,
(iv) $\Delta_4(a, b) = \max(a, b)$,
(v) $\Delta_5(a, b) = a + b - ab$,
(vi) $\Delta_6(a, b) = \min(a + b, 1)$.

Lemma 1.5.1. If a PM-Space contains two distinct points, then the condition (PM-5) cannot hold universally in the space under the choice $\Delta_4 = \max$.

Lemma 1.5.2. If a PM-Space is not a metric space and the condition (PM-5) holds universally in the space for some choice of $\Delta$ satisfying the condition (\Delta-1) to (\Delta-5), then the function $\Delta$ has the property that there exists a number $a, 0 < a < 1$, such that $\Delta(a, 1) \leq a$.

Theorem 1.5.1. If the condition (PM-5) holds universally in a PM-Space and $\Delta$ is continuous, then for any $x > 0$, $\Delta(F_{p,q}(x), 1) \leq F_{p,q}(x)$.

In view of the preceding lemmas and the fact that the three weaker functions in our list of $\Delta'$s satisfy $\Delta(a, 1) = a$, we are led to replace the conditions (\Delta-1), (\Delta-4) and (\Delta-5) by the following conditions:

(i) (\Delta-6): $\Delta(a, 1) = a$ and $\Delta(0, 0) = 0$.

Thus far we also add the associativity condition:

(ii) (\Delta-7): $\Delta(\Delta(a, b), c) = \Delta(a, \Delta(b, c))$,

which permits the extension of condition (\Delta-6).

Definition 1.5.5. A Menger PM-Space is a PM-Space in which the condition (PM-5) holds universally for some choice of $\Delta$ satisfying (\Delta-2), (\Delta-3), (\Delta-6) and (\Delta-7).

Definition 1.5.6. A triangular norm (or a $t$-norm) is a 2-place function $\Delta : [0, 1] \times [0, 1] \rightarrow [0, 1]$ satisfying the condition (\Delta-2), (\Delta-3), (\Delta-6) and (\Delta-7).

For classical examples of continuous $t$-norm, one may recall $t$-norms $T_L, T_P$ and $T_M$ which are respectively defined as $T_L(a, b) = \max\{a + b - 1, 0\}$, $T_P(a, b) = ab$ and $T_M(a, b) = \min\{a, b\}$.

The following lemma shows that, in determining whether or not a PM-Space is a Menger PM-Space, only triple of distinct points are to be considered.
Lemma 1.5.3. If the points \( p, q, r \) are not all distinct, then the condition (PM-5) holds for the triple \( p, q, r \) under any choice of \( \Delta \) satisfying (\( \Delta \)-2), (\( \Delta \)-3), (\( \Delta \)-6) and (\( \Delta \)-7).

1.6 Fuzzy metric spaces

It proved a turning point in the development of mathematics when the notion of fuzzy set was introduced by Zadeh [174] which laid the foundation of fuzzy mathematics. Since then many authors have extensively developed the theory of fuzzy sets and applications. Especially, Deng [36], Erceg [45], Kaleva and Seikkala [93], Kramosil and Michalek [104] and Xia and Guo [173] have introduced the concepts of fuzzy metric spaces in different ways. First ever attempt to define fuzzy metric spaces is due to Kramosil and Michalek [104].

A fuzzy metric space in the sense of Kramosil and Michalek [104] is defined as follows:

Definition 1.6.1. The 3-tuple \((X, M, \ast)\) is said to be a fuzzy metric space if \(X\) is an arbitrary set, \(\ast\) is a continuous \(t\)-norm and \(M\) is a fuzzy set on \(X^2 \times [0, \infty)\) satisfying the following conditions (for all \(x, y, z \in X\) and \(t, s > 0\)):

(i) \(M(x, y, 0) = 0\);

(ii) \(M(x, y, t) = 1 \forall t > 0\) iff \(x = y\);

(iii) \(M(x, y, t) = M(y, x, t)\);

(iv) \(M(x, z, t + s) \geq M(x, y, t) \ast M(y, z, s)\);

(v) \(M(x, y, .) : [0, \infty) \to [0, 1]\) is left continuous.

In what follows, we refer such spaces as KM-fuzzy metric spaces.

Lemma 1.6.1. ([57]) For every \(x, y \in X\), the mapping \(M(x, y, .)\) is nondecreasing on \((0, \infty)\).

Remark 1.6.1. ([49]) \(M(x, y, t)\) can be thought of as the degree of nearness between \(x\) and \(y\) with respect to \(t\). We identify \(x = y\) with \(M(x, y, t) = 1\), for \(t > 0\) and \(M(x, y, t) = 0\) with \(\infty\).

