Chapter 5

Some common fixed point theorems in fuzzy metric spaces

5.1 Introduction

The concept of fuzzy set was initiated by Zadeh [174] as a mathematical way to represent vagueness in everyday life. Thereafter, it was developed extensively by many authors which also include interesting applications of this theory in diverse areas. To use this concept in topology and analysis, several researchers have defined fuzzy metric space in several ways (e.g. [36, 49, 104]). George and Veeramani [49] modified the concept of fuzzy metric space introduced by Kramosil and Michalek [104] and also succeeded in inducing a Hausdorff topology on such fuzzy metric space which is often used in current researches these days. The strength of fuzzy mathematics lies in its noted and fruitful applications especially outside mathematics.

The origin of fixed point theory in fuzzy metric spaces is often traced back to the paper of Grabiec [57] wherein he extended classical fixed point theorems of Banach and Edelstein to complete and compact fuzzy metric spaces respectively. Subrahmanym [158] gave an extension of Jungck’s (cf. [85]) common fixed point theorem to fuzzy metric spaces. In this continuation, Vasuki [166] proved fuzzy analogue of a result due to Pant [124] in complete fuzzy metric space which has been further improved by Imdad and Ali [64] employing the notions of $R$-weak commutativity of type $(A_f), (A_g)$ and $(P)$.

In the recent past, weaker conditions of commutativity namely: weakly commuting mappings, compatible mappings, $R$-weakly commuting mappings, weakly compatible mappings and several other weak conditions of commutativity have been
extensively utilized to prove common fixed point theorems in fuzzy metric spaces (e.g. [64, 67, 117, 119, 158, 170]).

In 2002, Gregori and Sapena [56] introduced the notion of fuzzy contractive mapping and proved fixed point theorems in varied classes of complete fuzzy metric spaces in the senses of George and Veeramani [49], Kramosil and Michalek [104] and Grabiec [57]. Soon after, Mihet [113] proposed a fuzzy fixed point theorem for (weak) Banach contraction in \( M \)-complete fuzzy metric spaces. In this continuation, Mihet [114, 116] further enriched the fixed point theory for various contraction mappings in fuzzy metric spaces besides introducing variants of some new contraction mappings such as: Edelstein fuzzy contractive mappings, fuzzy \( \psi \)-contraction of \((\epsilon, \lambda)\) type etc. In the same spirit, Qiu et al. [135, 136] also obtained some common fixed point theorems for fuzzy mappings under suitable conditions. In 2010 Pacurar and Rus [123], introduced the concept of cyclic \( \phi \)-contraction and utilize the same to prove a fixed point theorem for cyclic \( \phi \)-contraction in the natural setting of complete metric spaces besides investigating several related problems in respect of fixed points.

Inspired from these ideas, we introduce the notion of cyclic weak \( \phi \)-contraction in a fuzzy metric space (in section 5.4). Furthermore, a fixed point theorem is established in \( G \)-complete fuzzy metric spaces in the sense of George and Veeramani [49] besides discussing some related problems.

Every metric space is a fuzzy metric space and is called a standard fuzzy metric space (cf. [49]). Here, we give some new and interesting examples of fuzzy metric spaces due to Gregori et al. [54]. For further details and more examples, one can see [54].

**Example 5.1.1.** ([54]) Let \( f : X \rightarrow \mathbb{R}^+ \) be a one-one function and let \( g : \mathbb{R}^+ \rightarrow [0, \infty) \) be an increasing continuous function. For fixed \( \alpha, \beta > 0 \), define \( M \) as

\[
M(x, y, t) = \left( \frac{(\min\{f(x), f(y)\})^\alpha + g(t)}{(\max\{f(x), f(y)\})^\alpha + g(t)} \right)^\beta.
\]

Then, \((X, M, *)\) is a fuzzy metric space on \( X \) wherein * is the product \( t \)-norm, i.e. \(* (a, b) = ab\).

**Example 5.1.2.** ([54]) Define a function \( M \) on \( X \times X \times I \) (wherein \((X, d)\) is a metric space) as

\[
M(x, y, t) = e^{-d(x, y)/g(t)}
\]

then \((X, M, *)\) is a fuzzy metric space on \( X \) wherein * is the product \( t \)-norm and \( g : \mathbb{R}^+ \rightarrow [0, \infty) \) is an increasing continuous function.

**Example 5.1.3.** ([54]) Let \((X, d)\) be a bounded metric space with \( d(x, y) < k \) (for all \( x, y \in X \) and \( k > 0 \)). Let \( g : \mathbb{R}^+ \rightarrow (k, \infty) \) be an increasing continuous function. Define a function \( M \) as

\[
M(x, y, t) = 1 - \frac{d(x, y)}{g(t)}
\]
then \((X, M, *)\) is a fuzzy metric space on \(X\) wherein \(*\) is a Lukasievicz \(t\)-norm, i.e. 
\[ *(a, b) = \max\{a + b - 1, 0\}. \]

**Definition 5.1.1.** (Grabiec [57], Vasuki and Veeramani [167]). Let \((X, M, *)\) be a fuzzy metric space. Then

(i) a sequence \(\{x_n\}\) in \(X\) is said to converge to \(x \in X\), denoted by \(x_n \to x\), iff 
\[ \lim_{n \to +\infty} M(x_n, x, t) = 1 \text{ for all } t > 0, \]
for each \(r \in (0, 1)\) and \(t > 0\), there exists a \(n_0 \in \mathbb{N}\) such that 
\[ M(x_n, x, t) > 1 - r \text{ for all } n \geq n_0, \]

(ii) a sequence \(\{x_n\}\) in \(X\) is a \(G\)–Cauchy sequence iff 
\[ \lim_{n \to +\infty} M(x_{n+p}, x_n, t) = 1 \]
for any \(p > 0\) and \(t > 0\),

(iii) the fuzzy metric space \((X, M, *)\) is called \(G\)–complete if every \(G\)–Cauchy sequence is convergent.

**Lemma 5.1.1.**[150] If \((X, M, *)\) is a KM-fuzzy metric space and \(\{x_n\}, \{y_n\}\) are sequences in \(X\) such that \(x_n \to x, y_n \to y\), then 
\[ M(x_{n+p}, x_n, t) \to 1 \]
for every continuity point \(t\) of \(M(x, y, .)\).

**Definition 5.1.2.**[119] A pair \((f, g)\) of self mappings of a KM (or GV)-fuzzy metric space \((X, M, *)\) is said to be compatible if for all \(t > 0\),
\[ \lim_{n \to \infty} M(fgx_n, gfx_n, t) = 1 \]
whenever \(\{x_n\}\) is a sequence in \(X\) such that \( \lim_{n \to \infty} fx_n = \lim_{n \to \infty} gx_n = z \) for some \(z \in X\).

**Definition 5.1.3.** A pair \((f, g)\) of self mappings of a KM (or GV)-fuzzy metric space \((X, M, *)\) is said be noncompatible if there exists at least one sequence \(\{x_n\}\) in \(X\) such that \( \lim_{n \to \infty} fx_n = \lim_{n \to \infty} gx_n = z \) for some \(z \in X\) but \( \lim_{n \to \infty} M(fgx_n, gfx_n, t) \neq 1 \) or nonexistent for at least one \(t > 0\).

Motivated from Aamri and Moutawakil [1], one can have the following (as utilized in Mihet [117]):

**Definition 5.1.4.** A pair \((f, g)\) of self mappings of a KM (or GV)-fuzzy metric space \((X, M, *)\) is said to satisfy the property (E.A) if there exists a sequence \(\{x_n\}\) in \(X\) such that for all \(t > 0\)
\[ \lim_{n \to \infty} M(fx_n, gx_n, t) = 1. \]
Clearly, a pair of nontrivial compatible as well as non-compatible mappings enjoys the property (E.A).

