Chapter 6

Common fixed point theorems in modified intuitionistic fuzzy metric spaces

6.1 Introduction

Atanassov [14] introduced and studied the concept of intuitionistic fuzzy set as a noted generalization of fuzzy set which has inspired intense research activity around this new notion (i.e. intuitionistic fuzzy set). Recently Park [129], using the idea of intuitionistic fuzzy sets, defined intuitionistic fuzzy metric spaces as a generalization of fuzzy metric spaces due to George and Veeramani [49] and also proved some basic results which include Baire’s theorem and uniform limit theorem besides some other core results. Thereafter, Saadati and Park [144] defined precompact sets in intuitionistic fuzzy metric spaces and proved that any subset of an intuitionistic fuzzy metric space is compact if and only if it is precompact and complete. They also defined topologically complete intuitionistic fuzzy metrizable spaces and proved that any $G_δ$ set in a complete intuitionistic fuzzy metric space is a topologically complete intuitionistic fuzzy metrizable space and vice versa. For more relevant work, one can be referred to [56, 129, 143, 144, 163].

One of the most important problems in fuzzy topology is to introduce appropriate concepts of intuitionistic fuzzy metric and intuitionistic fuzzy norm. These problems were investigated by Park [129] and Saadati and Park [144] respectively by introducing the notions of intuitionistic fuzzy metric and fuzzy norm. Intuitionistic fuzzy metric can be employed in modeling some phenomena wherein it is necessary to study the relationship between two probability functions as explained in [56]. Concretely speaking, it finds an application to two-slit experiment as the foundation of E-infinity theory in high energy particle physics which was first formulated by El Naschie in [43, 44]. Gregori et al. [56] pointed out that topologies generated

The contents of this chapter have been accepted for publication in Iranian Journal of Fuzzy Systems.
by fuzzy metric and intuitionistic fuzzy metric coincide. In view of this observation, Saadati et al. [145] modified the notion of intuitionistic fuzzy metric and defined the notion of modified intuitionistic fuzzy metric spaces with the help of continuous $t$-representable.

Before proving our results, we collect the relevant definitions and results.

**Lemma 6.1.1** [37]. Consider the set $L^*$ and operation $\leq_{L^*}$ defined by

$$L^* = \{(x_1, x_2) : (x_1, x_2) \in [0, 1]^2 \text{ and } x_1 + x_2 \leq 1\},$$

$$(x_1, x_2) \leq_{L^*} (y_1, y_2) \iff x_1 \leq y_1 \text{ and } x_2 \geq y_2 \in L^*. \text{ Then } (L^*, \leq_{L^*}) \text{ is a complete lattice.}$$

**Definition 6.1.1** [14]. An intuitionistic fuzzy set $A_{\zeta, \eta}$ in a universe $U$ is an object $A_{\zeta, \eta} = \{(\zeta_A(u), \eta_A(u)) | u \in U\}$, where, for all $u \in U$, $\zeta_A(u) \in [0, 1]$ and $\eta_A(u) \in [0, 1]$ are respectively called the membership degree and the non-membership degree of $u$ in $A_{\zeta, \eta}$ and furthermore $\zeta_A(u) + \eta_A(u) \leq 1$.

For every $z_i = (x_i, y_i) \in L^*$ and $c_i \in [0, 1]$ such that $\sum_{j=1}^{n} c_j = 1$, it is easy to verify that

$$c_1(x_1, y_1) + \cdots + c_n(x_n, y_n) = \sum_{j=1}^{n} c_j(x_j, y_j) = \left(\sum_{j=1}^{n} c_j x_j, \sum_{j=1}^{n} c_j y_j\right) \in L^*.$$  

We denote its units by $0_{L^*} = (0, 1)$ and $1_{L^*} = (1, 0)$. Classically, a triangular norm $\ast$ on $[0, 1]$ is defined as an increasing, commutative and associative mapping $\ast : [0, 1]^2 \rightarrow [0, 1]$ satisfying $\ast(1, x) = 1 \ast x = x$, for all $x \in [0, 1]$. A triangular co-norm $\diamondsuit$ is defined as an increasing, commutative and associative mapping $\diamondsuit : [0, 1]^2 \rightarrow [0, 1]$ satisfying $\diamondsuit(0, x) = 0 \diamondsuit x = x$, for all $x \in [0, 1]$. These definitions can be straightforwardly extended to $(L^*, \leq_{L^*})$ as follows:

**Definition 6.1.2** [37]. A triangular norm ($t$-norm) on $L^*$ is a mapping $T : L^* \times L^* \rightarrow L^*$ satisfying the following conditions:

(i) $T(x, 1_{L^*}) = x$,

(ii) $T(x, y) = T(y, x)$,

(iii) $T(x, T(y, z)) = T(T(x, y), z)$,

(iv) $x \leq_{L^*} x'$ and $y \leq_{L^*} y' \Rightarrow T(x, y) \leq_{L^*} T(x', y')$,

for all $x, y, z, x', y' \in L^*$.

**Definition 6.1.3** [37]. A continuous $t$-norm $T$ on $L^*$ is called continuous $t$-representable if and only if there exist a continuous $t$-norm $\ast$ and a continuous $t$-conorm $\diamondsuit$ on $[0, 1]$ such that, for all $x = (x_1, x_2)$, $y = (y_1, y_2) \in L^*$,

$$T(x, y) = (x_1 \ast y_1, x_2 \diamondsuit y_2).$$
Now, we define a sequence \( \{T^n\} \) recursively by \( \{T^1 = T\} \) and
\[
T^n(x^{(1)}, \ldots, x^{(n+1)}) = T(T^{n-1}(x^{(1)}, \ldots, x^{(n)}), x^{(n+1)})
\]
for \( n \geq 2 \) and \( x^{(i)} \in L^* \).

**Definition 6.1.4** [145]. Let \( M, N \) are fuzzy sets from \( X^2 \times (0, \infty) \) to \([0, 1]\) such that \( M(x, y, t) + N(x, y, t) \leq 1 \) for all \( x, y \in X \) and \( t > 0 \). The 3-tuple \((X, M_{M,N}, T)\) is said to be a modified intuitionistic fuzzy metric space (in short, modified IFMS) if \( X \) is an arbitrary non-empty set, \( T \) is a continuous \( t \)-representable and \( M_{M,N} \) is an intuitionistic fuzzy set from \( X^2 \times (0, \infty) \rightarrow L^* \) satisfying the following conditions (for every \( x, y \in X \) and \( t, s > 0 \)):

(i) \( M_{M,N}(x, y, t) >_L 0_{L^*} \),
(ii) \( M_{M,N}(x, y, t) = 1_{L^*} \) if and only if \( x = y \),
(iii) \( M_{M,N}(x, y, t) = M_{M,N}(y, x, t) \),
(iv) \( M_{M,N}(x, y, t + s) \geq_L T(M_{M,N}(x, y, t), M_{M,N}(x, y, s)) \),
(v) \( M_{M,N}(x, y, t) : (0, \infty) \rightarrow L^* \) is continuous.