In this context George and Veeramani [49] modify the above definition in order to introduce a Hausdorff topology on the fuzzy metric space.

A fuzzy metric space in the sense of George and Veeramani [49] is defined as follows:

Definition 1.6.2. The 3-tuple \((X, M, \ast)\) is said to be a fuzzy metric space if \(X\) is an arbitrary set, \(\ast\) is a continuous \(t\)-norm and \(M\) is a fuzzy set on \(X^2 \times (0, \infty)\) satisfying the following conditions (for all \(x, y \in X\) and \(t, s > 0\)): 
(i) \( M(x, y, t) > 0; \)
(ii) \( M(x, y, t) = 1 \) iff \( x = y; \)
(iii) \( M(x, y, t) = M(y, x, t); \)
(iv) \( M(x, z, t + s) \geq M(x, y, t) \ast M(y, z, s); \)
(v) \( M(x, y, .) : (0, \infty) \to [0, 1] \) is continuous.

In view of (i) and (ii), it is worth pointing out that \( 0 < M(x, y, t) < 1 \) (for all \( t > 0) \)
provided \( x \neq y \), (cf. [117]). In what follows, fuzzy metric spaces in the sense of
George and Veeramani [49] will be referred as GV-fuzzy metric spaces.

Example 1.6.1. Let \( X = \mathbb{R} \). Define \( a \ast b = ab \) and \( M(x, y, t) = \left[ e^{\left( \frac{|x-y|}{t} \right)} \right]^{-1} \) for
all \( x, y \in X \) and \( t > 0 \). Then \((X, M, \ast)\) is a fuzzy metric space.

Example 1.6.2. Let \( X = \mathbb{N} \). Define \( a \ast b = ab \) and for all \( t > 0 \)

\[
M(x, y, t) = \begin{cases} 
  x/y, & \text{if } x \leq y \\
  y/x, & \text{if } y \leq x.
\end{cases}
\]

Then \((X, M, \ast)\) is a fuzzy metric space.

However, Grabiec [57] was the first mathematician who proved Banach contraction principle and Edelstein [42] theorem in fuzzy metric spaces (in sense of Kramosil and Michalek [104]). He also defined Cauchy sequence in the following way.

Definition 1.6.3. A sequence \( \{x_n\} \) in a fuzzy metric space \((X, M, \ast)\) is Cauchy if
\( \lim_{n \to \infty} M(x_{n+p}, x_n, t) = 1 \) for each \( t > 0 \) and \( p > 0 \).

Theorem 1.6.1. ([57]) Let \((X, M, \ast)\) be a complete fuzzy metric space such that
\( \lim_{t \to \infty} M(x, y, t) = 1 \) for all \( x, y \in X \). Let \( T \) be a self mapping of \( X \) satisfying
\[
M(Tx, Ty, kt) \geq M(x, y, t) \quad \text{for all } x, y \in X, \ 0 < k < 1.
\]
Then \( T \) has a unique fixed point.

Theorem 1.6.2. ([57]) Let \((X, M, \ast)\) be a compact fuzzy metric space and \( T \) be
a self mapping of \( X \) satisfying
\[
M(Tx, Ty, .) > M(x, y, .) \quad \text{for all } x \neq y \in X.
\]
Then \( T \) has a unique fixed point.

Here it may be pointed out that the above function \( M \) is not a fuzzy metric with
the \( t \)-norm defined as \( a \ast b = \min\{a, b\}. \)
Definition 1.6.4. Let \((X, M, *)\) be a fuzzy metric space. One can define open ball \(B(x, r, t)\) as well as closed ball \(B[x, r, t]\) with center \(x \in X\) and radius \(r, 0 < r < 1, t > 0\) as
\[
B(x, r, t) = \{y \in X : M(x, y, t) > 1 - r\},
\]
\[
B[x, r, t] = \{y \in X : M(x, y, t) \geq 1 - r\}.
\]

Theorem 1.6.3. Every open (closed) ball is an open (a closed) set.

1.7 Intuitionistic fuzzy metric spaces

In an attempt to generalize the notion of fuzzy sets, Atanassov [13] introduced the notion of intuitionistic fuzzy sets which was subsequently developed by many authors (e.g. Coker [32], Atanassov [14, 15]). Intuitionistic fuzzy sets can be utilized in concrete specific consideration as an adequate tool to represent hesitancy concerning both membership and non-membership of an element to a set. To be more precise, a basic assumption of fuzzy set theory assumes that if we specify the degree of membership of an element in a fuzzy set as a real number from \([0, 1]\), say \(p\), then the degree of non-membership is automatically determined as \(1 - p\), need not be realistic in the context of intuitionistic fuzzy sets. In intuitionistic fuzzy set theory it is assumed that non-membership should not be more than \(1 - p\).