Also on the lines of Liu et al. [109], one can have the following:
Definition 5.1.5. Two pairs \((f, S)\) and \((g, T)\) of self mappings of a KM (or GV)-fuzzy metric space \((X, M, \ast)\) are said to satisfy the common property (E.A) if there exist two sequences \(\{x_n\}\) and \(\{y_n\}\) in \(X\) such that for all \(t > 0\)
\[
\lim_{n \to \infty} M(fx_n, Sx_n, t) = \lim_{n \to \infty} M(gy_n, Ty_n, t) = 1.
\]

The purpose of this chapter is to emphasize the role of the property (E.A) and the common property (E.A) to establish the existence of unique common fixed point for contractive mappings in fuzzy metric spaces. We prove our results in fuzzy metric spaces in the sense of Kramosil and Michalek [104] as well as George and Veeramani [49]. Our results generalize and extend several results from the existing literature.

The organization of this chapter is as follows: In Section 5.1, we have already presented relevant examples, definitions, results and also discussed the independence amongst certain weak conditions of commutativity. In Section 5.2, we prove some results in KM-fuzzy metric spaces (cf. [104]) while in Section 5.3, we prove similar results in GV-fuzzy metric spaces (cf. [49]). In Section 5.4, we introduce cyclic weak \(\phi\)-contraction in a fuzzy metric space and also prove some results using cyclic weak \(\phi\)-contraction. In the last section, we prove a general common fixed point theorem in the framework of non-Archimedean fuzzy metric spaces using cyclic weak \(\phi\)-contraction. Some related results and illustrative examples are also discussed.

Our results involve the class \(\Phi\) of all mappings \(\phi : [0, 1] \to [0, 1]\) satisfying the following properties:

\(\phi_1\) \(\phi\) is continuous and nondecreasing on \([0, 1]\);

\(\phi_2\) \(\phi(x) > x\) for all \(x \in (0, 1)\).

We note that if \(\phi \in \Phi\), then \(\phi(1) = 1\) and that \(\phi(x) \geq x \forall x \in [0, 1]\).

5.2 Results on KM-fuzzy metric spaces

We begin with the following observation.

Lemma 5.2.1. Let \(f, g, S\) and \(T\) be four self mappings of a KM-fuzzy metric space \((X, M, \ast)\) satisfying the following conditions:

(i) the pair \((f, S)\) (or \((g, T)\)) satisfies the property (E.A),

(ii) \(f(X) \subset T(X)\) (or \(g(X) \subset S(X)\)),

(iii) \(g(y_n)\) converges for every sequence \(\{y_n\}\) in \(X\) whenever \(T(y_n)\) converges (or \(f(y_n)\) converges for every sequence \(\{y_n\}\) in \(X\) whenever \(S(y_n)\) converges),

(iv) \(\forall x, y \in X, x \neq y, \exists t > 0\) such that \(0 < M(x, y, t) < 1\), wherein for some \(\phi \in \Phi\).
$M(fx, gy, t) \geq \phi(\min\{M(Sx, Ty, t), M(fx, Sx, t), M(gy, Ty, t), M(fx, Ty, t), M(Sx, gy, t)\})$ \hspace{1cm} (5.2.1.1)

then the pairs $(f, S)$ and $(g, T)$ share the common property (E.A).

**Proof.** Since the pair $(f, S)$ enjoys the property (E.A), there exists a sequence $\{x_n\}$ in $X$ such that 
\[
\lim_{n \to \infty} fx_n = \lim_{n \to \infty} Sx_n = z, \text{ for some } z \in X,
\]
which amounts to say that $\lim_{n \to \infty} M(fx_n, Sx_n, t) = 1$. Since $f(X) \subset T(X)$, for each $\{x_n\}$ there exists $\{y_n\}$ in $X$ such that $fx_n = Ty_n$ and hence $\lim_{n \to \infty} fx_n = \lim_{n \to \infty} Ty_n = z$. Thus in all, we have $fx_n \to z$, $Sx_n \to z$ and $Ty_n \to z$. Now, we assert that $\lim_{n \to \infty} M(gy_n, z, t) = 1$. Suppose $\lim_{n \to \infty} gy_n = l \neq z$, then using condition (5.2.1.1), we have
\[
M(fx_n, gy_n, t) \geq \phi(\min\{M(Sx_n, Ty_n, t), M(fx_n, Sx_n, t), M(gy_n, Ty_n, t), M(fx_n, Ty_n, t), M(Sx_n, gy_n, t)\})
\]
for all $n \in \mathbb{N}$ which on letting $n \to \infty$ and making use of Lemma 5.1.1, give rise
\[
M(z, l, t) \geq \phi(M(z, l, t)).
\]

As $z \neq l$, we have $0 < M(z, l, t_0) < 1$ for some $t_0 > 0$. Since $M(z, l, .)$ is left-continuous and nondecreasing, it can have (at the most) countably many points of discontinuity. Let on contrary that $t_0$ is a point of continuity of $M(z, l, .)$, then (in view of condition $\phi_2$), one would get $\phi(M(z, l, t_0)) > M(z, l, t_0)$, which is a contradiction, implying thereby $z = l$. This shows that the pairs $(f, S)$ and $(g, T)$ share the common property (E.A).

**Remark 5.2.1.** The converse of Lemma 5.2.1 is not true in general. For a counter example, one can utilize Example 5.3.1.

Our next result is a common fixed point theorem via common property (E.A).

**Theorem 5.2.1.** Let $f, g, S$ and $T$ be four self mappings of a KM-fuzzy metric space $(X, M, *)$ satisfying the condition (5.2.1.1). Suppose that

(i) the pairs $(f, S)$ and $(g, T)$ share the common property (E.A),

(ii) $S(X)$ and $T(X)$ are closed subsets of $X$.

Then pair $(f, S)$ as well as $(g, T)$ have a coincidence point. Moreover, $f, g, S$ and $T$ have a unique common fixed point in $X$ provided both the pairs $(f, S)$ and $(g, T)$ are weakly compatible.

**Proof.** Since the pairs $(f, S)$ and $(g, T)$ share the common property (E.A), therefore there exist two sequences $\{x_n\}$ and $\{y_n\}$ in $X$ such that 
\[
\lim_{n \to \infty} fx_n = \lim_{n \to \infty} Sx_n = \lim_{n \to \infty} gy_n = \lim_{n \to \infty} Ty_n = z, \text{ for some } z \in X.
\]
Since $S(X)$ is a closed subset of $X$, therefore $\lim_{n \to \infty} Sx_n = z \in S(X)$. Hence, there exists a point $u \in X$ such that $Su = z$. Now we assert that $M(fu, z, t) = 1$. If not, then using condition (5.2.1.1), we have

$$M(fu, gy_n, t) \geq \phi \min\{M(Su, Ty_n, t), M(fu, Su, t), M(gy_n, Ty_n, t), M(fu, Ty_n, t), M(Su, gy_n, t)\}$$

for all $n \in \mathbb{N}$, which on letting $n \to \infty$ and making use of Lemma 5.1.1, give rise

$$M(z, fu, t) \geq \phi(M(z, fu, t)).$$

Since $z \neq fu$, therefore $0 < M(z, fu, t_0) < 1$ for some $t_0 > 0$. As $M(z, fu, ..)$ is left-continuous and nondecreasing, it can assume (at the most) countable points of discontinuity. If we assume that $t_0$ is a continuity point of $M(z, fu, ..)$, then (in view of condition $\phi_2$), one gets $\phi(M(z, fu, t_0)) > M(z, fu, t_0)$, which is a contradiction. Therefore, $z = fu$ so that $fu = z = Su$. This shows that $u$ is a coincidence point of the pair $(f, S)$.

As $T(X)$ is a closed subset of $X$, therefore $\lim_{n \to \infty} Ty_n = z \in T(X)$. Also, there exists a point $w \in X$ such that $Tw = z$. Now we assert that $M(gw, z, t) = 1$. If not, then using condition (5.2.1.1), we have

$$M(fu, gw, t) \geq \phi(\min\{M(Su, Tw, t), M(fu, Su, t), M(gw, Tw, t), M(fu, Tw, t), M(Su, gw, t)\})$$

or

$$M(z, gw, t) \geq \phi(M(z, gw, t)).$$

As $z \neq gw$, then $0 < M(z, gw, t_0) < 1$ for some $t_0 > 0$. As $M(z, gw, ..)$ is left-continuous and nondecreasing, it has only (at the most) countable points of discontinuity. Now, one may suppose that $t_0$ is a continuity point of $M(z, gw, ..)$, then (in view of condition $\phi_2$) one gets $\phi(M(z, gw, t_0)) > M(z, gw, t_0)$, which is a contradiction. Therefore, $z = gw$ so that $gw = z = Tw$. Hence $w$ is a coincidence point of the pair $(g, T)$.

