Chapter 3

Unified common fixed point theorems for expansive type mappings in symmetric spaces

3.1 Introduction

A metrical common fixed point theorem is often comprised of conditions on commutativity, continuity, completeness and contraction besides suitable containment of range of one map into the range of the other. To prove new results, the researchers of this domain are required to improve one or more of these conditions. With a view to improve the commutativity conditions in such results, Sessa [152] initiated the idea of weak commutativity which was received well by the researchers of this direction. In process, several conditions of weak commutativity were introduced and utilized to prove new common fixed point theorems whose lucid survey (up to 2001) is available in Murthy [120]. In the last few years the notion of weak compatibility due to Jungck [90] has been extensively utilized to prove new results as it is a minimal condition merely requiring the commutativity at the set of coincidence points of the pair. Wang et al. [171] proved some fixed point theorems on expansion mappings corresponding to certain expansive condition whose earliest noted generalization is contained in Khan et al. [100]. In recent years a multitude of expansive type results are established which include Rhoades [142] and Taniguchi [160], Kang [94] and some others.

In the following lines, we collect the background material to make our presentation as self contained as possible.

Definition 3.1.1. [71] Two finite families of self mappings \( \{f_i\}^{m}_{i=1} \) and \( \{g_k\}^{n}_{k=1} \) of a set \( X \) are said to be pairwise commuting if:

(i) \( f_i f_j = f_j f_i \quad i,j \in \{1,2,\ldots,m\}, \)

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(ii) \( g_k g_l = g_l g_k \)  \( k, l \in \{1, 2, \ldots, n\} \),

(iii) \( f_i g_k = g_k f_i \)  \( i \in \{1, 2, \ldots, m\} \) and \( k \in \{1, 2, \ldots, n\} \).

In this chapter, we prove general common fixed point theorems (Section 3.3) via absorbing property of expansive mappings under a relatively improved implicit function (Section 3.2) on symmetric (resp. semi-metric) spaces which generalizes several relevant results contained in [3, 40, 76, 94, 100, 130, 131, 134] besides several other ones. Our results generalize several fixed point theorems in following respects:

(i) The class of expansive definitions covered by our implicit function is relatively wider one as it requires merely two conditions to satisfy.

(ii) The condition on completeness of the space is lightened to closedness of subspaces.

(iii) The conditions of containment of the ranges amongst involved maps are completely relaxed.

(iv) The class of underlying metric spaces is enlarged to class of symmetric spaces.

### 3.2 Implicit functions

Popa [131] initiated the idea of implicit functions instead of contraction conditions to prove fixed point theorems. Motivated by Ali and Imdad [3], Imdad and Khan [76] and Popa [131, 134], we define a new class of implicit functions to prove common fixed point theorems for expansive type mappings. In order to describe the implicit function, let \( \Phi \) be the set of all continuous functions \( F : \mathbb{R}^6_+ \rightarrow \mathbb{R} \) satisfying the following conditions:

\[
(F_1) : F(t, 0, t, 0, 0, t) < 0, \text{ for all } t > 0,
\]

\[
(F_2) : F(t, 0, 0, t, t, 0) < 0, \text{ for all } t > 0.
\]

**Example 3.2.1.** Define \( F(t_1, t_2, \ldots, t_6) : \mathbb{R}^6_+ \rightarrow \mathbb{R} \) as

\[
F(t_1, t_2, \ldots, t_6) = t_1 - k \max\{t_2, t_3, t_4, t_5, t_6\}, \quad \text{where } k > 1.
\]

**Example 3.2.2.** Define \( F(t_1, t_2, \ldots, t_6) : \mathbb{R}^6_+ \rightarrow \mathbb{R} \) as

\[
F(t_1, t_2, \ldots, t_6) = k \min\{t_2, t_3, t_4, t_5, t_6\} - t_1, \quad \text{where } 0 \leq k < 1.
\]

**Example 3.2.3.** Define \( F(t_1, t_2, \ldots, t_6) : \mathbb{R}^6_+ \rightarrow \mathbb{R} \) as

\[
F(t_1, t_2, \ldots, t_6) = \phi(\min\{t_2, t_3, t_4, t_5, t_6\}) - t_1
\]
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where \( \phi : \mathbb{R}^+ \rightarrow \mathbb{R} \) is an lower semi-continuous function such that \( \phi(0) = 0 \) and \( \phi(t) < t \) for all \( t > 0 \).