In this case \( M_{M,N} \) is called a modified intuitionistic fuzzy metric. Here,
\[
M_{M,N}(x, y, t) = (M(x, y, t), N(x, y, t)).
\]

**Remark 6.1.1** [163]. In an intuitionistic fuzzy metric space \((X, M_{M,N}, T), M(x, y, .)\) is non-decreasing and \( N(x, y, .) \) is non-increasing for all \( x, y \in X \).

**Example 6.1.1.** Let \((X, d)\) be a metric space. Denote \( T(a, b) = (a_1b_1, \min\{a_2 + b_2, 1\}) \) for all \( a = (a_1, a_2) \) and \( b = (b_1, b_2) \in L^* \) and let \( M \) and \( N \) be fuzzy sets on \( X^2 \times (0, \infty) \). Then an intuitionistic fuzzy metric can be defined as
\[
M_{M,N}(x, y, t) = (M(x, y, t), N(x, y, t)) = \left( \frac{h^n}{ht^n + md(x, y)}, \frac{md(x, y)}{ht^n + md(x, y)} \right)
\]
for all \( h, m, n, t \in \mathbb{R^+} \) so that \((X, M_{M,N}, T)\) is a modified IFMS.

**Example 6.1.2.** Let \( X = \mathbb{N} \). Define \( T(a, b) = (\max\{0, a_1 + b_1 - 1\}, a_2 + b_2 - a_2b_2) \) for all \( a = (a_1, a_2) \) and \( b = (b_1, b_2) \in L^* \) and let \( M \) and \( N \) be fuzzy sets on \( X^2 \times (0, \infty) \). Then \( M_{M,N}(x, y, t) \) defined as (for all \( x, y \in X \) and \( t > 0 \)):
\[
M_{M,N}(x, y, t) = (M(x, y, t), N(x, y, t)) = \begin{cases} \left( \frac{x}{y}, \frac{y-x}{y} \right) & \text{if } x \leq y \\ \left( \frac{y}{x}, \frac{x-y}{x} \right) & \text{if } y \leq x \end{cases}
\]
is an intuitionistic fuzzy metric so that \((X, M_{M,N}, T)\) is a modified IFMS.
Chapter 6: Common fixed point theorems in modified intuitionistic...

Definition 6.1.5 [37]. A negator on $L^*$ is a decreasing mapping $N : L^* \rightarrow L^*$ satisfying $N(0_{L^*}) = 1_{L^*}$ and $N(1_{L^*}) = 0_{L^*}$. A negator on $[0,1]$ is a decreasing mapping $N : [0,1] \rightarrow [0,1]$ satisfying $N(0) = 1$ and $N(1) = 0$. In what follows, $N_s$ denotes the standard negator on $[0,1]$ defined as $N_s(x) = 1 - x$ for all $x \in [0,1]$.

Definition 6.1.6. Let $(X, \mathcal{M}_{M,N}, \mathcal{T})$ be a modified IFMS. For $t > 0$, define the open ball $B(x, r, t)$ with center $x \in X$ and radius $0 < r < 1$, as

$$B(x, r, t) = \{ y \in X : \mathcal{M}_{M,N}(x, y, t) > L^*(N_s(r), r) \}.$$ 

A subset $A \subseteq X$ is called open if for each $x \in A$, there exist $t > 0$ and $r < 1$ such that $B(x, r, t) \subseteq A$. Let $\tau_{\mathcal{M}_{M,N}}$ denotes the family of all open subsets of $X$. Then, $\tau_{\mathcal{M}_{M,N}}$ is called the topology induced by intuitionistic fuzzy metric $\mathcal{M}_{M,N}$. Notice that this topology is Hausdorff (see Remark 3.3 and Theorem 3.5 in [129]).

Definition 6.1.7. A sequence $\{x_n\}$ in a modified IFMS $(X, \mathcal{M}_{M,N}, \mathcal{T})$ is called a Cauchy sequence if for each $0 < \epsilon < 1$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$\mathcal{M}_{M,N}(x_n, y_m, t) > L^*(N_s(\epsilon), \epsilon)$$

and for each $n, m \geq n_0$ where $N_s$ is the standard negator. The sequence $\{x_n\}$ is said to be convergent to $x \in X$ in the intuitionistic fuzzy metric space $(X, \mathcal{M}_{M,N}, \mathcal{T})$ and is generally denoted by $x_n \rightarrow x$ if $\mathcal{M}_{M,N}(x_n, x, t) \rightarrow 1_{L^*}$ whenever $n \rightarrow \infty$ for every $t > 0$. An IFMS is said to be complete if and only if every Cauchy sequence is convergent.

Lemma 6.1.2 [46]. Let $\mathcal{M}_{M,N}$ be an intuitionistic fuzzy metric. Then, for any $t > 0$, $\mathcal{M}_{M,N}(x, y, t)$ is non-decreasing with respect to $t$ in $(L^*, \leq_{L^*})$, for all $x, y \in X$.

Definition 6.1.8. Let $(X, \mathcal{M}_{M,N}, \mathcal{T})$ be a modified IFMS. Then $\mathcal{M}_{M,N}$ is said to be continuous on $X \times X \times (0, \infty)$ if

$$\lim_{n \rightarrow \infty} \mathcal{M}_{M,N}(x_n, y_n, t_n) = \mathcal{M}_{M,N}(x, y, t),$$

whenever a sequence $\{(x_n, y_n, t_n)\}$ in $X \times X \times (0, \infty)$ converges to a point $(x, y, t) \in X \times X \times (0, \infty)$ i.e.

$$\lim_{n \rightarrow \infty} \mathcal{M}_{M,N}(x_n, x, t) = \lim_{n \rightarrow \infty} \mathcal{M}_{M,N}(y_n, y, t) = 1_{L^*}$$

and

$$\lim_{n \rightarrow \infty} \mathcal{M}_{M,N}(x, y, t_n) = \mathcal{M}_{M,N}(x, y, t).$$

Lemma 6.1.3 [145]. Let $(X, \mathcal{M}_{M,N}, \mathcal{T})$ be a modified IFMS. Then $\mathcal{M}_{M,N}$ is continuous function on $X \times X \times (0, \infty)$.

Definition 6.1.9. Let $f$ and $g$ be mappings from a modified IFMS $(X, \mathcal{M}_{M,N}, \mathcal{T})$ into itself. Then the mappings are said to be compatible if

$$\lim_{n \rightarrow \infty} \mathcal{M}_{M,N}(fgx_n,gfx_n, t) = 1_{L^*}, \quad \forall \ t > 0.$$
whenever \( \{x_n\} \) is a sequence in \( X \) such that 
\[
\lim_{n \to \infty} f x_n = \lim_{n \to \infty} g x_n = x \in X.
\]

**Definition 6.1.10.** Let \( f \) and \( g \) be mappings from a modified IFMS \((X, \mathcal{M}_{M,N}, \mathcal{T})\) into itself. Then the mappings are said to be noncompatible if there exists at least one sequence \( \{x_n\} \) in \( X \) such that 
\[
\lim_{n \to \infty} f x_n = \lim_{n \to \infty} g x_n = x \in X \text{ but } \lim_{n \to \infty} \mathcal{M}_{M,N}(f g x_n, g f x_n, t) \neq 1_L \text{ or nonexistent for at least one } t > 0.
\]

**Remark 6.1.2.** Every pair of compatible self mappings \( f \) and \( g \) of a modified IFMS \((X, \mathcal{M}_{M,N}, \mathcal{T})\) is weakly compatible. But converse is not true as substantiated by following example.