Since $fu = Su$ and the pair $(f, S)$ is weakly compatible, therefore $fz = fSu = Sfu = Sz$. Now we need to show that $z$ is a common fixed point of the pair $(f, S)$. To accomplish this, we assert that $M(fz, z, t) = 1$. If not, then using condition (5.2.1.1), we have

$$M(fz, gw, t) \geq \phi(\min\{M(Sz, Tw, t), M(fz, Sz, t), M(gw, Tw, t), M(fz, Tw, t), M(Sz, gw, t)\})$$

implying thereby

$$M(fz, z, t) \geq \phi(\min\{M(fz, z, t), 1, 1, M(fz, z, t), M(fz, z, t)\}) = \phi(M(fz, z, t)).$$
If \( fz \neq z \), then \( 0 < M(fz, z, t_0) < 1 \) for some \( t_0 > 0 \). As \( M(fz, z, \cdot) \) is left-continuous and nondecreasing, it has only (at the most) countable points of discontinuity. If we suppose that \( t_0 \) is a continuity point of \( M(fz, z, \cdot) \), then (in view of condition \( \phi_2 \)) it follows that \( \phi(M(fz, z, t_0)) > M(fz, z, t_0) \), which is a contradiction. Therefore, \( M(fz, z, t) = 1 \) so that \( fz = z \) which shows that \( z \) is a common fixed point of the pair \((f, S)\).

As \( gw = Tw \) and the pair \((g, T)\) is weakly compatible, therefore \( gz = gTw = Tgw = Tz \). Next, we show \( z \) is a common fixed point of the pair \((g, T)\). To do this, we assert that \( M(gz, z, t) = 1 \). If not, then using condition (5.2.1.1), we have

\[
M(fu, gz, t) \geq \phi(\min\{M(Su, Tz, t), M(fu, Su, t), M(gz, Tz, t),
M(fu, Tz, t), M(Su, gz, t)\})
\]

implying thereby

\[
M(gz, z, t) \geq \phi(\{\min\{M(gz, z, t), 1, M(gz, z, t), M(gz, z, t)\}\}) = \phi(M(gz, z, t)).
\]

As earlier, one obtains \( gz = z \) which shows that \( z \) is a common fixed point of the pair \((g, T)\). Hence \( z \) is a common fixed point of \( f, g, S \) and \( T \). Uniqueness of the common fixed point is an easy consequence of condition (5.2.1.1) and condition \( \phi_2 \). This concludes the proof.

**Remark 5.2.2.** Theorem 5.2.1 generalizes relevant results of Mihet [117] to two pair of mappings including some other ones (e.g. [64, 67, 166, 170]) to KM-fuzzy metric space as Theorem 5.2.1 never requires conditions on the continuity of the involved mappings, completeness of the space (or subspace) and containment amongst range sets of involved mappings besides noted improvements in commutativity requirements.

**Theorem 5.2.2.** The conclusions of Theorem 5.2.1 remain true if the condition (ii) (of Theorem 5.2.1) is replaced by following:

(iii') \( f(X) \subset T(X) \) and \( g(X) \subset S(X) \).

As a corollary of Theorem 5.2.2, we can have the following result which is also a variant of Theorem 5.2.1.

**Corollary 5.2.1.** The conclusions of Theorems 5.2.1 and 5.2.2 remain true if the conditions (i) and (iii') are replaced by following:

(iii'') \( f(X) \) and \( g(X) \) are closed subsets of \( X \) provided \( f(X) \subset T(X) \) and \( g(X) \subset S(X) \).

**Theorem 5.2.3.** Let \( f, g, S \) and \( T \) be four self mappings of a KM-fuzzy metric space \((X, M, \ast)\) satisfying the conditions (i-iv) of Lemma 5.2.1. Suppose that

(v) \( S(X) \) (or \( T(X) \)) is a closed subset of \( X \).
Then pair \((f, S)\) as well as \((g, T)\) have a coincidence point. Moreover, \(f, g, S\) and \(T\) have a unique common fixed point in \(X\) provided that the pairs \((f, S)\) and \((g, T)\) are weakly compatible.

**Proof.** In view of Lemma 5.2.1, the pairs \((f, S)\) and \((g, T)\) share the common property (E.A), i.e. there exist two sequences \(\{x_n\}\) and \(\{y_n\}\) in \(X\) such that

\[
\lim_{n \to \infty} fx_n = \lim_{n \to \infty} Sx_n = \lim_{n \to \infty} gy_n = \lim_{n \to \infty} Ty_n = z, \text{ for some } z \in X.
\]

As \(S(X)\) is a closed subset of \(X\), on the lines of the proof of Theorem 5.2.1, one can show that the pair \((f, S)\) has a coincidence point, say \(u\). i.e. \(fu = z = Su\). Since \(f(X) \subset T(X)\) and \(fu \in f(X)\), there exists \(w \in X\) such that \(z = fu = Tw\). Now we assert that \(M(gw, z, t) = 1\). If not, then using condition (5.2.1.1), we have

\[
M(fu, gw, t) \geq \phi(\min\{M(Su, Tw, t), M(fu, Su, t), M(gw, Tw, t), M(fu, Tw, t), M(Su, gw, t)\})
\]

or

\[
M(z, gw, t) \geq \phi(\min\{1, M(gw, z, t), 1, 1, M(z, gw, t)\}) = \phi(M(z, gw, t)).
\]

Applying the analogous arguments as in Theorem 5.2.1, one can easily show that \(gw = z = Tw\). The rest of the proof can be completed on the lines of Theorem 5.2.1. This completes the proof.

By choosing \(f, g, S\) and \(T\) suitably, one can deduce the following corollary for a pair of mappings.

**Corollary 5.2.2.** Let \(f\) and \(S\) be two self mappings of a KM-fuzzy metric space \((X, M, *)\) satisfying the following conditions:

(i) the pair \((f, S)\) satisfies the property (E.A),

(ii) \(S(X)\) is a closed subset of \(X\),

(iii) \(\forall x, y \in X, x \neq y, \exists t > 0 \text{ such that } 0 < M(x, y, t) < 1\), wherein for some \(\phi \in \Phi\)

\[
M(fx, fy, t) \geq \phi(\min\{M(Sx, Sy, t), M(fx, Sx, t), M(fy, Sy, t), M(fx, Sy, t), M(Sx, fy, t)\}).
\]

Then pair \((f, S)\) has a coincidence point. Moreover, \(f\) and \(S\) have a unique common fixed point in \(X\) provided the pair \((f, S)\) is weakly compatible.

**Remark 5.2.3.** Above corollary extends and generalizes certain relevant results involving a pair of mappings (e.g. [64, 67, 124, 166, 170]).
As an application of Theorem 5.2.1, we have the following result for four finite families of self mappings.

**Theorem 5.2.4.** Let \( \{f_1, f_2, \ldots, f_m\}, \{g_1, g_2, \ldots, g_p\}, \{S_1, S_2, \ldots, S_n\} \) and \( \{T_1, T_2, \ldots, T_q\} \) be four finite families of self mappings of a KM-fuzzy metric space \((X, M, \ast)\) with \( f = f_1 f_2 \cdots f_m, g = g_1 g_2 \cdots g_p, S = S_1 S_2 \cdots S_n \) and \( T = T_1 T_2 \cdots T_q \) satisfying condition (5.2.1.1) and pairs \( (f, S) \) and \( (g, T) \) share the common property (E.A). If \( S(X) \) and \( T(X) \) are closed subsets of \( X \), then the pairs \( (f, S) \) and \( (g, T) \) have a coincidence point each.