Example 3.2.4. Define \( F(t_1, t_2, \cdots, t_6) : \mathbb{R}_+^6 \rightarrow \mathbb{R} \) as

\[
F(t_1, t_2, \cdots, t_6) = t_1 - a t_2 - b t_3 - c t_4 - e t_5 - f t_6,
\]

where \( a, b, c, e, f > 0 \) with \( c + e > 1, f + b > 1 \).

Example 3.2.5. Define \( F(t_1, t_2, \cdots, t_6) : \mathbb{R}_+^6 \rightarrow \mathbb{R} \) as

\[
F(t_1, t_2, \cdots, t_6) = t_1 - k \left[ \max \{ t_2^2, t_3 t_4, t_5 t_6, t_3 t_6, t_4 t_5 \} \right]^{\frac{1}{2}}, \quad \text{where } k > 1.
\]

Example 3.2.6. Define \( F(t_1, t_2, \cdots, t_6) : \mathbb{R}_+^6 \rightarrow \mathbb{R} \) as

\[
F(t_1, t_2, \cdots, t_6) = \begin{cases} 
  t_1 - a t_2 - \beta \frac{t_2^2 + t_6^2}{t_5 + t_6} - \gamma (t_3 + t_4), & \text{if } t_5 + t_6 \neq 0 \\
  t_1, & \text{if } t_5 + t_6 = 0
\end{cases}
\]

where \( \alpha, \gamma > 0 \) and \( \beta > 1 \).

Example 3.2.7. Define \( F(t_1, t_2, \cdots, t_6) : \mathbb{R}_+^6 \rightarrow \mathbb{R} \) as

\[
F(t_1, t_2, \cdots, t_6) = t_1 - a t_2 - b t_3 - c t_4 - \max \{ t_5, t_6 \},
\]

where \( a, b, c > 0 \)

Example 3.2.8. Define \( F(t_1, t_2, \cdots, t_6) : \mathbb{R}_+^6 \rightarrow \mathbb{R} \) as

\[
F(t_1, t_2, \cdots, t_6) = k \min \{ t_2, \max \{ t_3, t_4 \}, \max \{ t_5, t_6 \} \} - t_1,
\]

where \( 0 < k < 1 \).

Example 3.2.9. Define \( F(t_1, t_2, \cdots, t_6) : \mathbb{R}_+^6 \rightarrow \mathbb{R} \) as

\[
F(t_1, t_2, \cdots, t_6) = t_1 - k \min \{ t_1 + t_2, t_3 + t_5, t_4 + t_6 \},
\]

where \( k > 1 \).

Example 3.2.10. Define \( F(t_1, t_2, \cdots, t_6) : \mathbb{R}_+^6 \rightarrow \mathbb{R} \) as

\[
F(t_1, t_2, \cdots, t_6) = a t_2 + b (t_3 + t_4) + c (t_5 + t_6) - t_1,
\]

where \( a, b, c \geq 0 \) and \( b + c < 1 \).

Example 3.2.11. Define \( F(t_1, t_2, \cdots, t_6) : \mathbb{R}_+^6 \rightarrow \mathbb{R} \) as

\[
F(t_1, t_2, \cdots, t_6) = \min \{ (t_2 + t_3)/2, k (t_4 + t_5)/2, t_6 \} - t_1,
\]
where $0 \leq k < 1$.

**Example 3.2.12.** Define $F(t_1, t_2, \ldots, t_6) : \mathbb{R}_+^6 \to \mathbb{R}$ as

$$F(t_1, t_2, \ldots, t_6) = \min\{at_2, b(t_3 + t_5)/2, (t_4 + t_6)\} - t_1,$$

where $0 \leq a < 1$, $1 \leq b < 2$.

**Example 3.2.13.** Define $F(t_1, t_2, \ldots, t_6) : \mathbb{R}_+^6 \to \mathbb{R}$ as

$$F(t_1, t_2, \ldots, t_6) = at_2^2 + t_3t_4 + bt_5^2 + ct_6^2 - t_1^2,$$

where $a \geq 0$ and $0 \leq b, c < 1$.

**Example 3.2.14.** Define $F(t_1, t_2, \ldots, t_6) : \mathbb{R}_+^6 \to \mathbb{R}$ as

$$F(t_1, t_2, \ldots, t_6) = k(t_2^3 + t_3^3 + t_4^3 + t_5^3 + t_6^3) - t_1^3,$$

where $0 \leq k < 1/3$.

**Example 3.2.15.** Define $F(t_1, t_2, \ldots, t_6) : \mathbb{R}_+^6 \to \mathbb{R}$ as

$$F(t_1, t_2, \ldots, t_6) = t_1^2 - at_1^2t_2 - bt_1t_4t_5 - ct_1t_3t_6 - dt_3t_5t_6,$$

where $b, c > 1$.