**Example 6.1.3.** Let \((X, \mathcal{M}_{M,N}, \mathcal{T})\) be a modified IFMS, where \( X = [0, 2] \) and 
\[
\mathcal{M}_{M,N}(x, y, t) = \left( \frac{t}{t + |x - y|}, \frac{|x - y|}{t + |x - y|} \right)
\]
for all \( t > 0 \) and \( x, y \in X \), and \( \mathcal{T}(a, b) = (a b_1, \min\{a_2 + b_2, 1\}) \) for all \( a = (a_1, a_2) \) and \( b = (b_1, b_2) \in L^* \). Define self-mappings \( f \) and \( g \) on \( X \) as follows:
\[
f(x) = \begin{cases} 
2 & \text{if } 0 \leq x \leq 1 \\
\frac{x}{2} & \text{if } 1 < x \leq 2
\end{cases}
\]
\[
g(x) = \begin{cases} 
2 & \text{if } x = 1 \\
\frac{x + 3}{5} & \text{if } x \neq 1.
\end{cases}
\]

Then we have \( g(1) = f(1) = 2 \) and \( g(2) = f(2) = 1 \). Also \( g f(1) = f g(1) = 1 \) and \( g f(2) = f g(2) = 2 \), thus pair \((f, g)\) is weakly compatible. Moreover, \( f x_n = 1 - \frac{1}{10^n} \)
and \( g x_n = 1 - \frac{1}{10^n} \). Thus \( f x_n \to 1, \ g x_n \to 1 \). Further \( g f x_n = \frac{4}{5} - \frac{1}{20^n}, \ f g x_n = 2 \). Now
\[
\lim_{n \to \infty} \mathcal{M}_{M,N}(f g x_n, g f x_n, t) = \lim_{n \to \infty} \mathcal{M}_{M,N}(2, \frac{4}{5} - \frac{1}{20^n}, t) = \left( \frac{t}{t + \frac{1}{5}}, \frac{\frac{6}{5}}{t + \frac{1}{5}} \right) < 1_L \text{ for all } t > 0.
\]

Motivated by Aamri and Moutawakil [1], we define the following:

**Definition 6.1.11.** Let \( f \) and \( g \) be two self mappings of a modified IFMS \((X, \mathcal{M}_{M,N}, \mathcal{T})\). We say that \( f \) and \( g \) have the property (E.A) if there exists a sequence \( \{x_n\} \) in \( X \) such that for all \( t > 0 \)
\[
\lim_{n \to \infty} \mathcal{M}_{M,N}(f x_n, u, t) = \lim_{n \to \infty} \mathcal{M}_{M,N}(g x_n, u, t) = 1_L.
\]

**Example 6.1.4.** Let \((X, \mathcal{M}_{M,N}, \mathcal{T})\) be a modified IFMS, where \( X = \mathbb{R} \) and 
\[
\mathcal{M}_{M,N}(x, y, t) = \left( \frac{t}{t + |x - y|}, \frac{|x - y|}{t + |x - y|} \right)
\]
for all \( t > 0 \) and \( x, y \in X \). Define self-mappings \( f \) and \( g \) on \( X \) as follows:
\[
f x = 2x + 1 \quad \text{and} \quad g x = x + 2.
\]
Consider the sequence \( \{x_n = 1 + \frac{1}{n}, \ n = 1, 2, \ldots \} \). Thus we have

\[
\lim_{n \to \infty} \mathcal{M}_{M,N}(f x_n, 3, t) = \lim_{n \to \infty} \mathcal{M}_{M,N}(g x_n, 3, t) = 1_{L^*}
\]

for every \( t > 0 \). Then \( f \) and \( g \) share the property (E.A).

In the next example, we show that there do exist pair of mappings which do not share the property (E.A).

**Example 6.1.5.** Let \((X, \mathcal{M}_{M,N}, T)\) be a modified IFMS, where \( X = \mathbb{R} \) and \( \mathcal{M}_{M,N}(x, y, t) = \left( \frac{t}{t+|x-y|}, \frac{|x-y|}{t+|x-y|} \right) \) for all \( t > 0 \) and \( x, y \in X \). Define self-mappings \( f \) and \( g \) on \( X \) as \( f x = x + 1 \) and \( g x = x + 2 \). Suppose there exists a sequence \( \{x_n\} \) such that

\[
\lim_{n \to \infty} \mathcal{M}_{M,N}(f x_n, u, t) = \lim_{n \to \infty} \mathcal{M}_{M,N}(g x_n, u, t) = 1_{L^*}
\]

for some \( u \in X \). Then

\[
\lim_{n \to \infty} \mathcal{M}_{M,N}(f x_n, u, t) = \lim_{n \to \infty} \mathcal{M}_{M,N}(x_n + 1, u, t) = \lim_{n \to \infty} \mathcal{M}_{M,N}(x_n, u - 1, t) = 1_{L^*}
\]

\[
\lim_{n \to \infty} \mathcal{M}_{M,N}(g x_n, u, t) = \lim_{n \to \infty} \mathcal{M}_{M,N}(x_n + 2, u, t) = \lim_{n \to \infty} \mathcal{M}_{M,N}(x_n, u - 2, t) = 1_{L^*},
\]

which show that \( x_n \to u - 1 \) and \( x_n \to u - 2 \), which is a contradiction. Hence \( f \) and \( g \) do not share the property (E.A).

Motivated by Liu et al. [109], we also define the following:

**Definition 6.1.12.** Two pairs \((f, S)\) and \((g, T)\) of self mappings of a modified IFMS \((X, \mathcal{M}_{M,N}, T)\) are said to satisfy the common property (E.A) if there exist two sequences \( \{x_n\} \) and \( \{y_n\} \) in \( X \) such that

\[
\lim_{n \to \infty} \mathcal{M}_{M,N}(f x_n, u, t) = \lim_{n \to \infty} \mathcal{M}_{M,N}(S x_n, u, t) = \lim_{n \to \infty} \mathcal{M}_{M,N}(g y_n, u, t) = \lim_{n \to \infty} \mathcal{M}_{M,N}(T y_n, u, t) = 1_{L^*}
\]

for some \( u \in X \) and \( t > 0 \).

The purpose of this chapter is to introduce an implicit function and common property (E.A) in modified IFMS and utilize the same to prove some common fixed point theorems in modified IFMS.

