Chapter 1

Introduction

1.1 Special Functions

This thesis investigates the mathematical functions which arise in Analysis and Applied mathematical problems - the so called Special functions.

The algebraic aspect of the theory of special functions have not significantly changed since the nineteenth century.

Paul Turan remarked that special functions would be more appropriately label ‘useful functions’.

Because of their remarkable properties, special functions have been used for several centuries, since they have numerous applications in astronomy, trigonometric functions which have been studied for over a thousand years. Even the series expansions for sine and cosine, as well as the arc tangent were known for long time ago from the fourteen century. Since then the subject of special functions has been continuously developed with contribution of several mathematicians including Euler, Legendre, Laplace, Gauss, Kummer, Riemann and Ramanujan. In the past several years the discoveries of new special functions and applications of this kind of functions to new areas of mathematics have initiated a great interest of this field. These discoveries include work in combinatorics, initiated by Schutzeberg and Foata. Moreover, in recent years, particular cases of long familiar special functions have been clearly defined and applied as orthogonal polynomials.
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The special functions have been studied in several volumes by the collective of mathematicians formed by Andrew, G.E., Askey, R. and Roy, R. (Cambridge University Press, Encyclopedia of Mathematics and its Applications).

There are important results from the past that must be included in this field because they are so useful. Then, there are recent developments that should be brought to the attention of those who could use them: we would wish to help educate the new generation of mathematicians and scientists so they can further develop and apply this subject. Specialized texts dealing with some of these developments have recently appeared: Petkovitek, Wilf and Zeilberger (1996), Macdonald (1995), Heckman and Schlicktrull (1994) and Vilenkin and Kliniyk (1992).

Several important facts about hypergeometric series were first found by Euler, Pfaff and Gauss. This last mathematician fully recognized their significance and gave a systematic account of these. A half century after Gauss, Riemann developed hypergeometric functions from a different point of view, which made available the basic formulas with a minimum of computations.

Another approach to hypergeometric functions using contour integrals was presented by the English mathematician Barnes, E.W. in the first decade of the last century.

Hypergeometric functions have two very significant properties that add to their usefulness. They satisfy certain identities for special values of the function and they have transformation formulas.

The gamma functions and beta integrals dealt with an essential understanding of hypergeometric functions.

The gamma function was introduced into mathematics by Euler in 1720 when he solved the problem of extending the factorial function to all real or complex numbers.

There are extensions of gamma and beta functions that are also very important.

The theory of special functions with its numerous beautiful formulas is very well
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suited to an algorithmic approach to mathematics. In the nineteenth century it was
the ideal of Eisenstein and Kronecker to express and develop mathematical results
by means of formulas. Before them, this attitude was common and best exemplified
in the works of Euler, Jacobi and Gauss.

In the twentieth century, mathematics moved from this approach toward a more
abstract and existential methods. In fact agreeing with Hardy that Ramanujan came
100 years too late, Littlewood wrote that the great day of formulae seem to be over
(mentioned by Littlewood, 1986).

However, with the advent of computers and the consequent three of computational
mathematics formulas are now once again playing a larger role in mathematics. We
mention that beautiful, interesting and important formulas have been discovered since
Ramanujans time. These formulas are proving fertile and fruitful.

There are hundreds of special functions used in applied mathematics and comput-
ing sciences. A special function is a real or complex valued function of one or more
real or complex variables which is specified so completely that its numerical values
could in principle be tabulated. Besides elementary functions such as $x^n$, $e^x$, $\log x$,
and $\sin x$, ‘higher’ functions, both transcendental (such as Bessel functions) and al-
gebraic (such as various polynomials) come under the category of special functions.
Infact special functions are solutions of a wide class of mathematically and physically
relevant functional equations. As far as the origin of special functions is concerned
the special function of mathematical physics arises in the solution of partial differ-
tential equations governing the behavior of certain physical quantities. Probably the
most frequently occurring equation of this type in all physics is Laplace’s equation