**Example 3.2.16.** Define $F(t_1, t_2, \ldots, t_6) : \mathbb{R}_+^6 \to \mathbb{R}$ as

$$F(t_1, t_2, \ldots, t_6) = t_1^2 - at_1^2t_2 - bt_3^2 - ct_4^2 - dt_5t_6,$$

where $b, c > 1$.

**Example 3.2.17.** Define $F(t_1, t_2, \ldots, t_6) : \mathbb{R}_+^6 \to \mathbb{R}$ as

$$F(t_1, t_2, \ldots, t_6) = \frac{at_3t_4 + bt_5t_6 + ct_2^2}{t_5 + t_6 + t_2} - t_1, \quad \text{if } t_5 + t_6 + t_2 \neq 0$$

where $b + c > 3$.

Since verification of requirements ($F_1$ and $F_2$) for Examples 3.2.1-3.2.17 are easy, details are not included.

### 3.3 Main results

We begin with the following observation.

**Lemma 3.3.1.** Let $X$ be a nonempty set equipped with a continuous symmetric (semi-metric) $d$. If $f, g, S, T : X \to X$ are four mappings which satisfy the conditions:

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(i) the pair \( (f, S) \) (or \( (g, T) \)) satisfies the property \((E.A)\),

(ii) \( f(X) \subset T(X) \) (or \( g(X) \subset S(X) \)),

(iii) for all \( x, y \in X(x \neq y) \) and \( F \in \Phi \) wherein \( F \) satisfies condition \((F_2)\)

\[
F(d(fx, gy), d(Sx, Ty), d(fx, Sx), d(gy, Ty), d(Sx, gy), d(Ty, fx)) > 0
\]  

\((3.1.1.1)\)

whenever, one of \( d(fx, gy), d(gy, Ty) \) and \( d(Sx, gy) \) is positive. Then the pairs \( (f, S) \) and \( (g, T) \) share the common property \((E.A)\).

**Proof.** If the pair \( (f, S) \) enjoys the property \((E.A)\), then there exists a sequence \( \{x_n\} \) in \( X \) such that

\[
\lim_{n \to \infty} fx_n = \lim_{n \to \infty} Sx_n = t, \quad \text{for some } t \in X.
\]

Since \( f(X) \subset T(X) \), therefore for each \( \{x_n\} \) there exists \( \{y_n\} \) in \( X \) such that \( fx_n = Ty_n \). Thus, \( \lim_{n \to \infty} Ty_n = \lim_{n \to \infty} fx_n = t \) and in all we have \( fx_n \to t, Sx_n \to t \) and \( Ty_n \to t \). Now, we assert that \( gy_n \to t \). If not, then using \((3.1.1.1)\), we have

\[
F(d(fx_n, gy_n), d(Sx_n, Ty_n), d(fx_n, Sx_n), d(gy_n, Ty_n), d(Sx_n, gy_n), d(Ty_n, fx_n)) > 0
\]

which on making \( n \to \infty \), gives rise

\[
F(d(t, \lim_{n \to \infty} gy_n), 0, 0, d(\lim_{n \to \infty} gy_n, t), d(t, \lim_{n \to \infty} gy_n), 0) \geq 0,
\]

a contradiction to \((F_2)\). Hence \( \lim_{n \to \infty} gy_n \to t \) which shows that the pairs \( (f, S) \) and \( (g, T) \) share the common property \((E.A)\).

**Remark 3.3.1.** The converse of Lemma 3.3.1 is not true in general. For a counter example, one can utilize Example 3.5.1 to be furnished in the concluding section.

Now, we state and prove our main result for two pairs of pointwise absorbing mappings satisfying earlier described implicit functions.

**Theorem 3.3.1.** Let \( f, g, S \) and \( T \) be four self mappings defined on a nonempty set \( X \) equipped with a symmetric (semi-metric) \( d \) which enjoys \((1C)\) and \((HE)\) which satisfy the inequality \((3.1.1.1)\) wherein \( F \in \Phi \) satisfies \((F_1)\) and \((F_2)\), whenever, one of \( d(fx, gy), d(fx, Sx), d(gy, Ty) \) and \( d(Sx, gy) \) is positive. Suppose that:

(i) the pairs \( (f, S) \) and \( (g, T) \) share the common property \((E.A)\) and

(ii) \( S(X) \) and \( T(X) \) are closed subsets of \( X \).