The organization of this chapter is as follows: In preceding section (i.e. Section 6.1), we have already presented some relevant definitions, examples, results besides defining the property (E.A) and the common property (E.A) in modified IFMS. In Section 6.2, we define a new implicit function to enhance the domain of applicability which includes several well known contraction conditions such as: \( \phi \)-type contraction, quasi-contraction and others besides admitting new unknown contraction conditions which are used to prove a general common fixed point theorem for two pairs of weakly compatible self mappings satisfying the common property (E.A) in modified IFMS while in section 6.3, we prove some results in modified IFMS using the property (E.A) and the common property (E.A). Last section is devoted to two illustrative examples to the results provided in Section 6.3.
6.2 Implicit functions

Motivated by Ali and Imdad [3], we introduce an implicit function as follows: Let $\Psi$ be the set of all upper semi-continuous functions $F(t_1, t_2, \cdots, t_6) : (L^*)^6 \to L^*$, satisfying the following conditions (for all $u, 0, 1 \in L^*$ where $u = (u_1, u_2), 0 = 0_{L^*} = (0, 1)$ and $1 = 1_{L^*} = (1, 0)$):

$(F_1) : F(u, 1, u, 1, u, 1) <_{L^*} 0$, for all $u >_{L^*} 0$,

$(F_2) : F(u, 1, 1, u, u, 1) <_{L^*} 0$, for all $u >_{L^*} 0$,

$(F_3) : F(u, u, 1, 1, u, u) <_{L^*} 0$, for all $u >_{L^*} 0$.

The following examples satisfy $(F_1), (F_2)$, and $(F_3)$.

Example 6.2.1. Define $F(t_1, t_2, \cdots, t_6) : L^*^6 \to L^*$ as

$$F(t_1, t_2, \cdots, t_6) = t_1 - \alpha \min\{t_2, t_3, t_4, t_5, t_6\}, \quad \text{where } \alpha > 1.$$  

Example 6.2.2. Define $F(t_1, t_2, \cdots, t_6) : L^*^6 \to L^*$ as

$$F(t_1, t_2, \cdots, t_6) = t_1^2 - c_1 \min\{t_2^2, t_3^2, t_4^2\} - c_2 \min\{t_3 t_6, t_4 t_5\},$$

where $c_1, c_2 > 0$, $c_1 + c_2 > 1$ and $c_1 \geq 1$.

Example 6.2.3. Define $F(t_1, t_2, \cdots, t_6) : L^*^6 \to L^*$ as

$$F(t_1, t_2, \cdots, t_6) = t_1^3 - a \min\{t_1 t_2, t_1 t_3 t_4, t_5 t_6^2, t_5 t_6^2\},$$

where $a > 1$.

Example 6.2.4. Define $F(t_1, t_2, \cdots, t_6) : L^*^6 \to L^*$ as

$$F(t_1, t_2, \cdots, t_6) = t_1^3 - a \frac{t_2^2 t_4^2 + t_5^2 t_6^2}{t_2 + t_3 + t_4},$$

where $a \geq \frac{3}{2}$.

Example 6.2.5. Define $F(t_1, t_2, \cdots, t_6) : L^*^6 \to L^*$ as

$$F(t_1, t_2, \cdots, t_6) = (1 + pt_2) t_1 - p \min\{t_3 t_4, t_5 t_6\} - \psi(\min\{t_2, t_3, t_4, t_5, t_6\}),$$

where $p \geq 0$ and $\psi : L^* \to L^*$ is a continuous function such that $\psi(t) >_{L^*} t$ for all $t \in L^* \setminus \{0, 1\}$.

Example 6.2.6. Define $F(t_1, t_2, \cdots, t_6) : L^*^6 \to L^*$ as

$$F(t_1, t_2, \cdots, t_6) = t_1^2 - a \frac{t_2^2 + t_3^2 + t_4^2}{t_5 + t_6}.$$
where \( a \geq 2 \).

**Example 6.2.7.** Define \( F(t_1, t_2, \cdots, t_6) : L^6 \to L^* \) as

\[
F(t_1, t_2, \cdots, t_6) = t_1 - \psi(\min\{t_2, t_3, t_4, t_5, t_6\}),
\]

where \( \psi : L^* \to L^* \) is a continuous function such that \( \psi(t) > L^* \) for all \( t \in L^* \setminus \{0, 1\} \).

**Example 6.2.8.** Define \( F(t_1, t_2, \cdots, t_6) : L^6 \to L^* \) as

\[
F(t_1, t_2, \cdots, t_6) = t_1 - a \frac{t_2^2 t_4^2}{t_2 + t_5 + t_6},
\]

where \( a \geq 3 \).

**Example 6.2.9.** Define \( F(t_1, t_2, \cdots, t_6) : L^6 \to L^* \) as

\[
F(t_1, t_2, \cdots, t_6) = t_1^2 - a \min\{t_2^2, t_3^2, t_4^2\} - b \frac{t_5}{t_5 + t_6},
\]

where \( a \geq 1 \) and \( b > 0 \).

**Example 6.2.10.** Define \( F(t_1, t_2, \cdots, t_6) : L^6 \to L^* \) as

\[
F(t_1, t_2, \cdots, t_6) = t_1^2 - a \min\{t_2^2, t_5^2, t_6^2\} - b \frac{t_3}{t_3 + t_4},
\]

where \( a \geq 1 \) and \( b > 0 \).

**Example 6.2.11.** Define \( F(t_1, t_2, \cdots, t_6) : L^6 \to L^* \) as

\[
F(t_1, t_2, \cdots, t_6) = t_1 - a_1 t_2 - a_2 t_3 - a_3 t_4 - a_4 t_5 - a_5 t_6,
\]

where \( a_1, a_2, a_3, a_4, a_5 > 0, a_2 + a_5 \geq 1, a_3 + a_4 \geq 1 \) and \( a_1 + a_4 + a_5 \geq 1 \).

### 6.3 Main results

We begin with the following lemma.

**Lemma 6.3.1.** Let \( f, g, S \) and \( T \) be self mappings of a modified IFMS \((X, M_{M,N}, T)\) satisfying the following conditions:

(i) the pair \((f, S)\) (or \((g, T)\)) satisfies the property (E.A),

(ii) \( f(X) \subset T(X) \) (or \( g(X) \subset S(X) \)),

(iii) \( g(y_n) \) converges for every sequence \( y_n \) in \( X \) whenever \( T(y_n) \) converges (or \( f(y_n) \) converges for every sequence \( y_n \) in \( X \) whenever \( S(y_n) \) converges),
(iv) for all $x, y \in X$, $F \in \Psi$

\[
F(\mathcal{M}_{M,N}(fx, gy, t), \mathcal{M}_{M,N}(Sx, Ty, t), \mathcal{M}_{M,N}(fx, Sx, t), \mathcal{M}_{M,N}(gy, Ty, t), \\
\mathcal{M}_{M,N}(fx, Ty, t), \mathcal{M}_{M,N}(Sx, gy, t)) \geq L^* 0
\]

then the pairs $(f, S)$ and $(g, T)$ share the common property (E.A).