Moreover, \( f_i, S_k, g_r \) and \( T_t \) have a unique common fixed point provided the pairs of families \( \{(f_i), \{S_k\}\} \) and \( \{(g_r), \{T_t\}\} \) commute pairwise, where \( i \in \{1, \ldots, m\}, \ k \in \{1, \ldots, n\}, \ r \in \{1, \ldots, p\} \) and \( t \in \{1, \ldots, q\} \).

**Proof.** Proof follows on the lines of the corresponding result contained in Imdad et al. [71].

By setting \( f_1 = f_2 = \cdots = f_m = G, \ g_1 = g_2 = \cdots = g_p = H, \ S_1 = S_2 = \cdots = S_n = I \) and \( T_1 = T_2 = \cdots = T_q = J \) in Theorem 5.2.4, we deduce the following result for iterates of mappings:

**Corollary 5.2.3.** Let \( G, H, I \) and \( J \) be four self mappings of a KM-fuzzy metric space \((X, M, \ast)\), pairs \( (G^m, I^n) \) and \( (H^p, J^q) \) share the common property (E.A) and satisfy the condition: \( \forall \ x, y \in X, \ x \neq y, \ \exists \ t > 0 \) such that \( 0 < M(x, y, t) < 1, \) where \( \phi \in \Phi \)

\[
M(G^m x, H^p y, t) \geq \phi(\min\{M(I^n x, J^q y, t), M(G^m x, I^n x, t), M(H^p y, J^q y, t), M(G^m x, J^q y, t), M(I^n x, H^p y, t)\})
\]

where \( m, n, p \) and \( q \) are positive integers. If \( I^n(X) \) and \( J^q(X) \) are closed subsets of \( X \), then \( G, H, I \) and \( J \) have a unique common fixed point provided the pair \( (G, I) \) as well as \( (H, J) \) is commuting.

**Remark 5.2.4.** Theorem 5.2.4 is a partial generalization of Theorem 5.2.1 as commutativity requirements in Theorem 5.2.4 are relatively stronger than weak compatibility in Theorem 5.2.1. Furthermore, we can prove more results similar to Theorems 5.2.2-5.2.3 and Corollaries 5.2.1-5.2.2 in respect of Theorem 5.2.4 and Corollary 5.2.3.

Next, we state and prove Grabiec-type common fixed point theorems for four self mappings. First, we state the following lemma without proof as proof is an easy consequence of one of the properties of fuzzy metric.

**Lemma 5.2.2.** If \( M(x, y, k t) \geq M(x, y, t) \) for all \( x, y \in X, \ t > 0 \) and with \( k \in (0, 1) \), then \( x = y \).

**Theorem 5.2.5.** Let \( f, g, S \) and \( T \) be four self mappings of a KM-fuzzy metric space \((X, M, \ast)\) satisfying the conditions (i) and (ii) of Theorem 5.2.1, and for all
$x, y \in X$, $t > 0$ and for some $k \in (0, 1)$

$$M(fx, gy, kt) \geq \min\{M(Sx, Ty, t), M(fx, Sx, t), M(gy, Ty, t), M(fx, Ty, t), M(Sx, gy, t)\}.$$  \hspace{1cm} (5.2.5.1)

Then pair $(f, S)$ as well as $(g, T)$ have a coincidence point. Moreover, $f, g, S$ and $T$ have a unique common fixed point in $X$ provided both the pairs $(f, S)$ and $(g, T)$ are weakly compatible.

**Proof.** In view of (i), there exist two sequences $\{x_n\}, \{y_n\}$ in $X$ and some $z \in X$ such that

$$\lim_{n \to \infty} fx_n = \lim_{n \to \infty} Sx_n = \lim_{n \to \infty} gy_n = \lim_{n \to \infty} Ty_n = z.$$  

Since $S(X)$ is a closed subset of $X$, therefore $\lim_{n \to \infty} Sx_n = z \in S(X)$. Hence there exists some $u \in X$ such that $Su = z$. Now, we have to assert that $fu = Su$. Using (5.2.5.1), we have

$$M(fu, gy_n, kt) \geq \min\{M(Su, Ty_n, t), M(fu, Su, t), M(gy_n, Ty_n, t), M(fu, Ty_n, t), M(Su, gy_n, t)\}$$

which on making $n \to \infty$ and using Lemma 5.1.1, give rise

$$M(fu, Su, kt) \geq \min\{1, M(fu, Su, t), 1, M(fu, Su, t), 1\}$$

or

$$M(fu, Su, kt) \geq M(fu, Su, t).$$

In view of Lemma 5.2.2, $fu = Su$ which shows that $u$ is a coincidence point of the pair $(f, S)$.

Since $T(X)$ is also a closed subset of $X$, then $\lim_{n \to \infty} Ty_n = z \in T(X)$. Therefore there exists a point $w \in X$ such that $Tw = z$. To accomplish that $gw = z$, using (5.2.5.1), we have

$$M(fu, gw, kt) \geq \min\{M(Su, Tw, t), M(fu, Su, t), M(gw, Tw, t), M(fu, Tw, t), M(Su, gw, t)\}$$

or

$$M(z, gw, kt) \geq \min\{1, 1, M(gw, z, t), 1, M(z, gw, t)\} = M(z, gw, t).$$

Appealing again to Lemma 5.2.2, we have $gw = z$, and hence $Tw = gw = z$. This shows that $w$ is a coincidence point of the pair $(g, T)$.

As $fu = Su$ and the pair $(f, S)$ is weakly compatible, therefore $fz = fSu = Sfu = Sz$. Using (5.2.5.1), we have

$$M(fz, gw, kt) \geq \min\{M(Sz, Tw, t), M(fz, Sz, t), M(gw, Tw, t), M(fz, Tw, t), M(fz, Tw, t),$$

$$M(fz, gw, kt) \geq \min\{M(Sz, Tw, t), M(fz, Sz, t), M(gw, Tw, t), M(fz, Tw, t),$$

$$M(fz, gw, kt) \geq \min\{M(Sz, Tw, t), M(fz, Sz, t), M(gw, Tw, t), M(fz, Tw, t),$$

$$M(Sz, gw, t)\}. $$

or

$$M(z, gw, kt) \geq \min\{1, 1, M(gw, z, t), 1, M(z, gw, t)\} = M(z, gw, t).$$

Appealing again to Lemma 5.2.2, we have $gw = z$, and hence $Tw = gw = z$. This shows that $w$ is a coincidence point of the pair $(g, T)$.
\begin{align*}
M(Sz, gw, t) \} \\
\text{or} \\
M(fz, z, kt) \geq \min\{M(fz, z, t), 1, 1, M(fz, z, t), M(fz, z, t)\} = M(fz, z, t).
\end{align*}

By Lemma 5.2.2, we have \( fz = z \) which shows that \( z \) is a common fixed point of \( f \) and \( S \).

Since the pair \((g, T)\) is weakly compatible and \( gz = Tz \), using (5.2.5.1) and Lemma 5.2.2, one can easily show that \( z \) is common fixed point of \( g \) and \( T \). Hence both pairs have common fixed point \( z \). Uniqueness of common fixed point \( z \) is an easy consequence of condition (5.2.5.1). This completes the proof.

Remark 5.2.5. Theorem 5.2.5 generalizes some common fixed point theorems due to Grabiec [57], Mishra et al. [119], Subrahmanyam [158], Vijayaraju and Sajath [170], and extends some relevant results of Ali and Imdad [3] to fuzzy metric spaces which also include quasi-contractions.

Remark 5.2.6. Results similar to Lemma 5.2.1, Theorems 5.2.2-5.2.4 and Corollaries 5.2.1-5.2.3 can be proved in respect of contraction condition (5.2.5.1) which generalize and extend several results form the literature, but we avoid the details due to repetition.

\section{5.3 Results on GV-fuzzy metric spaces}

We begin with the following observation.

If \((X, M, \ast)\) is a GV-fuzzy metric space, then some of the hypotheses in the preceding results can be relaxed.