Then the pairs \( (f, S) \) and \( (g, T) \) have a coincidence point each. Moreover, \( f, g, S \) and \( T \) have a common fixed point provided the pairs \( (f, S) \) and \( (g, T) \) are pointwise absorbing.
Proof. In view of (i), there exist two sequences \( \{x_n\} \) and \( \{y_n\} \) in \( X \) such that

\[
\lim_{n \to \infty} f x_n = \lim_{n \to \infty} S x_n = \lim_{n \to \infty} g y_n = \lim_{n \to \infty} T y_n = t
\]

for some \( t \in X \).

Since \( S(X) \) is a closed subset of \( X \), therefore \( \lim_{n \to \infty} S x_n = t \in S(X) \). Thus, there exists \( u \in X \) such that \( Su = t \). Now, we assert that \( fu = Su \). If it is not so, then \( d(fu, Su) > 0 \). On using (3.1.1.1), we have

\[
F(d(fu, gy_n), d(Su, Ty_n), d(fu, Su), d(gy_n, Ty_n), d(Su, gy_n), d(Ty_n, fu)) > 0
\]

which on making \( n \to \infty \), (besides using (1C) and (HE)) gives rise

\[
F(d(fu, t), d(Su, t), d(fu, Su), d(t, t), d(Su, t), d(t, fu)) \geq 0
\]

or

\[
F(d(fu, Su), 0, d(fu, Su), 0, 0, d(Su, fu)) \geq 0,
\]

which contradicts \((F_1)\) as long as \( d(fu, Su) > 0 \). Hence \( fu = Su \) which shows that \( u \) is a coincidence point of the pair \((f, S)\).

Since \( T(X) \) is a closed subset of \( X \), therefore \( \lim_{n \to \infty} Ty_n = t \in T(X) \) and henceforth \( Tw = t \) for some \( w \in X \). Suppose \( d(Tw, gw) > 0 \), then on using (3.1.1.1), one gets

\[
F(d(fx_n, gw), d(Sx_n, Tw), d(fx_n, Sx_n), d(gw, Tw), d(Sx_n, gw), d(Tw, fx_n)) > 0
\]

which on making \( n \to \infty \), (besides using (1C) and (HE)) gives rise

\[
F(d(t, gw), d(t, Tw), d(t, t), d(gw, Tw), d(t, gw), d(Tw, t)) \geq 0
\]

or

\[
F(d(Tw, gw), 0, 0, d(gw, Tw), d(Tw, gw), 0) \geq 0,
\]

which contradicts \((F_2)\) as long as \( d(Tw, gw) > 0 \). This shows that \( w \) is a coincidence point of the pair \((g, T)\).

As the pairs \((f, S)\) and \((g, T)\) are pointwise absorbing, one can write

\[
Su = Sfu, \quad fu = fSu, \quad Tw = Tgw, \quad gw = gTw,
\]

so that

\[
fu = Sfu, \quad fu = ffu \quad \text{and} \quad gw = Tgw, \quad gw = ggw
\]

which show that \( fu \ (fu = gw) \) is a common fixed point of \( f, g, S \) and \( T \). This concludes the proof.

**Theorem 3.3.2.** The conclusions of Theorem 3.3.1 remain true if condition (ii) (of Theorem 3.1) is replaced by the following besides retaining the rest of the hypotheses.
(ii') \( f(X) \subseteq T(X) \) and \( g(X) \subseteq S(X) \).

As a corollary of Theorem 3.3.1, we can have the following result which is also a variant of Theorem 3.3.1.

Corollary 3.3.1. The conclusions of Theorems 3.3.1 and 3.3.2 remain true if the conditions (ii) and (ii') are replaced by following.

(iii') \( f(X) \) and \( g(X) \) are closed subsets of \( X \) provided \( f(X) \subseteq T(X) \) and \( g(X) \subseteq S(X) \).

Remark 3.3.2. Theorem 3.3.1 generalizes relevant results of Imdad and Khan [76], Pathak and Tiwari [130] besides some other ones.

Theorem 3.3.3. Let \( f, g, S \) and \( T \) be four self mappings defined on nonempty set \( X \) equipped with a continuous symmetric (semi-metric) \( d \) satisfying the inequality (3.1.1.1) wherein \( F \in \Phi \) satisfies \((F_1)\) and \((F_2)\), whenever, one of \( d(fx, gy), d(fx, Sx), d(gy, Ty) \) and \( d(Sx, gy) \) is positive. Suppose that:

(i) the pair \((f, S)\) (or \((g, T)\)) enjoys the property \((E.A)\),
(ii) \( f(X) \subseteq T(X) \) (or \( g(X) \subseteq S(X) \)), and
(iii) \( S(X) \) (or \( T(X) \)) is closed subset of \( X \).