As opposed to a fuzzy set in \(X\), given by
\[
A' = \{\langle x, \mu_{A'}(x) \rangle : x \in X\}
\]
where \(\mu_{A'} : X \to [0, 1]\) is the membership function of the set \(A'\), an intuitionistic fuzzy set \(A\) is given by
\[
A = \{\langle x, \mu_A(x), \nu_A(x) \rangle : x \in X\}
\]
where \(\mu_A : X \to [0, 1]\) and \(\nu_A : X \to [0, 1]\) such that \(0 \leq \mu_A(x) + \nu_A(x) \leq 1\) and \(\mu_A(x), \nu_A(x) \in [0, 1]\) denote the degree of membership and degree of non-membership of \(x \in A\), respectively. Obviously, each fuzzy set can be realized as an intuitionistic fuzzy set in the following manner:
\[
A = \{\langle x, \mu_{A'}(x), 1 - \mu_{A'}(x) \rangle : x \in X\}.
\]

Definition 1.7.1. ([111]) A binary operation \(\Diamond : [0, 1] \times [0, 1] \to [0, 1]\) is a continuous t-conorm if

(i) \(\Diamond\) is commutative and associative;

(ii) \(\Diamond\) is continuous;

(iii) \(a \Diamond 0 = a\) for all \(a \in [0, 1]\);
(iv) \(a \diamond b \leq c \diamond d\) whenever \(a \leq c\) and \(b \leq d\) for all \(a, b, c, d \in [0, 1]\).

The concepts of t-norms and t-conorms are known as the axiomatic skeletons which are respectively utilized in characterizing fuzzy intersections and unions.

In 2004, Park [129] introduced and discussed a corresponding notion of intuitionistic fuzzy metric space which is designed using the idea of intuitionistic fuzzy sets due to Atanassov [13] and the concept of GV-fuzzy metric spaces. Truly speaking, Park’s notion is useful in modeling some phenomena wherein it is necessary to study relationship between two probability functions. This definition due to Park [129] is as follows:

**Definition 1.7.2.** ([129]) A 5-tuple \((X, M, N, *, \diamond)\) is said to be an intuitionistic fuzzy metric space if \(X\) is an arbitrary set, * is a continuous \(t\)-norm, \(\diamond\) is a continuous \(t\)-conorm and \(M, N\) are fuzzy sets on \(X^2 \times (0, \infty)\) satisfying the following conditions (for all \(x, y, z \in X\) and \(s, t > 0\)):

1. \(M(x, y, t) + N(x, y, t) \leq 1\),
2. \(M(x, y, t) > 0\),
3. \(M(x, y, t) = 1\) if and only if \(x = y\),
4. \(M(x, y, t) = M(y, x, t)\),
5. \(M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)\),
6. \(M(x, y, t) : (0, \infty) \to (0, 1]\) is continuous,
7. \(N(x, y, t) > 0\),
8. \(N(x, y, t) = 0\) if and only if \(x = y\),
9. \(N(x, y, t) = N(y, x, t)\),
10. \(N(x, y, t) \diamond N(y, z, s) \geq N(x, z, t + s)\),
11. \(N(x, y, t) : (0, \infty) \to (0, 1]\) is continuous.

Then \((M, N)\) is called an intuitionistic fuzzy metric on \(X\). The functions \(M(x, y, t)\) and \(N(x, y, t)\) denote the degree of nearness and degree of non-nearness between \(x\) and \(y\) with respect to \(t\), respectively.

**Remark 1.7.1.** Every fuzzy metric space \((X, M, *)\) is an intuitionistic fuzzy metric space of the form \((X, M, 1 - M, *, \diamond)\) such that \(t\)-norm * and \(t\)-conorm \(\diamond\) are inter related as \(x \diamond y = 1 - ((1 - x) * (1 - y))\) for any \(x, y \in X\).