Lemma 5.3.1. Let \( f, g, S \) and \( T \) be four self mappings of a GV-fuzzy metric space \((X, M, \ast)\) satisfying the conditions (i)-(iii) of Lemma 5.2.1 and (iv) for some \( \phi \in \Phi \) and some \( t > 0 \)

\begin{align*}
M(fx, gy, t) \geq \phi(\min\{M(Sx, Ty, t), M(fx, Sx, t), M(gy, Ty, t), M(fx, Ty, t), M(Sx, gy, t)\}), \quad (5.3.1.1)
\end{align*}

then the pairs \((f, S)\) and \((g, T)\) share the common property (E.A).

Proof. As the pair \((f, S)\) enjoys the property (E.A), there exists a sequence \( \{x_n\} \) in \( X \) such that

\[ \lim_{n \to \infty} fx_n = \lim_{n \to \infty} Sx_n = z, \quad \text{for some } z \in X, \]

implying thereby \( \lim_{n \to \infty} M(fx_n, Sx_n, t) = 1 \). Since \( f(X) \subset T(X) \), therefore for each \( \{x_n\} \) there exists \( \{y_n\} \) in \( X \) such that \( fx_n = Ty_n \). Therefore, \( \lim_{n \to \infty} fx_n = \lim_{n \to \infty} Ty_n = \)
Thus in all, we have \( f x_n \to z, \ S x_n \to z \) and \( T y_n \to z \). In view of (iii), \( \{ g y_n \} \) also converges. Suppose \( \lim_{n \to \infty} g y_n = l \neq z \). Then using condition (5.3.1.1), we have

\[
M(f x_n, g y_n, t) \geq \phi(\min\{M(S x_n, T y_n, t), M(g y_n, T y_n, t), M(f x_n, S x_n, t), M(f x_n, T y_n, t), M(S x_n, g y_n, t)\})
\]

which on making \( n \to \infty \), gives rise

\[
M(z, l, t) \geq \phi(M(z, l, t)).
\]

As \( z \neq l \) implies \( 0 < M(z, l, t) < 1 \), henceforth \( \phi(M(z, l, t)) > M(z, l, t) \), which is a contradiction. Therefore, \( z = l \) which shows that the pairs \( (f, S) \) and \( (g, T) \) share the common property \( (E.A) \).

**Theorem 5.3.1.** Let \( f, g, S \), and \( T \) be four self mappings of a GV-fuzzy metric space \((X, M, *)\) satisfying the condition (5.3.1.1). Suppose that

(i) the pairs \( (f, S) \) and \( (g, T) \) share the common property \( (E.A) \),

(ii) \( S(X) \) and \( T(X) \) are closed subsets of \( X \).

Then pair \( (f, S) \) as well as \( (g, T) \) have a coincidence point. Moreover, \( f, g, S \), and \( T \) have a unique common fixed point in \( X \) provided both the pairs \( (f, S) \) and \( (g, T) \) are weakly compatible.

**Proof.** Since the pairs \( (f, S) \) and \( (g, T) \) share the common property \( (E.A) \), there exist two sequences \( \{x_n\} \) and \( \{y_n\} \) in \( X \) such that

\[
\lim_{n \to \infty} f x_n = \lim_{n \to \infty} S x_n = \lim_{n \to \infty} g y_n = \lim_{n \to \infty} T y_n = z, \quad \text{for some } z \in X.
\]

Since \( S(X) \) is a closed subset of \( X \), therefore \( \lim_{n \to \infty} S x_n = z \in S(X) \). Also, there exists a point \( u \in X \) such that \( S u = z \). Now we assert that \( M(f u, z, t) = 1 \). If not, then using condition (5.3.1.1), we have

\[
M(f u, g y_n, t) \geq \phi(\min\{M(S u, T y_n, t), M(g y_n, T y_n, t), M(f u, S u, t), M(f u, T y_n, t), M(S u, g y_n, t)\}).
\]

Letting \( n \to \infty \), one gets

\[
M(z, f u, t) \geq \phi(M(z, f u, t)).
\]

As \( z \neq f u \) implies \( 0 < M(z, f u, t) < 1 \), henceforth \( \phi(M(z, f u, t)) > M(z, f u, t) \), which is a contradiction. Therefore, \( z = f u \) so that \( f u = z = S u \). Hence \( u \) is a coincidence point of the pair \( (f, S) \).

Since \( T(X) \) is also a closed subset of \( X \), therefore \( \lim_{n \to \infty} T y_n = z \in T(X) \). Also, there exists a point \( w \in X \) such that \( T w = z \). Now we assert that \( M(g w, z, t) = 1 \). If not, then using condition (5.3.1.1), we have

\[
M(f u, g w, t) \geq \phi(\min\{M(S u, T w, t), M(f u, S u, t), M(g w, T w, t), M(f u, T w, t), M(S u, g w, t)\}).
\]
M(Su, gw, t))

or

\[ M(z, gw, t) \geq \phi(M(z, gw, t)). \]

If \( z \neq gw \), then \( 0 < M(z, gw, t) < 1 \) and henceforth \( \phi(M(z, gw, t)) > M(z, gw, t) \), which is a contradiction. Therefore, \( z = gw \) so that \( gw = z = Tw \). Thus, \( w \) is a coincidence point of the pair \((g, T)\).

Since \( fu = Su \) and the pair \((f, S)\) is weakly compatible, therefore \( fz = fSu = Sfu = Sz \). Now, we need to show that \( z \) is a common fixed point of the pair \((f, S)\). To accomplish this, we assert that \( M(fz, z, t) = 1 \). If not, then using condition (5.3.1.1), we have

\[ M(fz, gw, t) \geq \phi(\min\{M(Sz, Tw, t), M(gw, Tw, t), M(fz, Sz, t), M(fz, Tw, t), M(Sz, gw, t)\}) \]

implying thereby

\[ M(fz, z, t) \geq \phi(\min\{M(fz, z, t), 1, 1, M(fz, z, t), M(fz, z, t)\}) \]

or

\[ M(fz, z, t) \geq \phi(M(fz, z, t)). \]

If \( fz \neq z \), then \( 0 < M(fz, z, t) < 1 \) and henceforth \( \phi(M(fz, z, t)) > M(fz, z, t) \), which is a contradiction. Therefore, \( M(fz, z, t) = 1 \) so that \( fz = z \) which shows that \( z \) is a common fixed point of the pair \((f, S)\).

As \( gw = Tw \) and the pair \((g, T)\) is weakly compatible, therefore \( gz = gTw = Tgw = Sz \). Next, we show that \( z \) is a common fixed point of the pair \((g, T)\). To do this, we assert that \( M(gz, z, t) = 1 \). If not, then using condition (5.3.1.1), we have

\[ M(fu, gz, t) \geq \phi(\min\{M(Su, Tz, t), M(gz, Tz, t), M(fu, Su, t), M(fu, Tz, t), M(Su, gz, t)\}) \]

implying thereby

\[ M(gz, z, t) \geq \phi(\min\{M(gz, z, t), 1, 1, M(gz, z, t), M(gz, z, t)\}) \]

or

\[ M(gz, z, t) \geq \phi(M(gz, z, t)). \]

If \( gz \neq z \), then \( 0 < M(gz, z, t) < 1 \) and henceforth \( \phi(M(gz, z, t)) > M(gz, z, t) \), which is a contradiction. Therefore, \( M(gz, z, t) = 1 \) so that \( gz = z \) which shows that \( z \) is a common fixed point of the pair \((g, T)\). Uniqueness of the common fixed point is an easy consequence of the condition (5.3.1.1) in view of the condition \( \phi_2 \). This concludes the proof.
Remark 5.3.1. Theorem 5.3.1 generalizes relevant results contained in Mihet [117], Imdad and Ali [64, 66], Vijayaraju and Sajath [170] and several others to GV-fuzzy metric spaces. Our results can also be viewed as fuzzy version of some metric fixed point theorems contained in Ali and Imdad [3].

Remark 5.3.2. Notice that results similar to Theorems 5.2.2-5.2.4, Corollaries 5.2.1-5.2.3 can also be outlined in GV-fuzzy metric spaces (in respect of contraction condition (5.3.1.1)), but we do not include the details due to repetition.

Now, we state a result similar to Theorem 5.2.5 in GV-fuzzy metric space under fewer conditions.