**Proof.** Suppose that the pair $(f, S)$ enjoys the property (E.A), there exists a sequence $\{x_n\}$ in $X$ such that

\[
\lim_{n \to \infty} fx_n = \lim_{n \to \infty} Sx_n = z,
\]

for some $z \in X$, i.e. \( \lim_{n \to \infty} \mathcal{M}_{M,N}(fx_n, Sx_n, t) = 1 \). Since $f(X) \subset T(X)$, for each $x_n$ there exists $y_n$ in $X$ such that $fx_n = Ty_n$. Therefore, $\lim_{n \to \infty} fx_n = \lim_{n \to \infty} Ty_n = z$. Thus in all we have $fx_n \to z$, $Sx_n \to z$ and $Ty_n \to z$. Moreover in view of (iii), $\{gy_n\}$ also converges. Now we assert that $\lim_{n \to \infty} \mathcal{M}_{M,N}(gy_n, z, t) = 1$. If not, then using inequality (6.3.1.1), we have

\[
F(\mathcal{M}_{M,N}(fx_n, gy_n, t), \mathcal{M}_{M,N}(Sx_n, Ty_n, t), \mathcal{M}_{M,N}(gy_n, Ty_n, t), \mathcal{M}_{M,N}(fx_n, Sx_n, t), \\
\mathcal{M}_{M,N}(fx_n, Ty_n, t), \mathcal{M}_{M,N}(Sx_n, gy_n, t)) \geq L^* 0
\]

which on making $n \to \infty$, gives rise

\[
F(\lim_{n \to \infty} \mathcal{M}_{M,N}(Ty_n, gy_n, t), 1, \lim_{n \to \infty} \mathcal{M}_{M,N}(gy_n, Ty_n, t), 1, 1, \lim_{n \to \infty} \mathcal{M}_{M,N}(Ty_n, gy_n, t)) \geq L^* 0
\]

which is a contradiction to $(F_1)$. Hence $\lim_{n \to \infty} \mathcal{M}_{M,N}(gy_n, Ty_n, t) = 1$, i.e. $\lim_{n \to \infty} gy_n = z$ which shows that the pairs $(f, S)$ and $(g, T)$ share the common property (E.A).

Our next result is a common fixed point theorem via the common property (E.A).

**Theorem 6.3.1.** Let $f, g, S$ and $T$ be self mappings of a modified IFMS $(X, \mathcal{M}_{M,N}, T)$ satisfying the condition (6.3.1.1). Suppose that

(i) the pairs $(f, S)$ and $(g, T)$ share the common property (E.A) and

(ii) $S(X)$ and $T(X)$ are closed subsets of $X$.

Then pair $(f, S)$ as well as $(g, T)$ have a coincidence point. Moreover, $f, g, S$ and $T$ have a unique common fixed point in $X$ provided both the pairs $(f, S)$ and $(g, T)$ are weakly compatible.

**Proof.** Since the pairs $(f, S)$ and $(g, T)$ share the common property (E.A), there exist two sequences $\{x_n\}$ and $\{y_n\}$ in $X$ such that

\[
\lim_{n \to \infty} fx_n = \lim_{n \to \infty} Sx_n = \lim_{n \to \infty} gy_n = \lim_{n \to \infty} Ty_n = z,
\]

for some $z \in X$. 

\[97\]
As $S(X)$ is a closed subset of $X$, therefore $\lim_{n \to \infty} Sx_n = z \in S(X)$ and hence there exists a point $u \in X$ such that $Su = z$. Now we assert that $M_{M,N}(fu, z, t) = 1$. If not, then using inequality (6.3.1.1), we have

$$F(M_{M,N}(fu, gy_n, t), M_{M,N}(Su, Ty_n, t), M_{M,N}(gy_n, Ty_n, t), M_{M,N}(fu, Su, t), M_{M,N}(fu, Ty_n, t), M_{M,N}(Su, gy_n, t)) \geq L \ast 0$$

which on making $n \to \infty$, gives rise

$$F(M_{M,N}(fu, z, t), 1, 1, M_{M,N}(fu, z, t), M_{M,N}(fu, z, t), 1) \geq L \ast 0$$

a contradiction to $(F_2)$ so that $M_{M,N}(fu, z, t) = 1$, i.e. $fu = z = Su$. Hence $u$ is a coincidence point of the pair $(f, S)$.

Since $T(X)$ is a closed subset of $X$, therefore $\lim_{n \to \infty} Ty_n = z \in T(X)$ and hence there exists a point $w \in X$ such that $Tw = z$. Now we assert that $M_{M,N}(gw, z, t) = 1$. If not, then using inequality (6.3.1.1), we have

$$F(M_{M,N}(fx_n, gw, t), M_{M,N}(Sx_n, Tw, t), M_{M,N}(gw, Tw, t), M_{x_n, Sx_n, t), M_{M,N}(fx_n, Tw, t), M_{M,N}(Sx_n, gw, t)) \geq L \ast 0$$

which on making $n \to \infty$, gives rise

$$F(M_{M,N}(z, gw, t), 1, M_{M,N}(gw, z, t), 1, 1, M_{M,N}(z, gw, t)) \geq L \ast 0$$

a contradiction to $(F_1)$, so that $M_{M,N}(gw, z, t) = 1$, i.e. $gw = z = Tw$. This shows that $w$ is a coincidence point of the pair $(g, T)$.

Since $fu = Su$ and the pair $(f, S)$ is weakly compatible, therefore $fz = fSu = Sfu = Sz$. Now we need to show that $z$ is a common fixed point of the pair $(f, S)$. To accomplish this, we assert that $M_{M,N}(fz, z, t) = 1$. If not, then using inequality (6.3.1.1), we have

$$F(M_{M,N}(fz, gw, t), M_{M,N}(Sz, Tw, t), M_{M,N}(gw, Tw, t), M_{M,N}(fz, Sz, t), M_{M,N}(fz, Tw, t), M_{M,N}(Sz, gw, t)) \geq L \ast 0$$

or

$$F(M_{M,N}(fz, z, t), M_{M,N}(fz, z, t), 1, 1, M_{M,N}(fz, z, t), M_{M,N}(fz, z, t)) \geq L \ast 0$$

which is a contradiction to $(F_3)$, so that $M_{M,N}(fz, z, t) = 1$ implying thereby $fz = z$ which shows that $z$ is a common fixed point of the pair $(f, S)$.

Also $gw = Tw$ and the pair $(g, T)$ is weakly compatible, therefore $gz = gTw = Tgw = Tz$. Next, we have to show that $z$ is a common fixed point of the pair $(g, T)$. To do this, we assert that $M(gz, z, t) = 1$. If not, then using inequality (6.3.1.1), we have

$$F(M_{M,N}(fu, gz, t), M_{M,N}(Su, Tz, t), M_{M,N}(gz, Tz, t), M_{M,N}(fu, Su, t), M_{M,N}(fu, Su, t), M_{M,N}(gz, Tz, t), M_{M,N}(fu, Su, t), M_{M,N}(fu, Su, t)) \geq L \ast 0$$
\[ M_{M,N}(fu, Tz, t), M_{M,N}(Su, gz, t) \geq L^* 0 \]

or
\[ F(M_{M,N}(z, gz, t), M_{M,N}(z, gz, t), 1, 1, M_{M,N}(z, gz, t), M_{M,N}(z, gz, t)) \geq L^* 0 \]

which is a contradiction to \((F_3)\) so that \(M_{M,N}(gz, z, t) = 1\), i.e. \(gz = z\) which shows that \(z\) is a common fixed point of the pair \((g, T)\). Hence \(z\) is a common fixed point of \(f, g, S\) and \(T\). Uniqueness of the common fixed point is an easy consequence of the inequality \((6.3.1.1)\) (in view of condition \((F_3)\)).