$$\nabla^2 \psi = 0 \quad (1.1.1)$$

satisfied by a certain function $\psi$ describing the physical situation under discussion.
The mathematical problem consists of finding those functions which satisfy equation
(1.1.1) and also satisfy certain prescribed conditions on the surfaces bounding the
region being considered. For example, if $\psi$ denotes the electrostatic potential of a
system, $\psi$ will be constant over any conducting surface. The shape of these boundaries often makes it desirable to work in curvilinear coordinates $q_1$, $q_2$, $q_3$ instead of in rectangular Cartesian coordinates $x$, $y$, $z$. In this case we have relations

$$x = x(q_1, q_2, q_3), \quad y = y(q_1, q_2, q_3), \quad z = z(q_1, q_2, q_3). \quad (1.1.2)$$

expressing the Cartesian coordinates in terms of the curvilinear coordinates. If equations (1.1.2) are such that

$$\frac{\partial x}{\partial q_1} \frac{\partial x}{\partial q_1} + \frac{\partial y}{\partial q_1} \frac{\partial y}{\partial q_1} + \frac{\partial z}{\partial q_1} \frac{\partial z}{\partial q_1} = 0.$$

when $i \neq j$ we say that the coordinates $q_1$, $q_2$, $q_3$ are orthogonal curvilinear coordinates. The element of length $dl$ is then given by

$$dl^2 = h_1^2 dq_1^2 + h_2^2 dq_2^2 + h_3^2 dq_3^2. \quad (1.1.3)$$

where

$$h_i^2 = \left( \frac{\partial x}{\partial q_1} \right)^2 + \left( \frac{\partial y}{\partial q_1} \right)^2 + \left( \frac{\partial z}{\partial q_1} \right)^2. \quad (1.1.4)$$

and it can easily be shown that

$$\nabla^2 \psi = \frac{1}{h_1 h_2 h_3} \left\{ \frac{\partial}{\partial q_1} \left( \frac{h_2 h_3}{h_1} \frac{\partial \psi}{\partial q_1} \right) + \frac{\partial}{\partial q_2} \left( \frac{h_3 h_1}{h_2} \frac{\partial \psi}{\partial q_2} \right) + \frac{\partial}{\partial q_3} \left( \frac{h_1 h_2}{h_3} \frac{\partial \psi}{\partial q_3} \right) \right\}. \quad (1.1.5)$$

One method of solving Laplace’s equation consists of finding solutions of the type

$$\psi = Q_1(q_1)Q_2(q_2)Q_3(q_3),$$

by substituting from (1.1.5) into (1.1.1). We then find that

$$\frac{1}{Q_1} \left\{ \frac{\partial}{\partial q_1} \left( \frac{h_2 h_3}{h_1} \frac{\partial Q_1}{\partial q_1} \right) + \frac{1}{Q_2} \frac{\partial}{\partial q_2} \left( \frac{h_3 h_1}{h_2} \frac{\partial Q_2}{\partial q_2} \right) + \frac{1}{Q_3} \frac{\partial}{\partial q_3} \left( \frac{h_1 h_2}{h_3} \frac{\partial Q_3}{\partial q_3} \right) \right\} = 0.$$

If further, it so happens that

$$\frac{h_2 h_3}{h_1} = f_1(q_1)F_1(q_2, q_3),$$
etc., then this last equation reduces to the form

$$F_1(q_2, q_3) \frac{1}{Q_1} \frac{d}{dq_1} \left\{ f_1(q_1) \frac{dQ_1}{dq_1} \right\} + F_2(q_1, q_3) \frac{1}{Q_2} \frac{d}{dq_2} \left\{ f_2(q_2) \frac{dQ_2}{dq_2} \right\} + F_3(q_1, q_2) \frac{1}{Q_3} \frac{d}{dq_3} \left\{ f_3(q_3) \frac{dQ_3}{dq_3} \right\} = 0.$$ 

Now, in certain circumstances, it is possible to find three functions $g_1(q_1)$, $g_2(q_2)$, $g_3(q_3)$ with the property that

$$F_1(q_2, q_3)g_1(q_1) + F_2(q_3, q_1)g_2(q_2) + F_3(q_1, q_2)g_3(q_3) = 0.$$ 

When this is so, it follows immediately that the solution of Laplace’s equation (1.1.1) reduces to the solution of three self-adjoint ordinary linear differential equations

$$\frac{d}{dq_i} \left\{ f_i \frac{dQ_i}{dq_i} \right\} g_i Q_i = 0, \quad (i = 1, 2, 3). \quad (1.1.6)$$

It is the study of differential equations of this kind which leads to the special functions of mathematical physics. The adjective “special” is used in this connection because here we are not, as in analysis, concerned with the general properties of functions, but only with the properties of functions which arise in the solution of special problems.