**Remark 1.7.2.** In intuitionistic fuzzy metric space \(X, M(x, y, .)\) is non decreasing and \(N(x, y, .)\) is non increasing for all \(x, y \in X\).
Example 1.7.1. Let \((X, d)\) be a metric space. Denote \(a \ast b = ab\) and \(a \triangledown b = \min \{a, a + b\}\) for all \(a, b \in [0, 1]\) and let \(M\) and \(N\) be fuzzy sets on \(X^2 \times (0, \infty)\) defined as follows:

\[
M(x, y, t) = \frac{ht^n}{ht^n + md(x, y)}, \quad N(x, y, t) = \frac{d(x, y)}{kt^n + md(x, y)}
\]

for all \(h, k, m, n \in \mathbb{R}^+\). Then \((X, M, N, \ast, \triangledown)\) is an intuitionistic fuzzy metric space called induced intuitionistic fuzzy metric space.

Remark 1.7.3. Note that the above example holds even with the \(t\)-norm \(a \ast b = \min \{a, b\}\) and the \(t\)-conorm \(a \triangledown b = \max \{a, b\}\) and hence \((M, N)\) is an intuitionistic fuzzy metric with respect to any continuous \(t\)-norm and \(t\)-conorm. In the above example by taking \(h = k = m = n = 1\), we get

\[
M(x, y, t) = \frac{t}{t + d(x, y)}, \quad N(x, y, t) = \frac{d(x, y)}{t + d(x, y)}.
\]

We call this intuitionistic fuzzy metric induced by a metric \(d\) as standard intuitionistic fuzzy metric.

Example 1.7.2. Let \(X = \mathbb{N}\). Define \(a \ast b = \max \{0, a + b - 1\}\) and \(a \triangledown b = a + b - ab\) for all \(a, b \in [0, 1]\) and let \(M\) and \(N\) be fuzzy sets on \(X^2 \times (0, \infty)\) as follows:

\[
M(x, y, t) = \begin{cases} 
  x/y, & \text{if } x \leq y \\
  y/x, & \text{if } y \leq x
\end{cases} \quad \text{and} \quad N(x, y, t) = \begin{cases} 
  \frac{y-x}{y}, & \text{if } x \leq y \\
  \frac{x-y}{x}, & \text{if } y \leq x
\end{cases}
\]

for all \(x, y \in X\) and \(t > 0\). Then \((X, M, N, \ast, \triangledown)\) is an intuitionistic fuzzy metric space.

Remark 1.7.4. Notice that in the foregoing example, \(t\)-norm \(\ast\) and \(t\)-conorm \(\triangledown\) are not interrelated and there exists no metric \(d\) on \(X\) satisfying

\[
M(x, y, t) = \frac{t}{t + d(x, y)}, \quad N(x, y, t) = \frac{d(x, y)}{t + d(x, y)}
\]

where \(M(x, y, t)\) and \(N(x, y, t)\) are as defined in the above example. Also note that the above functions \((M, N)\) is not an intuitionistic fuzzy metric with the \(t\)-norm and \(t\)-conorm defined as \(a \ast b = \min \{a, b\}\) and \(a \triangledown b = \max \{a, b\}\).

In 2006, Alaca et al. [2] using the idea of intuitionistic fuzzy sets, redefined the notion of intuitionistic fuzzy metric spaces (as defined by Park [129]) with the help of continuous \(t\)-norms and continuous \(t\)-conorms as a generalization of KM-fuzzy metric spaces. Further, they introduced the notion of Cauchy sequences in intuitionistic fuzzy metric spaces and extended the well known fixed point theorems of Banach [18] and Edelstein [42] to intuitionistic fuzzy metric spaces following the idea of Grabiec [57]. Their definition runs as follows:
Definition 1.7.3. ([2]) A 5-tuple \((X, M, N, \ast, \diamondsuit)\) is said to be an intuitionistic fuzzy metric space if \(X\) is an arbitrary set, \(\ast\) is a continuous \(t\)-norm, \(\diamondsuit\) is a continuous \(t\)-conorm and \(M, N\) are fuzzy sets on \(X^2 \times [0, \infty)\) satisfying the following conditions:

(i) \(M(x, y, t) + N(x, y, t) \leq 1,\)
(ii) \(M(x, y, 0) = 0,\)
(iii) \(M(x, y, t) = 1\) if and only if \(x = y,\)
(iv) \(M(x, y, t) = M(y, x, t),\)
(v) \(M(x, y, t) \ast M(y, z, s) \leq M(x, z, t + s),\)
(vi) \(M(x, y, \cdot) : [0, \infty) \to [0, 1]\) is left continuous,
(vii) \(\lim_{t \to \infty} M(x, y, t) = 1,\)
(viii) \(N(x, y, 0) = 1,\)
(ix) \(N(x, y, t) = 0\) if and only if \(x = y,\)
(x) \(N(x, y, t) = N(y, x, t),\)
(xi) \(N(x, y, t) \ast N(y, z, s) \geq N(x, z, t + s),\)
(xii) \(N(x, y, \cdot) : [0, \infty) \to [0, 1]\) is right continuous,
(xiii) \(\lim_{t \to \infty} N(x, y, t) = 0,\)

for all \(x, y, z \in X\) and \(s, t > 0\). Then \((M, N)\) is called an intuitionistic fuzzy metric on \(X\). The functions \(M(x, y, t)\) and \(N(x, y, t)\) denote the degree of nearness and the degree of non-nearness between \(x\) and \(y\) with respect to \(t\), respectively.