**Remark 6.3.1.** Theorem 6.3.1 extends relevant results of Ali and Imdad [3], Imdad and Ali [64, 67] and other results to modified IFMS.

**Theorem 6.3.2.** The conclusions of Theorem 6.3.1 remain true if the condition \((ii)\) (of Theorem 6.3.1) is replaced by following.

\((ii')\) \(f(X) \subset T(X)\) and \(g(X) \subset S(X)\).

As a corollary of Theorem 6.3.2, we can have the following result which is also a variant of Theorem 6.3.1.

**Corollary 6.3.1.** The conclusions of Theorems 6.3.1 and 6.3.2 remain true if the conditions \((iii)\) (of Theorem 6.3.1) and \((ii')\) (of Theorem 6.3.2) are replaced by following.

\((ii'')\) \(f(X)\) and \(g(X)\) are closed subsets of \(X\) provided \(f(X) \subset T(X)\) and \(g(X) \subset S(X)\).

**Theorem 6.3.3.** Let \(f, g, S\) and \(T\) be self mappings of a modified IFMS \((X, M_{M,N}, T)\) satisfying the conditions \((i-iv)\) of Lemma 6.3.1. Suppose that

\((v)\) \(S(X)\) (or \(T(X)\)) is a closed subset of \(X\).

Then pairs \((f, S)\) as well as \((g, T)\) have a coincidence point. Moreover, \(f, g, S\) and \(T\) have a unique common fixed point in \(X\) provided both the pairs \((f, S)\) and \((g, T)\) are weakly compatible.

**Proof.** In view of Lemma 6.3.1, the pairs \((f, S)\) and \((g, T)\) share the common property \((E.A)\), i.e. there exist two sequences \(\{x_n\}\) and \(\{y_n\}\) in \(X\) such that

\[ \lim_{n \to \infty} f x_n = \lim_{n \to \infty} S x_n = \lim_{n \to \infty} g y_n = \lim_{n \to \infty} T y_n = z, \text{ for some } z \in X. \]

As \(S(X)\) is a closed subset of \(X\), on the lines of Theorem 6.3.1, one can show that the pair \((f, S)\) has a coincidence point, say \(u\), i.e. \(fu = Su\). Since \(fu \in f(X)\) and \(f(X) \subset T(X)\), then there exists \(w \in X\) such that \(fu = Tw\). Now we assert that \(M_{M,N}(gw, z, t) = 1\). If not, then using inequality \((6.3.1.1)\), we have

\[ F(M_{M,N}(fx_n, gw, t), M_{M,N}(Sx_n, Tw, t), M_{M,N}(gw, Tw, t), M_{M,N}(fx_n, Sx_n, t), \]

\[ M_{M,N}(fu, Tz, t), M_{M,N}(Su, gz, t)) \geq L^* 0 \]
\[ \mathcal{M}_{M,N}(fx_n, Tw, t), \mathcal{M}_{M,N}(Sx_n, gw, t) \geq_{L^*} 0 \]

which on making \( n \to \infty \), gives rise

\[ F(\mathcal{M}_{M,N}(z, gw, t), 1, \mathcal{M}_{M,N}(gw, z, t), 1, \mathcal{M}_{M,N}(z, gw, t)) \geq_{L^*} 0 \]

which is a contradiction to \((F_1)\), hence \( \mathcal{M}_{M,N}(gw, z, t) = 1 \), i.e. \( gw = z = Tw \). Therefore, \( w \) is a coincidence point of the pair \((g, T)\). The rest of the proof can be completed on the lines of Theorem 6.3.1.

By choosing \( f, g, S \) and \( T \) suitably, one can deduce a result for a pair of mappings.

**Corollary 6.3.2.** Let \( f \) and \( S \) be self mappings of a modified IFMS \((X, \mathcal{M}_{M,N}, T)\) satisfying the following conditions:

(i) the pair \((f, S)\) satisfies the property (E.A),

(ii) \( S(X) \) is a closed subset of \( X \) and

(iii) for all \( x, y \in X \), \( F \in \Psi \) and \( t > 0 \)

\[ F(\mathcal{M}_{M,N}(fx, fy, t), \mathcal{M}_{M,N}(Sx, Sy, t), \mathcal{M}_{M,N}(fx, Sx, t), \mathcal{M}_{M,N}(fy, Sy, t), \mathcal{M}_{M,N}(fx, Sx, t), \mathcal{M}_{M,N}(fx, Ty, t)) \geq_{L^*} 0, \]

then pair \((f, S)\) has a coincidence point. Moreover, \( f \) and \( S \) have a unique common fixed point in \( X \) provided the pair \((f, S)\) is weakly compatible.

**Remark 6.3.2.** Above corollary generalizes certain relevant results involving pair of mappings from the existing literature (e.g. [64, 67]).

**Corollary 6.3.3.** The conclusions of Theorem 6.3.1 remain true if inequality (6.3.1.1) is replaced by one of the following conditions. For all \( x, y \in X \) and \( t > 0 \),

(i) \[ \mathcal{M}_{M,N}(fx, gy, t) \geq_{L^*} \alpha \min\{\mathcal{M}_{M,N}(Sx, Ty, t), \mathcal{M}_{M,N}(gy, Sx, t), \mathcal{M}_{M,N}(fx, Sx, t), \mathcal{M}_{M,N}(fx, Ty, t), \mathcal{M}_{M,N}(fx, Sx, t)\}, \quad \text{where } \alpha > 1. \]

(ii) \[ \mathcal{M}_{M,N}(fx, gy, t)^2 \geq_{L^*} c_1 \min\{\mathcal{M}_{M,N}(Sx, Ty, t)^2, \mathcal{M}_{M,N}(gy, Ty, t)^2, \mathcal{M}_{M,N}(fx, Sx, t)^2, \mathcal{M}_{M,N}(fx, Ty, t), \mathcal{M}_{M,N}(gy, Ty, t)\}, \mathcal{M}_{M,N}(Sx, gy, t)\}, \]

where \( c_1, c_2 > 0 \), \( c_1 + c_2 \geq 1 \) and \( c_1 \geq 1 \).

(iii) \[ \mathcal{M}_{M,N}(fx, gy, t)^3 \geq_{L^*} a \min\{\mathcal{M}_{M,N}(fx, gy, t)^2, \mathcal{M}_{M,N}(Sx, Ty, t)^2, \mathcal{M}_{M,N}(fx, Sx, t), \mathcal{M}_{M,N}(fx, Ty, t)\}, \mathcal{M}_{M,N}(Sx, gy, t)\}, \mathcal{M}_{M,N}(fx, Ty, t)\mathcal{M}_{M,N}(Sx, gy, t)\}, \]

where \( a > 1 \).
(iv) $\mathcal{M}_{M,N}(fx,gy,t)^3 \geq L^* a \frac{\mathcal{M}_{M,N}(gy,Ty,t)^2 \mathcal{M}_{M,N}(fx,Sx,t)^2 + \mathcal{M}_{M,N}(fx,Ty,t)^2 \mathcal{M}_{M,N}(Sx,gy,t)^2}{\mathcal{M}_{M,N}(Sx,Ty,t) + \mathcal{M}_{M,N}(gy,Ty,t) + \mathcal{M}_{M,N}(fx,Sx,t)}$, where $a > \frac{3}{2}$.