To take a particular case, consider the cylindrical polar coordinates $(Q, \varphi, z)$ defined by the equations

$$x = Q \cos \varphi, \quad y = Q \sin \varphi, \quad z = z$$

for which $h_1 = 1$, $h_2 = Q$, $h_3 = 1$.

From equation (1.1.5) we see that, for these coordinates, Laplace’s equation is of the form

$$\frac{\partial^2 \psi}{\partial Q^2} + \frac{1}{Q} \frac{\partial \psi}{\partial Q} + \frac{1}{Q^2} \frac{\partial^2 \psi}{\partial \varphi^2} + \frac{\partial^2 \psi}{\partial z^2} = 0. \quad (1.1.7)$$

If we now make the substitution

$$\psi = R(Q)\Phi(\varphi)Z(z), \quad (1.1.8)$$
we find that equation (1.1.7) may be written in the form

\[
\frac{1}{R} \left( \frac{\partial^2 R}{\partial Q^2} + \frac{1}{Q} \frac{\partial R}{\partial Q} \right) + \frac{1}{Q^2 \Phi} \frac{\partial^2 \Phi}{\partial \varphi^2} + \frac{1}{Z} \frac{\partial^2 Z}{\partial z^2} = 0.
\]  

(1.1.9)

This shows that if \( \Phi, \ Z, \ R \) satisfy the equations

\[
\frac{\partial^2 \Phi}{\partial \varphi^2} + n^2 \Phi = 0, \quad (1.1.9a)
\]

\[
\frac{\partial^2 Z}{\partial z^2} + m^2 Z = 0, \quad (1.1.9b)
\]

\[
\frac{\partial^2 R}{\partial Q^2} + \frac{1}{Q} \frac{\partial R}{\partial Q} + \left( m^2 - \frac{n^2}{Q^2} \right) R = 0, \quad (1.1.9c)
\]

respectively, then the function (1.1.8) is a solution of Laplace’s equation (1.1.7). The study of these ordinary differential equations will lead us to the special functions appropriate to this coordinate system. For instance, equation (1.1.9a) may be taken as the equation defining the circular functions. In this context \( \sin(n\varphi) \) is defined as that solution of (1.1.9a) which has value 0 when \( \varphi = 0 \) and \( \cos(n\varphi) \) as that which has value 1 when \( \varphi = 0 \). Similarly equation (1.1.9b) defines the exponential functions. In actual practice we do not proceed in this way merely because we have already encountered these functions in another context and from their familiar properties studied their relation to equations (1.1.9a) and (1.1.9b). The situation with respect to equation (1.1.9c) is different; we cannot express its solution in terms of the elementary functions of analysis, as we were able to do with the other two equations. In this case we define new functions in terms of the solutions of this equation and by investigating the series solutions of the equations derive the properties of the functions so defined. Equation (1.1.9c) is called Bessel’s equation and solutions of it are called Bessel functions. Bessel functions are of great importance in theoretical physics. The study of special functions grew up with the calculus and is consequently one of the oldest branches of analysis. It flourished in the nineteenth century as part of the theory of complex variables. In the second half of the twentieth century it has received a new impetus from a connection with Lie groups and a connection with averages of elementary functions. The history of special functions is closely tied to the problem of terrestrial and celestial mechanics that were solved in the eighteenth and nineteenth centuries,
the boundary-value problems of electromagnetism and heat in the nineteenth, and
the eigenvalue problems of quantum mechanics in the twentieth.