Theorem 5.3.2. Let \( f, g, S \) and \( T \) be four self mappings of a GV-fuzzy metric space \((X, M, \ast)\) satisfying the conditions (i) and (ii) of Theorem 5.3.1. Also for all \( x, y \in X \), with some \( k \in (0, 1) \) and some \( t > 0 \),

\[
M(fx, gy, kt) \geq \min\{M(Sx, Ty, t), M(fx, Sx, t), M(gy, Ty, t), M(fx, Ty, t), M(Sx, gy, t)\},
\]

(5.3.2.1)

holds. Then pair \((f, S)\) as well as \((g, T)\) have a coincidence point. Moreover, \( f, g, S \) and \( T \) have a unique common fixed point in \( X \) provided both the pairs \((f, S)\) and \((g, T)\) are weakly compatible.

Proof. Proof can be completed on the lines of the proof of Theorem 5.2.5.

Remark 5.3.3. Notice that results similar to Theorems 5.2.2-5.2.4, Corollaries 5.2.1-5.2.3 can also be outlined in respect of contraction condition (5.3.2.1) in GV-fuzzy metric spaces, but we omit the details due to repetition.

Now, we furnish an example to demonstrate the validity of the hypotheses of Theorem 5.3.1.

Example 5.3.1. Let \((X, M, \ast)\) be a GV-fuzzy metric space wherein \( X = [0, 1] \), \( a \ast b = ab \) with

\[
M(x, y, t) = \frac{t}{t + |x - y|} \text{ for all } t > 0.
\]

Define self mappings \( f, g, S \) and \( T \) on \( X \) by

\[
\begin{align*}
f(x) &= \begin{cases} 
1 & \text{if } x \in [0, 1] \cap Q \\
\frac{1}{2} & \text{if } x \not\in [0, 1] \cap Q,
\end{cases} \\
g(x) &= \begin{cases} 
1 & \text{if } x \in [0, 1] \cap Q \\
\frac{1}{4} & \text{if } x \not\in [0, 1] \cap Q,
\end{cases} \\
S(x) = T(x) &= \begin{cases} 
1 & \text{if } x \in [0, 1] \cap Q \\
0 & \text{if } x \not\in [0, 1] \cap Q.
\end{cases}
\end{align*}
\]
Then \( f(X) = \{1, \frac{1}{2}\} \not\subset \{0, 1\} = S(X) \) and \( g(X) = \{1, \frac{1}{2}\} \not\subset \{1, 0\} = T(X) \).

Now, if we take \( x \in [0, 1] \cap Q \) and \( y \not\in [0, 1] \cap Q \), then (for all \( x, y \in X \) and \( t > 0 \)) we have

\[
M(fx, gy, t) \geq \phi\left( \min\left\{ M(Sx, Ty, t), M(gy, Ty, t), M(fx, Sx, t), M(fx, Ty, t), M(Sx, gy, t) \right\} \right)
\]
or

\[
M(1, \frac{1}{4}, t) \geq \phi\left( \min\left\{ M(1, 0, t), M(\frac{1}{4}, 0, t), M(1, 1, t), M(1, 0, t), M(1, \frac{1}{4}, t) \right\} \right)
\]
or

\[
\frac{t}{t + \frac{3}{4}} \geq \phi\left( \min\left\{ \frac{t}{t + 1}, \frac{t}{t + \frac{1}{4}}, \frac{t}{t + 1}, \frac{t}{t + \frac{3}{4}} \right\} \right)
\]

which is true for all \( t > 0 \) (here \( \phi(s) = \sqrt{s} \)). Thus, all the conditions of Theorem 5.3.1 are satisfied. Notice that 1 is the unique common fixed point of all the involved mappings.

### 5.4 Results on cyclic weak \( \phi \)-contractions

In this section, we introduce the notion of cyclic weak \( \phi \)-contraction in a fuzzy metric space. Using this kind of contraction, we prove a fixed point theorem for \( G \)-complete fuzzy metric spaces in the sense of George and Veeramani [49]. Moreover, some problems related to the fixed point are discussed. Some examples are also presented to support the concept defined in this chapter.

**Definition 5.4.1.** (Di Bari and Vetro [39]) Let \((X, M, \ast)\) be a fuzzy metric space. The fuzzy metric \( M \) is triangular if it satisfies the condition

\[
\left( \frac{1}{M(x, y, t)} - 1 \right) \leq \left( \frac{1}{M(x, z, t)} - 1 \right) + \left( \frac{1}{M(y, z, t)} - 1 \right)
\]

for every \( x, y, z \in X \) and every \( t > 0 \).

**Definition 5.4.2.** (Pacurar [123]) Let \( X \) be a nonempty set, \( m \) a positive integer and \( f : X \rightarrow X \) an operator. By definition, \( X = \bigcup_{i=1}^{m} X_i \) is a cyclic representation of \( X \) with respect to \( f \) if

(i) \( X_i, \ i = 1, 2, \ldots, m \) are nonempty sets;

(ii) \( f(X_1) \subset X_2, \ldots, f(X_{m-1}) \subset X_m; f(X_m) \subset X_1. \)
Example 5.4.1. Let $X = \mathbb{R}$. Assume $A_1 = A_3 = [-2, 0]$ and $A_2 = A_4 = [0, 2]$, so that $Y = \bigcup_{i=1}^{4} A_i = [-2, 2]$. Define $f : Y \to Y$ such that $f(x) = -\frac{x}{2}$, for all $x \in Y$. Clearly, $Y = \bigcup_{i=1}^{4} A_i$ is a cyclic representation of $Y$.

Inspired by Karapinar [96], we introduce the notion of cyclic weak $\phi$-contraction in fuzzy metric spaces.

Definition 5.4.3. Let $(X, M, *)$ be a fuzzy metric space, $A_1, A_2, \ldots, A_m$ be closed subsets of $X$ and $Y = \bigcup_{i=1}^{m} A_i$. An operator $f : Y \to Y$ is called a cyclic weak $\phi$-contraction if the following conditions hold:

(i) $Y = \bigcup_{i=1}^{m} A_i$ is a cyclic representation of $Y$ with respect to $f$;

(ii) there exists a continuous, non-decreasing function $\phi : [0, \infty) \to [0, \infty)$ with $\phi(r) > 0$ for $r > 0$ and $\phi(0) = 0$, such that

$$\left( \frac{1}{M(fx, fy, t)} - 1 \right) \leq \left( \frac{1}{M(x, y, t)} - 1 \right) - \phi \left( \frac{1}{M(x, y, t)} - 1 \right) \cdots (5.4.3.1)$$

for any $x \in A_i$, $y \in A_{i+1}$ ($i = 1, 2, \ldots, m$, where $A_{m+1} = A_1$) and each $t > 0$.

Theorem 5.4.1. Let $(X, M, *)$ be a fuzzy metric space, $A_1, A_2, \ldots, A_m$ be closed subsets of $X$ and $Y = \bigcup_{i=1}^{m} A_i$ be $G$-complete. Suppose that $\phi : [0, +\infty) \to [0, +\infty)$ is a continuous, non-decreasing function with $\phi(r) > 0$ for each $r \in (0, +\infty)$ and $\phi(0) = 0$. If $f : Y \to Y$ is a cyclic weak $\phi$-contraction, then $f$ has a unique fixed point $z \in \bigcap_{i=1}^{m} A_i$.

Proof. Let $x_0 \in Y = \bigcup_{i=1}^{m} A_i$ and set $x_n = fx_{n-1}$ ($n \geq 1$). Clearly, we get

$$M(x_n, x_{n+1}, t) = M(fx_{n-1}, fx_n, t)$$

for any $t > 0$. Besides, for any $n \geq 0$, there exists $i_n \in \{1, 2, \ldots, m\}$ such that $x_n \in A_{i_n}$ and $x_{n+1} \in A_{i_{n+1}}$. Then by (5.4.3.1) for $t > 0$, we have

$$\left( \frac{1}{M(x_n, x_{n+1}, t)} - 1 \right) = \left( \frac{1}{M(fx_{n-1}, fx_n, t)} - 1 \right) \leq \left( \frac{1}{M(x_{n-1}, x_n, t)} - 1 \right) - \phi \left( \frac{1}{M(x_{n-1}, x_n, t)} - 1 \right) \leq \left( \frac{1}{M(x_{n-1}, x_n, t)} - 1 \right)$$

which implies that $M(x_n, x_{n+1}, t) \geq M(x_{n-1}, x_n, t)$ for all $n \geq 1$ and so $\{M(x_{n-1}, x_n, t)\}$ is a non-decreasing sequence of positive real numbers in $(0, 1]$. 