(v) $(1+p\mathcal{M}_{M,N}(Sx,Ty,t))\mathcal{M}_{M,N}(fx,gy,t) \geq L^* p \min\{\mathcal{M}_{M,N}(fx,Sx,t)\mathcal{M}_{M,N}(gy,Ty,t), \mathcal{M}_{M,N}(Sx,gy,t)\} + \psi(\min\{\mathcal{M}_{M,N}(Sx,Ty,t), \mathcal{M}_{M,N}(Sx,gy,t)\})$, where $p \geq 0$ and $\psi : L^* \rightarrow L^*$ is a continuous function such that $\psi(t) > L^* t$ for all $t \in L^* \setminus \{0, 1\}$.

(vi) $\mathcal{M}_{M,N}(fx,gy,t)^2 \geq L^* a \frac{\mathcal{M}_{M,N}(Sx,Ty,t)^2 + \mathcal{M}_{M,N}(gy,Ty,t)^2 + \mathcal{M}_{M,N}(fx,Sx,t)^2}{\mathcal{M}_{M,N}(Sx,Ty,t) + \mathcal{M}_{M,N}(Sx,ty,t)}$, where $a > 2$.

(vii) $\mathcal{M}_{M,N}(fx,gy,t) \geq L^* \psi(\min\{\mathcal{M}_{M,N}(Sx,Ty,t), \mathcal{M}_{M,N}(gy,Ty,t), \mathcal{M}_{M,N}(Sx,gy,t)\})$, where $\psi : L^* \rightarrow L^*$ is a continuous function such that $\psi(t) > L^* t$ for all $t \in L^* \setminus \{0, 1\}$.

(viii) $\mathcal{M}_{M,N}(fx,gy,t)^3 \geq L^* a \frac{\mathcal{M}_{M,N}(gy,Ty,t)^2 \mathcal{M}_{M,N}(fx,Sx,t)^2}{\mathcal{M}_{M,N}(Sx,Ty,t) + \mathcal{M}_{M,N}(Sx,ty,t)}$, where $a > 3$.

(ix) $\mathcal{M}_{M,N}(fx,gy,t)^2 \geq L^* a \min\{\mathcal{M}_{M,N}(Sx,Ty,t)^2, \mathcal{M}_{M,N}(gy,Ty,t)^2\}$

\[ \mathcal{M}_{M,N}(fx,Sx,t)^2 + b \frac{\mathcal{M}_{M,N}(fx,Ty,t)}{\mathcal{M}_{M,N}(Sx,gy,t)} \]

where $a \geq 1$ and $b > 0$.

(x) $\mathcal{M}_{M,N}(fx,gy,t)^2 \geq L^* a \min\{\mathcal{M}_{M,N}(Sx,Ty,t)^2, \mathcal{M}_{M,N}(fx,Ty,t)^2\}$

\[ \mathcal{M}_{M,N}(gy,Sx,t)^2 + b \frac{\mathcal{M}_{M,N}(gy,Ty,t)}{\mathcal{M}_{M,N}(Sx,Sx,t) + \mathcal{M}_{M,N}(gy,Ty,t)} \]

where $a \geq 1$ and $b > 0$.

(xi) $\mathcal{M}_{M,N}(fx,gy,t) \geq L^* a_1 \mathcal{M}_{M,N}(Sx,Ty,t) + a_2 \mathcal{M}_{M,N}(gy,Ty,t)$

\[ + a_3 \mathcal{M}_{M,N}(fx,Sx,t) + a_4 \mathcal{M}_{M,N}(fx,Ty,t) + a_5 \mathcal{M}_{M,N}(gy,Sx,t) \]

where $a_1, a_2, a_3, a_4, a_5 > 0$, $a_2 + a_5 \geq 1$, $a_3 + a_4 \geq 1$ and $a_1 + a_4 + a_5 \geq 1$. 

Proof. The proof follows from Theorem 6.3.1 and Examples 6.2.1–6.2.11.

Remark 6.3.3. Corollaries corresponding to contraction conditions (i-xi) are new results as these results never require conditions on the containment of ranges of involved mappings as utilized by earlier authors. Some contraction conditions embodied in the above corollary are well known, and extend and generalize corresponding relevant results (e.g., [2, 3, 64, 67, 127, 143, 163, 170]).

As an application of Theorem 6.3.1, we can have the following result for four finite families of self mappings.

Theorem 6.3.4. Let \( \{f_1, f_2, \ldots, f_m\}, \{g_1, g_2, \ldots, g_p\}, \{S_1, S_2, \ldots, S_n\} \) and \( \{T_1, T_2, \ldots, T_q\} \) be four finite families of self mappings of a modified IFMS \((X, \mathcal{M}_{M,N}, T)\) with \( f = f_1 f_2 \cdots f_m, g = g_1 g_2 \cdots g_p, S = S_1 S_2 \cdots S_n \) and \( T = T_1 T_2 \cdots T_q \) satisfying inequality \((6.3.1.1)\) and the pairs \((f, S)\) and \((g, T)\) share the common property \((E.A)\). If \( S(X) \) and \( T(X) \) are closed subsets of \( X \), then the pair \((f, S)\) and \((g, T)\) have a coincidence point each.

Moreover, \( f_i, S_k, g_r \) and \( T_t \) have a unique common fixed point provided the pairs of families \((\{f_i\}, \{S_k\})\) and \((\{g_r\}, \{T_t\})\) commute pairwise where \( i \in \{1, \ldots, m\}, k \in \{1, \ldots, n\}, r \in \{1, \ldots, p\} \) and \( t \in \{1, \ldots, q\} \).

Proof. Proof follows on the lines of the corresponding result contained in Imdad et al. [71].

By setting \( f_1 = f_2 = \cdots = f_m = G, g_1 = g_2 = \cdots = g_p = H, S_1 = S_2 = \cdots = S_n = I \) and \( T_1 = T_2 = \cdots = T_q = J \) in Theorem 6.3.4, we deduce the following:

Corollary 6.3.4. Let \( G, H, I \) and \( J \) be four self mappings of a modified IFMS \((X, \mathcal{M}_{M,N}, T)\), pairs \((G^m, I^n)\) and \((H^p, J^q)\) share the common property \((E.A)\) and satisfying the condition for all \( x, y \in X, F \in \Psi \) and \( t > 0 \)

\[
F(\mathcal{M}_{M,N}(G^m x, H^p y, t), \mathcal{M}_{M,N}(I^n x, J^q y, t), \mathcal{M}_{M,N}(G^m x, I^n x, t), \mathcal{M}_{M,N}(H^p y, J^q y, t), \mathcal{M}_{M,N}(G^m x, J^q y, t), \mathcal{M}_{M,N}(I^n x, H^p y, t)) \geq L \cdot 0,
\]

where \( m, n, p \) and \( q \) are positive integers. If \( I^n(X) \) and \( J^q(X) \) are closed subsets of \( X \), then \( G, H, I \) and \( J \) have a unique common fixed point provided \( GI = IG \) and \( HJ = JH \).