Then the pairs \((f, S)\) and \((g, T)\) have a coincidence point each. Moreover, if the pairs \((f, S)\) and \((g, T)\) are pointwise absorbing, then \( f, g, S \) and \( T \) have a common fixed point.

Proof. In view of Lemma 3.3.1, the pairs \((f, S)\) and \((g, T)\) share the common property \((E.A)\) i.e. there exist two sequences \(\{x_n\}\) and \(\{y_n\}\) in \(X\) such that

\[
\lim_{n \to \infty} fx_n = \lim_{n \to \infty} Sx_n = \lim_{n \to \infty} gy_n = \lim_{n \to \infty} Ty_n = t \in X.
\]

If \( S(X) \) is a closed subset of \( X \), then proceeding on the lines of the proof of Theorem 3.3.1, one can show that the pair \((f, S)\) has coincidence point, say \( u \), i.e. \( fu = Su \). Since \( f(X) \subseteq T(X) \) and \( fu \in f(X) \), there exists \( w \in X \) such that \( fu = Tw \). Now we assert that \( gw = Tw \). If not, then on using (3.1.1.1), we have

\[
F(d(fx_n, gw), d(Sx_n, Tw), d(fx_n, Sx_n), d(gw, Tw), d(Sx_n, gw), d(Tw, fx_n)) > 0
\]

which on making \( n \to \infty \), gives rise

\[
F(d(t, gw), d(t, Tw), d(t, t), d(gw, Tw), d(t, gw), d(Tw, t)) \geq 0
\]

or

\[
F(d(Tw, gw), 0, 0, d(gw, Tw), d(Tw, gw), 0) \geq 0,
\]

a contradiction to \((F_2)\). Hence \( gw = Tw \), which shows that \( w \) is a coincidence point of the pair \((g, T)\). Rest of the proof can be completed on the lines of Theorem 3.3.1. This concludes the proof of the theorem.
By choosing \( f, g, S \) and \( T \) suitably, one can deduce corollaries for a pair or triode of mappings. The details of two possible corollaries for a triode of mappings are not included. However, as a sample, we outline the following natural result for a pair of self mappings.

**Corollary 3.3.2.** Let \( f \) and \( S \) be two self mappings defined on a nonempty set \( X \) equipped with a continuous symmetric (semi-metric) \( d \) which satisfy the following conditions:

(i) the pair \((f, S)\) enjoys the property \((E.A)\),

(ii) for all \( x, y \in X (x \neq y) \) and \( F \in \Phi \) (wherein \((F)\) satisfies \((F_1)\) and \((F_2)\)),

\[
F(d(fx, fy), d(Sx, Sy), d(fx, Sx), d(fy, Sy), d(fx, Sx)) > 0,
\]

whenever, one of \( d(fx, gy), d(fx, Sx), d(gy, Ty) \) and \( d(Sx, gy) \) is positive and

(iii) \( S(X) \) is a closed subset of \( X \),

then the pair \((f, S)\) has a coincidence point. Moreover, \( f \) and \( S \) have a common fixed point provided the pair \((f, S)\) is pointwise absorbing.

**Corollary 3.3.3.** The conclusions of Theorem 3.3.1 remain true if inequality (3.1.1.1) is replaced by one of the following expansion type conditions. For all \( x, y \in X (x \neq y) \),

(i) \( d(fx, gy) > k \max\{d(Sx, Ty), d(fx, Sx), d(gy, Ty), d(fx, Ty), d(gy, Sx)\} \),

where \( k > 1 \).

(ii) \( k \min\{d(Sx, Ty), d(fx, Sx), d(gy, Ty), d(fx, Ty), d(gy, Sx)\} > d(fx, gy) \),

where \( 0 < k < 1 \).

(iii) \( \phi \max\{d(Sx, Ty), d(fx, Sx), d(gy, Ty), d(fx, Ty), d(gy, Sx)\} > d(fx, gy) \),

where \( \phi : \mathbb{R}^+ \rightarrow \mathbb{R} \) is an lower semi-continuous function such that \( \phi(0) = 0 \) and \( \phi(t) > t \) for all \( t > 0 \).

(iv) \( d(fx, gy) > a d(Sx, Ty) + b d(fx, Sx) + c d(gy, Ty) + ed(fx, Ty) + f d(gy, Sx) \),

where \( a, b, c, e, f > 0 \) with \( a + e > 1, \ f + b > 1 \).

(v) \( d(fx, gy) > k \max\{d(Sx, Ty)^2, d(fx, Sx)d(gy, Ty), d(fx, Ty)d(gy, Sx), d(fx, Sx)d(fx, Ty), d(gy, Ty)d(gy, Sx)\}^{\frac{1}{2}} \),

where \( k > 1 \).