Let \( S(t) = \lim_{n \to +\infty} M(x_{n-1}, x_n, t) \). Now, we show that \( S(t) = 1 \) for all \( t > 0 \). If not, there exists some \( t > 0 \) such that \( S(t) < 1 \). Then, on making \( n \to +\infty \) in preceding inequality, we obtain

\[
\left( \frac{1}{S(t)} - 1 \right) \leq \left( \frac{1}{S(t)} - 1 \right) - \phi \left( \frac{1}{S(t)} - 1 \right)
\]

which is a contradiction. Therefore \( M(x_n, x_{n+1}, t) \to 1 \) as \( n \to +\infty \).

Now, for each positive integer \( p \), we have

\[
M(x_n, x_{n+p}, t) \geq M(x_n, x_{n+1}, t/p) \cdot M(x_{n+1}, x_{n+2}, t/p) \cdots M(x_{n+p-1}, x_{n+p}, t/p).
\]

It follows that

\[
\lim_{n \to +\infty} M(x_n, x_{n+p}, t) \geq 1 \cdot 1 \cdots 1 = 1
\]

so that \( \{x_n\}_{n \geq 0} \) is a \( G \)-Cauchy sequence. As \( Y \) is \( G \)-complete, then there exists \( y \in Y \) such that \( \lim_{n \to +\infty} x_n = y \). On the other hand, by the condition (i) of Definition 5.4.3, it follows that the iterative sequence \( \{x_n\} \) has an infinite number of terms in \( A_i \) for each \( i = 1, 2, \ldots, m \). Since \( Y \) is \( G \)-complete, from each \( A_i, i = 1, 2, \ldots, m \), one can extract a subsequence of \( \{x_n\} \) that converges to \( y \). In virtue of the fact that each \( A_i, i = 1, 2, \ldots, m \) is closed, we conclude that \( y \in \bigcap_{i=1}^{m} A_i \) and thus \( \bigcap_{i=1}^{m} A_i \neq \emptyset \).

Obviously, \( \bigcap_{i=1}^{m} A_i \) is closed and \( G \)-complete. Now, consider the restriction of \( f \) on \( \bigcap_{i=1}^{m} A_i \), i.e. \( f | \bigcap_{i=1}^{m} A_i : \bigcap_{i=1}^{m} A_i \to \bigcap_{i=1}^{m} A_i \) which satisfies the assumptions of Corollary 4 in [169] and thus, \( f | \bigcap_{i=1}^{m} A_i \) has a unique fixed point in \( \bigcap_{i=1}^{m} A_i \), say \( z \), which is obtained by iteration from the starting point \( x_0 \in Y \). To this end, we have to show that \( x_n \to z \) as \( n \to +\infty \). Then, by (5.4.3.1), we have

\[
\left( \frac{1}{M(x_n, z, t)} - 1 \right) \leq \left( \frac{1}{M(x_{n-1}, z, t)} - 1 \right) - \phi \left( \frac{1}{M(x_{n-1}, z, t)} - 1 \right)
\]

Now, letting \( n \to +\infty \), we get

\[
\left( \frac{1}{M(y, z, t)} - 1 \right) \leq \left( \frac{1}{M(y, z, t)} - 1 \right) - \phi \left( \frac{1}{M(y, z, t)} - 1 \right)
\]

which is a contradiction if \( M(y, z, t) < 1 \), and so, we conclude that \( z = y \). Obviously, \( z \) is the unique fixed point of \( f \).

**Remark 5.4.1.** Theorem 5.4.1 extends and generalizes the related results of [96, 123, 136] in fuzzy metric spaces via cyclic weak \( \phi \)-contraction.

**Example 5.4.2.** Let \( X = \mathbb{R} \) and \( M(x, y, t) = \frac{t}{t + |x - y|} \), for all \( x, y \in X, t > 0 \).

Assume \( A_1 = A_2 = \cdots = A_m = [0, 1] \), so that \( Y = \bigcup_{i=1}^{m} A_i = [0, 1] \) and define
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Let \( f : Y \to Y \) such that \( f(x) = \frac{x^2}{4} \). Furthermore, if \( \phi : [0, \infty) \to [0, \infty) \) is defined by \( \phi(s) = \frac{s}{2} \), we have

\[
\left( \frac{1}{M(fx, fy, t)} - 1 \right) = \frac{|x^2 - y^2|}{4t} \leq \frac{|x - y|}{2t} = \left( \frac{1}{M(x, y, t)} - 1 \right) - \phi \left( \frac{1}{M(x, y, t)} - 1 \right).
\]

Then \( f \) is a cyclic weak \( \phi \)-contraction. All the conditions of Theorem 5.4.1 are satisfied and \( f \) has a unique fixed point \( z = 0 \in \bigcap_{i=1}^{m} A_i \).

**Theorem 5.4.2.** If we add to the hypotheses of Theorem 5.4.1 the following condition:

(i) there exists a sequence \( \{y_n\} \) in \( Y \) such that \( M(y_n, fy_n, t) \to 1 \) as \( n \to +\infty \) for any \( t > 0 \),

then \( y_n \to z \) as \( n \to +\infty \), provided the fuzzy metric \( M \) is triangular, and \( z \) is the unique fixed point of \( f \) in \( Y \).

**Proof.** By Theorem 5.4.1, we have that \( z \in \bigcap_{i=1}^{m} A_i \) is the unique fixed point of \( f \).

Now, from the triangularity of \( M \) and (5.4.3.1), we have

\[
\left( \frac{1}{M(y_n, z, t)} - 1 \right) \leq \left( \frac{1}{M(y_n, fy_n, t)} - 1 \right) + \left( \frac{1}{M(fy_n, fz, t)} - 1 \right) \\
\leq \left( \frac{1}{M(y_n, fy_n, t)} - 1 \right) + \left( \frac{1}{M(y_n, z, t)} - 1 \right) - \phi \left( \frac{1}{M(y_n, z, t)} - 1 \right)
\]

which is equivalent to

\[
\phi \left( \frac{1}{M(y_n, z, t)} - 1 \right) \leq \left( \frac{1}{M(y_n, fy_n, t)} - 1 \right).
\]

On making \( n \to \infty \), one gets

\[
\lim_{n \to +\infty} \phi \left( \frac{1}{M(y_n, z, t)} - 1 \right) = 0, \quad \text{since} \quad \lim_{n \to +\infty} \left( \frac{1}{M(y_n, fy_n, t)} - 1 \right) = 0.
\]

Owing to the properties of \( \phi \), we conclude that \( M(y_n, z, t) \to 1 \), which is equivalent to say that \( y_n \to z \) as \( n \to +\infty \).

**Theorem 5.4.3.** If we add to the hypotheses of Theorem 5.4.1 the following condition:
(ii) there exists a convergent sequence \( \{y_n\}_{n \in \mathbb{N}} \) in \( Y \) such that \( M(y_{n+1}, fy_n, t) \to 1 \) as \( n \to +\infty \) for any \( t > 0 \),

then there exists \( x \in Y \) such that \( M(y_n, f^nx, t) \to 1 \) as \( n \to +\infty \), provided the fuzzy metric \( M \) is triangular.