Remark 1.7.5. Since \(\ast\) and \(\diamondsuit\) are continuous, the limit is uniquely determined from (v) and (xi), respectively.

Now, we present some preliminary definitions and results especially in respect of intuitionistic fuzzy metric spaces (in the sense of Park [129]) needed in our subsequent discussion.

Definition 1.7.4. Let \((X, M, N, \ast, \diamondsuit)\) be an intuitionistic fuzzy metric space. Let \(r \in (0, 1), \ t > 0 \) and \(x \in X\). The set \(B(x, r, t) = \{y \in X : M(x, y, t) > 1 - r, \ N(x, y, t) < r\}\) is called an open ball with center \(x\) and radius \(r\) with respect to \(t\).

Theorem 1.7.1. Every open ball \(B(x, r, t)\) is an open set.

Remark 1.7.6. Let \((X, M, N, \ast, \diamondsuit)\) be an intuitionistic fuzzy metric space. Define \(\tau_{(M, N)} = \{A \subset X : \text{for each } x \in A, \text{ there exists } t > 0 \text{ and } r \in (0, 1) \text{ such that }\)


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$B(x, r, t) \subset A$. Then $\tau_{(M,N)}$ is a topology on $X$, (induced by the intuitionistic fuzzy metric $(M,N)$).

**Remark 1.7.7.** (i) From Theorem 1.7.1 and Remark 1.7.6, every intuitionistic fuzzy metric $(M,N)$ on $X$ generates a topology $\tau_{(M,N)}$ on $X$ which has as a base the family of open sets of the form \( \{B(x, r, t) : x \in X, \ r \in (0,1), \ t > 0\}. \)

(ii) Since \( \{B(x, 1/n, 1/n) : n = 1, 2, ...\} \) is a local base at $x$, the topology $\tau_{(M,N)}$ is first countable.

**Theorem 1.7.2.** Every intuitionistic fuzzy metric space is Hausdorff.

**Remark 1.7.8.** Let $(X, d)$ be a fuzzy metric space. Let

$$M(x, y, t) = \frac{t}{t + d(x, y)}, \quad N(x, y, t) = \frac{d(x, y)}{kt + d(x, y)}, \quad k \in \mathbb{R}^+$$

be the intuitionistic fuzzy metric defined on $X$. Then the topology $\tau_d$ induced by the fuzzy metric $d$ and the topology $\tau_{(M,N)}$ induced by the intuitionistic fuzzy metric $(M,N)$ are the same (see [55]).

**Definition 1.7.5.** Let $(X, M, N, *, \diamond)$ be an intuitionistic fuzzy metric space. A subset $A$ of $X$ is said to be IF-bounded if there exist $t > 0$ and $r \in (0,1)$ such that $M(x, y, t) > 1 - r$ and $N(x, y, t) < r$ for all $x, y \in A$.

**Remark 1.7.9.** Let $(X, M, N, *, \diamond)$ be an intuitionistic fuzzy metric space induced by a metric $d$ on $X$. Then $A \subset X$ is IF-bounded if and only if it is bounded.

**Theorem 1.7.3.** Every compact subset $A$ of an intuitionistic fuzzy metric space $(X, M, N, *, \diamond)$ is IF-bounded.

From Remark 1.7.9 and Theorems 1.7.2 and 1.7.3, (Park [129]) pointed out that in an intuitionistic fuzzy metric space every compact set is closed and bounded.

The following theorem is useful.

**Theorem 1.7.4.** Let $(X, M, N, *, \diamond)$ be an intuitionistic fuzzy metric space and $\tau_{(M,N)}$ be the topology on $X$ induced by the fuzzy metric. Then for a sequence $\{x_n\}$ in $X$, $x_n \rightarrow x$ if and only if $M(x_n, x, t) \rightarrow 1$ and $N(x_n, x, t) \rightarrow 0$ as $n \rightarrow \infty$. 