(vi) \( d(fx, gy) > \begin{cases} \alpha d(Sx, Ty) + \beta \frac{d(fx, Ty)^2 + d(gy, Sx)^2}{d(fx, Ty) + d(gy, Sx)} + \gamma(d(fx, Sx) + d(gy, Ty)), & \text{if } d(fx, Ty) + d(gy, Sx) \neq 0 \\ 0, & \text{if } d(fx, Ty) + d(gy, Sx) = 0 \end{cases} \)

where \( \alpha, \gamma > 0 \) and \( \beta > 1 \).
(vii) \(d(fx, gy) > a \cdot d(Sx, Ty) + b \cdot d(fx, Sx) + c \cdot d(gy, Ty) + \max\{d(fx, Ty), d(gy, Sx)\}\), where \(a, b, c > 0\).

(viii) \(k \cdot \min\{d(Sx, Ty), \max\{d(fx, Sx), d(gy, Ty), \max\{d(fx, Ty), d(gy, Sx)\}\}\} > d(fx, gy),\) where \(0 < k < 1\).

(ix) \(d(fx, gy) > k \cdot \min\{d(fx, gy + d(Sx, Ty), d(fx, Sx) + d(fx, Ty), d(gy, Ty) + d(gy, Sx)\}\), where \(k > 1\).

(x) \(ad(Sx, Ty) + b(d(fx, Sx) + d(gy, Ty)) + c(d(Sx, gy) + d(Ty, fx)) > d(fx, gy),\) where \(a, b, c > 0\) and \(b + c < 1\).

(xi) \(\min\{(d(Sx, Ty) + d(fx, Sx)) / 2, k(d(gy, Ty) + d(Sx, gy)) / 2, d(Ty, fx)\} > d(fx, gy),\) where \(0 \leq k < 1\).

(xii) \(\min\{(a(Sx, Ty), b(d(fx, Sx) + d(Sx, gy)) / 2, (d(gy, Ty) + d(Ty, fx))\} > d(fx, gy),\) where \(0 \leq a < 1, 1 \leq b < 2\).

(xiii) \(ad(Sx, Ty)^2 + d(fx, Sx)d(gy, Ty) + bd(Sx, gy)^2 + cd(Ty, fx)^2 > d(fx, gy)^2,\) where \(a \geq 0\) and \(0 \leq b + c < 1\).

(xiv) \(k(d(Sx, Ty)^3 + d(fx, Sx)^3 + d(gy, Ty)^3 + d(Sx, gy)^3 + d(Ty, fx)^3) > d(fx, gy)^3,\) where \(0 \leq k < 1/3\).

(xv) \(d(fx, gy)^3 > ad(fx, gy)^2d(Sx, Ty) + bd(fx, gy)d(gy, Ty)d(Sx, gy) + cd(fx, gy)d(fx, Sx)d(Ty, fx) + dd(fx, Sx)d(Sx, gy)d(Ty, fx)\)

where \(b, c > 1\).

(xvi) \(d(fx, gy)^2 > ad(fx, gy)d(Sx, Ty) + bd(fx, Sx)^2 + cd(gy, Ty)^2 + dd(Sx, gy)d(Ty, fx)\)

where \(b, c > 1\).

(xvii)

\[
\frac{ad(fx, Sx)d(gy, Ty) + bd(fx, Ty)d(gy, Sx) + cd(Sx, Ty)^2}{d(fx, Ty) + d(gy, Sx) + d(Sx, Ty)} > d(fx, gy)
\]

if \(d(fx, Ty) + d(gy, Sx) + d(Sx, Ty) \neq 0\)

where \(b + c > 3\).

Proof. Proof follows from Theorem 3.3.1 and Examples 3.2.1-3.2.17.

Remark 3.3.3. Corollaries corresponding to conditions (i) to (xvii) are new results as these results never require any conditions on containment of ranges amongst involved mappings. Some expansive conditions listed in above corollary are well known and generalize certain relevant results of the existing literature (e.g. [76, 82, 94, 100, 130, 134]).
3.4 Results with unique common fixed point

If we add the condition \((F_3)\): \(F(t,t,0,0,t,t) \leq 0\), for all \(t > 0\), to our implicit functions, then implicit functions satisfying \((F_1)\), \((F_2)\) and \((F_3)\) ensure the uniqueness of common fixed point. Here, it can be pointed out that all preceding examples need not satisfy \(F_3\) (e.g. Example 2.17). However, we prove the following unique common fixed point theorem in symmetric spaces.