**Proof.** Once again, according to the proof of Theorem 5.4.1, we observe that for any initial point \( x \in Y \), \( z \in \bigcap_{i=1}^{m} A_i \) is the unique fixed point of \( f \). Moreover, for any \( t > 0, 0 < M(y_n, z, t) < 1 \) and \( 0 < M(y_{n+1}, z, t) < 1 \). Set \( y \) as a limit of a convergent sequence \( \{y_n\} \) in \( Y \). Now, from the triangularity of \( M(x, y, t) \) and (5.4.3.1), we have

\[
\left( \frac{1}{M(y_{n+1}, z, t)} - 1 \right) \leq \left( \frac{1}{M(y_{n+1}, fy_n, t)} - 1 \right) + \left( \frac{1}{M(fy_n, fz, t)} - 1 \right) - \phi \left( \frac{1}{M(y_n, z, t)} - 1 \right) \leq \left( \frac{1}{M(y_{n+1}, fy_n, t)} - 1 \right) + \left( \frac{1}{M(y_n, z, t)} - 1 \right) - \phi \left( \frac{1}{M(y_n, z, t)} - 1 \right).
\]

Then, on making \( n \to +\infty \) in the preceding inequality, we get

\[
\left( \frac{1}{M(y, z, t)} - 1 \right) \leq \left( \frac{1}{M(y, z, t)} - 1 \right) - \phi \left( \frac{1}{M(y, z, t)} - 1 \right).
\]

Clearly, above inequality is true if and only if \( \phi \left( \frac{1}{M(y, z, t)} - 1 \right) = 0 \). By the properties of the function \( \phi \), \( \left( \frac{1}{M(y, z, t)} - 1 \right) = 0 \) so that \( y = z \). Consequently, we have \( M(y_n, f^nx, t) \to 1 \) as \( n \to \infty \).

**Theorem 5.4.4.** Notice that Theorem 5.4.1 remains true if we suppose that \( \phi : [0, +\infty) \to [0, +\infty) \) is a lower semi-continuous function.

In our next result, we consider two fuzzy metric spaces. More precisely, we state and prove the following theorem.

**Theorem 5.4.5.** Let \((X, M, \ast)\) and \((X, M', \ast)\) be two fuzzy metric spaces, \( m \) be a positive integer, \( A_1, A_2, \ldots, A_m \) be non-empty closed subsets of \( X \) and \( Y = \bigcup_{i=1}^{m} A_i \). Suppose that

(i) \( Y = \bigcup_{i=1}^{m} A_i \) is a cyclic representation of \( Y \) with respect to \( f \);

(ii) \( M(x, y, t) \geq M'(x, y, t) \), for all \( x, y \in Y \);

(iii) \((Y, M, \ast)\) is a \( G \)-complete fuzzy metric space;

(iv) \( f : (Y, M, \ast) \to (Y, M, \ast) \) is continuous;
(v) \( f : (Y, M', *) \rightarrow (Y, M', *) \) is a cyclic weak \( \phi \)-contraction, where \( \phi : [0, +\infty) \rightarrow [0, +\infty) \) is a lower semi-continuous function with \( \phi(s) > 0 \) for all \( s \in (0, +\infty) \) and \( \phi(0) = 0 \).

Then \( \{ f^n x_0 \} \rightarrow z \in (Y, M, *) \), for all \( x_0 \in Y \) and \( z \) is a unique fixed point of \( f \).

**Proof.** Let \( x_0 \in Y \) as in Theorem 5.4.1. By (v), we deduce that \( \{ f^n x_0 \} \) is a \( G \)-Cauchy sequence in \( (Y, M', *) \). Moreover, by (ii), \( \{ f^n x_0 \} \) is a \( G \)-Cauchy sequence in \( (Y, M, *) \). Now, (iii) implies that \( \{ f^n x_0 \} \rightarrow z \in (Y, M, *) \) for any starting point \( x_0 \in Y \). Finally, by (iv) we get that \( z \) is a unique fixed point of \( f \).

### 5.5 A related result

In this section, we prove a result on non-Archimedean fuzzy metric space. In [115], Mihet proved a theorem which ensures the existence of a fixed point for fuzzy \( \psi \)-contractive mappings in the setting of complete non-Archimedean fuzzy metric spaces (see also [168]). In what follows, we state a result analogous to Theorem 5.4.1 in the framework of complete non-Archimedean fuzzy metric spaces and also illustrate the same using an example.

For the sake of completeness, we recall the following definition.

**Definition 5.5.1.** If, in Definition 1.5.2, the triangular inequality (iv) is replaced by the following condition

\[
\text{(NA)} \quad M(x, z, \max\{t, s\}) \geq M(x, y, t) \ast M(y, z, s), \quad \forall x, y, z \in X, \quad \forall \ t, s > 0,
\]

then the 3-tuple \( (X, M, *) \) is called a non-Archimedean fuzzy metric space. It is easy to check that the triangular inequality (NA) implies (iv) (of Definition 1.5.2) i.e. every non-Archimedean fuzzy metric space is itself a fuzzy metric space.

**Example 5.5.1.** Let \( (X, d) \) be an ordinary metric space and let \( \theta \) be a non-decreasing and continuous function from \( [0, +\infty) \) into \( (0, 1) \) such that \( \theta(0) = 0 \), \( \theta(t) > 0 \) for \( t > 0 \) and \( \lim_{t \to +\infty} \theta(t) = 1 \). Some examples of these functions are \( \theta(t) = t/(t+1), \\theta(t) = 1 - e^{-t} \). Let \( a \ast b \leq ab \) for all \( a, b \in [0, 1] \), and define \( M(x, y, t) = [\theta(t)]^d(x, y) \) for all \( x, y \in X \) and for each \( t > 0 \). It is easy to see that \( (X, M, *) \) is a non-Archimedean fuzzy metric space.

Following the lines of the proof of Theorem 5.4.1, one can prove the following:

**Theorem 5.5.1.** Let \( (X, M, *) \) be a non-Archimedean fuzzy metric space, \( A_1, A_2, \ldots, A_m \) be closed subsets of \( X \) and \( Y = \bigcup_{i=1}^{m} A_i \) be \( G \)-complete. Suppose that \( \phi : [0, +\infty) \rightarrow [0, +\infty) \) is a continuous, non-decreasing function with \( \phi(r) > 0 \) for each \( r \in (0, +\infty) \) and \( \phi(0) = 0 \). If \( f : Y \rightarrow Y \) is a cyclic weak \( \phi \)-contraction, then \( f \) has a unique fixed point \( z \in \bigcap_{i=1}^{m} A_i \).
Example 5.5.2. Let $X = \mathbb{R}$. Assume $A_1 = A_2 = A_3 = [0, 1]$, so that $Y = \bigcup_{i=1}^{3} A_i = [0, 1]$. Define $f : Y \to Y$ such that $fx = \frac{1 + x}{2}$, for all $x \in Y$. It is clear that $Y = \bigcup_{i=1}^{3} A_i$ is a cyclic representation of $Y$. Let

$$M(x, y, t) = \frac{\min\{x, y\}}{\max\{x, y\}}, \quad \text{for all } x, y \in X \text{ and } t > 0.$$ 

It is easy to show that $(X, M, \ast)$ is a non-Archimedean fuzzy metric space. Furthermore, if $\phi : [0, \infty) \to [0, \infty)$ is defined by $\phi(s) = \frac{s^2}{2}$. Assuming $x \geq y$, we have

$$\left( \frac{1}{M(fx, fy, t)} - 1 \right) = \frac{1 + x}{1 + y} - 1 = \frac{x - y}{1 + y} \leq \frac{1}{2} \left( \frac{x}{y} - 1 \right) = \left( \frac{1}{M(x, y, t)} - 1 \right) - \phi \left( \frac{1}{M(x, y, t)} - 1 \right).$$

Then $f$ is a cyclic weak $\phi$-contraction. All the conditions of Theorem 5.5.1 are satisfied and $f$ has a unique fixed point $z = 1 \in \bigcap_{i=1}^{n} A_i$.

Remark 5.5.2. Theorem 5.5.1. can be viewed as a generalization of Theorem 5.4.1, and so it extends and generalizes the related results of [96, 123, 136] in non-Archimedean fuzzy metric spaces via cyclic weak $\phi$-contraction.

Remark 5.5.3. Notice that results similar to Theorems 5.4.2-5.4.5 can also be outlined in respect to non-Archimedean fuzzy metric spaces, but we omit the details due to repetition.