6.4 Illustrative examples

Finally, we conclude this chapter with the following two examples. Example 6.4.1 demonstrates Theorem 6.3.1 besides exhibiting its superiority over earlier relevant results (e.g., [64, 67]).
Example 6.4.1. Let \((X, \mathcal{M}_{M,N}, \mathcal{T})\) be an intuitionistic fuzzy metric space, wherein 
\(X = (0, 128), \mathcal{T}(a,b) = (a_1b_1, \min\{a_2 + b_2, 1\})\) for all \(a = (a_1, a_2)\) and \(b = (b_1, b_2) \in L^*\) with 
\[
\mathcal{M}_{M,N}(x, y, t) = \left( \frac{t}{t + |x - y|}, \frac{|x - y|}{t + |x - y|} \right). 
\]
Define self mappings \(f, g, S\) and \(T\) on \(X\) by
\[
f(x) = \begin{cases} 
2x^4 - 1 & \text{if } 1 \leq x \leq 2 \\
7 & \text{otherwise,} 
\end{cases} \quad g(x) = \begin{cases} 
2x^6 - 1 & \text{if } 1 \leq x \leq 2 \\
3 & \text{otherwise,} 
\end{cases} 
S(x) = \begin{cases} 
x^2 & \text{if } 1 \leq x \leq 2 \\
2 & \text{otherwise} 
\end{cases} \quad \text{and} \quad T(x) = \begin{cases} 
x^3 & \text{if } 1 \leq x \leq 2 \\
4 & \text{otherwise.} 
\end{cases} 
\]
Define \(F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - \psi[\min\{t_2, t_3, t_4, t_5, t_6\}]\) where \(\psi(s) > L^* s\) for all \(s \in L^* \\{0, 1\} \text{ and } F \in \Phi.\)

Now, for all \(x, y \in X\) and \(t > 0,\) we have
\[
\psi \left[ \min \left\{ \mathcal{M}_{M,N}(Sx, Ty, t), \mathcal{M}_{M,N}(gy, Ty, t), \mathcal{M}_{M,N}(fx, Sx, t), \mathcal{M}_{M,N}(fx, Ty, t) \right\} \right] \leq L^* \mathcal{M}_{M,N}(fx, gy, t) 
\]
which demonstrates the verification of the esteemed implicit function. The remaining requirements of Theorem 6.3.1 can be easily verified. Notice that 1 is the unique common fixed point of \(f, g, S\) and \(T.\) This example cannot be used in the context of similar results contained in [64, 67] as those results require condition on containments amongst the ranges of the mappings. Notice that \(f(X) = [1, 31] \not\subset [1, 8] = T(X)\) whereas \(g(X) = [1, 127] \not\subset [1, 4] = S(X).\)

Next example shows an instance wherein Corollary 6.3.4 is applicable but Theorem 6.3.1 is not.

Example 6.4.2. Let \((X, \mathcal{M}_{M,N}, \mathcal{T})\) be an intuitionistic fuzzy metric space, wherein 
\(X = [0, 1], \mathcal{T}(a,b) = (a_1b_1, \min\{a_2 + b_2, 1\})\) for all \(a = (a_1, a_2)\) and \(b = (b_1, b_2) \in L^*\) with 
\[
\mathcal{M}_{M,N}(x, y, t) = \left( \frac{t}{t + |x - y|}, \frac{|x - y|}{t + |x - y|} \right). 
\]
Define self mappings \(f, g, S\) and \(T\) on \(X\) by
\[
f(x) = \begin{cases} 
1 & \text{if } x \in [0, 1] \cap Q \\
\frac{1}{2} & \text{if } x \not\in [0, 1] \cap Q, 
\end{cases} \quad g(x) = \begin{cases} 
1 & \text{if } x \in [0, 1] \cap Q \\
\frac{1}{4} & \text{if } x \not\in [0, 1] \cap Q. 
\end{cases} 
\]
Chapter 6: Common fixed point theorems in modified intuitionistic...

\[ S(x) = \begin{cases} 
1 & \text{if } x = 1 \\
0 & \text{if } x \in [0, 1) 
\end{cases} \quad \text{and} \quad T(x) = \begin{cases} 
1 & \text{if } x = 1 \\
\frac{1}{3} & \text{if } x \in [0, 1). 
\end{cases} \]

Then \( f^2(X) = \{1\} \subset \{0, 1\} = T^2(X) \) and \( g^2(X) = \{1\} \subset \{\frac{1}{3}, 1\} = S^2(X) \).

Define \( F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - \psi[\min\{t_2, t_3, t_4, t_5, t_6\}] \) where \( \psi(s) = \sqrt{s} \) for all \( s \in L^* \{0, 1\} \) and \( F \in \Psi \).

Now, for all \( x, y \in X \) and \( t > 0 \), we have

\[
\psi\left[ \min \left\{ \mathcal{M}_{M,N}(S^2x, T^2y, t), \mathcal{M}_{M,N}(g^2y, T^2y, t), \mathcal{M}_{M,N}(f^2x, S^2x, t), \mathcal{M}_{M,N}(f^2x, T^2y, t), \mathcal{M}_{M,N}(S^2x, g^2y, t) \right\} \right] \\
\leq L^* 1 = \mathcal{M}_{M,N}(1, 1, t) = \mathcal{M}_{M,N}(f^2x, g^2y, t)
\]

which demonstrates the verification of the esteemed implicit function. The remaining requirements of Corollary 6.3.4 can be easily verified. Notice that 1 is the unique common fixed point of \( f, g, S \) and \( T \).

However this implicit function does not hold for the maps \( f, g, S \) and \( T \) in respect of Theorem 6.3.1. Otherwise, with \( x = 0 \) and \( y = \frac{1}{\sqrt{2}} \), we get

\[
\psi\left[ \min \left\{ \mathcal{M}_{M,N}(Sx, Ty, t), \mathcal{M}_{M,N}(gy, Ty, t), \mathcal{M}_{M,N}(fx, Sx, t), \mathcal{M}_{M,N}(fx, Ty, t), \mathcal{M}_{M,N}(Sx, gy, t) \right\} \right] \\
= \psi\left[ \min \left\{ \mathcal{M}_{M,N}(0, \frac{1}{3}, t), \mathcal{M}_{M,N}(\frac{1}{4}, \frac{1}{3}, t), \mathcal{M}_{M,N}(1, 0, t), \mathcal{M}_{M,N}(1, \frac{1}{3}, t), \mathcal{M}_{M,N}(0, \frac{1}{4}, t) \right\} \right] \\
\leq L^* \mathcal{M}_{M,N}(1, \frac{1}{4}, t) = \mathcal{M}_{M,N}(fx, gy, t)
\]

which is not true for all \( t > 0 \) (e.g. \( t = \frac{1}{2} \)). Thus Corollary 6.3.4 is a partial generalization of Theorem 6.3.1 and can be situationally useful.