**Theorem 3.4.1.** Let \(f, g, S \) and \(T \) be self mappings defined on a symmetric (resp. semi-metric) space \((X,d)\) equipped with a symmetric \(d\) which enjoys \((1C)\) and \((HE)\) besides satisfying inequality (3.1.1.1) wherein every \(F \in \Phi\) satisfies \((F_1)\), \((F_2)\) and \((F_3)\) whenever, one of \(d(fx,gy), d(fx, Sx), d(gy, Ty)\) and \(d(Sx, gy)\) is positive. Suppose that:

(i) the pairs \((f,S)\) and \((g,T)\) share the common property \((E.A)\) and

(ii) \(S(X)\) and \(T(X)\) are closed subsets of \(X\).

Then the pairs \((f,S)\) and \((g,T)\) have a coincidence point each. Moreover, if the pairs \((f,S)\) and \((g,T)\) are pointwise absorbing, then \(f, g, S\) and \(T\) have a unique common fixed point.

**Proof.** In view of Theorem 3.3.1, \(f, g, S\) and \(T\) have a common fixed point. The uniqueness of the common fixed point is an easy consequence of the condition \((F_3)\).

**Remark 3.4.1.** In the additional presence of \((F_3)\), Theorem 3.3.2-3.3.3 and Corollary 3.3.1-3.3.3 ensure the uniqueness of common fixed point. But we avoid the details due to repetition.

As an application of Theorem 3.4.1, we have the following result for four finite families of self mappings.

**Theorem 3.4.2.** Let \(\{f_1, f_2, \ldots, f_m\}, \{g_1, g_2, \ldots, g_p\}, \{S_1, S_2, \ldots, S_n\}\) and \(\{T_1, T_2, \ldots, T_q\}\) be four finite families of self mappings defined of a symmetric (resp. semi-metric) space \((X,d)\) equipped with a symmetric \(d\) which enjoys \((1C)\) and \((HE)\) with \(f = f_1 f_2 \ldots f_m\), \(g = g_1 g_2 \ldots g_p\), \(S = S_1 S_2 \ldots S_n\) and \(T = T_1 T_2 \ldots T_q\) which satisfy condition (3.1.1.1) wherein every \(F \in \Phi\) satisfies \((F_1)\), \((F_2)\) and \((F_3)\). If \(I^n(X)\) and \(J^n(X)\) are closed subsets of \(X\) and the pairs \((f,S)\) and \((g,T)\) share the common property \((E.A)\), then

(i) the pair \((f,S)\) has a coincidence point,

(ii) the pair \((g,T)\) has a coincidence point.

Moreover \(f_1, g_k, S_r\) and \(T_t\) have a unique common fixed point provided the pair of families \(\{f_i\}\{S_r\}\) and \(\{g_k\}\{T_t\}\) commute pairwise, where \(i \in \{1,2,\ldots,m\}\), \(k \in \{1,2,\ldots,p\}\), \(r \in \{1,2,\ldots,n\}\) and \(t \in \{1,2,\ldots,q\}\).

**Proof.** Proof follows on the lines of corresponding result due to Imdad et al. [80, Theorem 2.2].
By setting \( f_1 = f_2 = \ldots = f_m = G, \ g_1 = g_2 = \ldots = g_p = H, \ S_1 = S_2 = \ldots = S_n = I \) and \( T_1 = T_2 = \ldots = T_q = J \) in Theorem 3.4.2, we deduce the following theorem involving iterates of mappings:

**Corollary 3.4.1.** Let \( G, H, I \) and \( J \) be self mappings defined on a symmetric (resp. semi-metric) space \((X, d)\) equipped with a symmetric \(d\) which enjoys \((1C)\) and \((HE)\) such that the pairs \((G^m, I^n)\) and \((H^p, J^q)\) share the common property \((E.A)\) and also satisfy the condition

\[
F(d(G^m x, H^p y), d(I^n x, J^q y), d(G^m x, I^n x), d(H^p y, J^q y), \]
\[
d(I^n x, H^p y), d(J^q y, G^m x)) > 0
\]

for all \(x, y \in X\) wherein every \(F \in \Phi\) satisfies \((F_1)\), \((F_2)\) and \((F_3)\) and \(m, n, p\) and \(q\) are fixed positive integers. If \(I^n(X)\) and \(J^q(X)\) are closed subsets of \(X\), then \(G, H, I\) and \(J\) have a unique common fixed point provided \(GI = IG\) and \(HJ = JH\).

**Remark 3.4.2.** By restricting four families as \(\{f_1, f_2\}, \{g_1, g_2\}, \{S_1\}\) and \(\{T_1\}\) in Theorem 3.4.1, we deduce a substantial but partial generalization of the main results of Imdad and Khan [73, 75] for expansive mappings as such result will yield stronger commutativity requirement besides relaxing continuity requirements and weakening completeness requirement of the space to the closedness of subspaces.

**Remark 3.4.3.** Corollary 3.4.1 is a slight but partial generalization of Theorem 3.4.1 as the commutativity requirements (i.e. \(GI = IG\) and \(HJ = JH\)) are relatively more stringent.

### 3.5 An illustrative example

In what follows, we furnish an example demonstrating the utility of Theorem 3.4.1 over the earlier results especially those contained in [35, 47, 76, 130, 160] besides some other one.

**Example 3.5.1.** Consider \(X = [2, 20]\) with symmetric \(d(x, y) = (x - y)^2\). Define self mappings \(f, g, S\) and \(T\) on \(X\) by

\[
f(x) = \begin{cases} 
2, & \text{if } x = 2 \text{ and } x > 5 \\
1, & \text{if } 2 < x \leq 5 
\end{cases} \quad g(x) = \begin{cases} 
2, & \text{if } x = 2 \text{ and } x > 5 \\
\frac{3}{2}, & \text{if } 2 < x \leq 5 
\end{cases} \\
S(x) = \begin{cases} 
2, & \text{if } x = 2 \\
5, & \text{if } 2 < x \leq 5 \\
\frac{x-1}{2}, & \text{if } x > 5 
\end{cases} \quad T(x) = \begin{cases} 
2, & \text{if } x = 2 \\
7, & \text{if } 2 < x \leq 5 \\
\frac{x+1}{3}, & \text{if } x > 5 
\end{cases}
\]
Consider sequences \( \{x_n = 5 + \frac{1}{n}\} \) and \( \{y_n = 5 + \frac{2}{n}\} \) in \( X \). Clearly,

\[
\lim_{n \to \infty} f x_n = \lim_{n \to \infty} S x_n = \lim_{n \to \infty} g y_n = \lim_{n \to \infty} T y_n = 2
\]

which shows that pairs \((f, S)\) and \((g, T)\) satisfy the common property \((E.A)\), \(f(X) = \{1, 2\} \not\subset [2, \frac{19}{2}] = S(X)\) and \(g(X) = \{\frac{3}{2}, 2\} \not\subset [2, 7] = T(X)\) moreover \(S(X) = [2, \frac{19}{2}]\) and \(T(X) = [2, 7]\) are closed subset of \( X \). Also define a continuous implicit function \( F : \mathbb{R}^6 \to \mathbb{R} \) such that \( F(t_1, t_2, \cdots, t_6) = k \min \{t_2, \max \{t_3, t_4\}, \max \{t_5, t_6\}\} - t_1 \), where \(0 < k < 1\) and \(F \in \Phi\).

By a routine calculation, one can verify the inequality (3.1.1.1): towards the verification of implicit function, let \(2 < x, y \leq 5\).

\[
k \min \left\{d(Sx, Ty), \max\{d(fx, Sx), d(gy, Ty)\}, \max\{d(fx, Ty), d(gy, Sx)\}\right\} > d(fx, gy)
\]

or

\[
k \min \left\{d(5, 7), \max\{d(1, 5), d(\frac{3}{2}, 7)\}, \max\{d(1, 7), d(\frac{3}{2}, 5)\}\right\} > d(1, \frac{3}{2})
\]

or

\[
k \min \left\{(5 - 7)^2, \max\{(1 - 5)^2, (\frac{3}{2} - 7)^2\}, \max\{(1 - 7)^2, (\frac{3}{2} - 5)^2\}\right\} > (1 - \frac{3}{2})^2
\]

or

\[
k \min \left\{4, \max\{16, 30.25\}, \max\{36, 12.25\}\right\} > \frac{1}{4}
\]

or

\[
k \min\{4, 30.25, 36\} > \frac{1}{4},
\]

hence inequality (3.1.1.1) is true for \(0 < k < 1\). Therefore, all the conditions of Theorem 3.4.1 are satisfied and 2 is a unique common fixed point of the pairs \((f, S)\) and \((g, T)\) which is their coincidence point as well.

Here it is worth noting that none of the theorems (with rare possible exceptions) can be used in the context of this example as Theorem 3.4.1 never requires any condition on the containment of ranges of mappings while completeness condition is replaced by closedness of subspaces. Moreover, the continuity requirements of involved mappings are completely relaxed.