Seventeenth-century England was the birthplace of special functions. John Wallis
at Oxford took two first steps towards the theory of the gamma function long before
Euler reached it. Wallis had also the first encounter with elliptic integrals while
using Cavalieri's primitive forerunner of the calculus. [It is curious that two kinds
of special functions encountered in the seventeenth century, Wallis' elliptic integral
and Newton's elementary symmetric functions, belongs to the class of hypergeometric
functions of several variables, which was not studied systematically nor even defined
formally until the end of the nineteenth century]. A more sophisticated calculus,
which made possible the real flowering of special functions, was developed by Newton
at Cambridge and by Leibnitz in Germany during the period 1665-1685. Taylor's
theorem was found by Scottish mathematician Gregory in 1670, although it was not
published until 1715 after rediscovery by Taylor.

In 1703 James Bernoulli solved a differential equation by an infinite series which
would now be called the series representation of a Bessel function. Although Bessel
functions were met by Euler and others in various mechanics problems, no systematic
study of the functions was made until 1824, and the principal achievements in the
eighteenth century were the gamma function and the theory of elliptic integrals. Euler
found most of the major properties of the gamma functions around 1730. In 1772
Euler evaluated the Beta-function integral in terms of the gamma function. Only
the duplication and multiplication theorems remained to be discovered by Legendre
and Gauss, respectively, early in the next century. Other significant developments
were the discovery of Vandermonde's theorem in 1722 and the definition of Legendre
polynomials and the discovery of their addition theorem by Laplace and Legendre
during 1782-1785. In a slightly different form the polynomials had already been met
by Liouville in 1722.

The golden age of special functions, which was centered in nineteenth century
German and France, was the result of developments in both mathematics and physics:
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the theory of analytic functions of a complex variable on one hand, and on the other hand, the field theories of physics (e.g. heat and electromagnetism) and their property of double periodicity was published by Abel in 1827. Elliptic functions grew up in symbiosis with the general theory of analytic functions and flourished throughout the nineteenth century, specially in the hands of Jacobi and Weierstrass.

Another major development was the theory of hypergeometric series which began in a systematic way (although some important results had been found by Euler and Pfaff) with Gauss’s memoir on the $_2F_1$ series in 1812, a memoir which was a landmark also on the path towards rigour in mathematics. The $_3F_2$ series was studied by Clausen (1928) and the $_1F_1$ series by Kummer (1836). The functions which Bessel considered in his memoir of 1824 are $_0F_1$ series; Bessel started from a problem in orbital mechanics, but the functions have found a place in every branch of mathematical physics, near the end of the century Appell (1880) introduced hypergeometric functions of two variables, and Lauricella generalized them to several variables in 1893.

The subject was considered to be part of pure mathematics in 1900, applied mathematics in 1950. In physical science special functions gained added importance as solutions of the Schrodinger equation of quantum mechanics, but there were important developments of a purely mathematical nature also. In 1907 Barnes used gamma function to develop a new theory of Gauss’s hypergeometric function $_2F_1$. Various generalizations of $_2F_1$ were introduced by Horn, Kampé de Fériet, MacRobert, and Mijer. From another new view point, that of a differential difference equation discussed much earlier for polynomials by Appell (1880), Truesdell (1948) made a partly successful effort at unification by fitting a number of special functions into a single framework.

1.2 Definitions, Notations and Results Used

Frequently occurring definitions, notations and miscellaneous results used in this thesis are as given below:
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The Gamma, Beta and Related Functions

The gamma function was introduced into mathematics by Euler in 1720 when he solved the problem of expanding the factorial function to all real or complex numbers. This problem was apparently suggested by Daniel Bernoulli and Goldbach. The Gamma function has several equivalent definitions, most of which are due to Euler,\n
\[ \Gamma(z) = \begin{cases} \int_0^\infty t^{z-1}e^{-t} \, dt, & \text{Re}(z) > 0 \\ \frac{\Gamma(z+1)}{z}, & \text{Re}(z) < 0, \ z \neq 0, -1, -2, \ldots \end{cases} \]  

(1.2.1)

The relation (1.2.1), yields the useful result

\[ \Gamma(n+1) = n!, \ n = 0, 1, 2, \ldots \]

which shows that gamma function is the generalization of factorial function.

The Beta Function

The Beta function \( B(p, q) \) is function of two complex variables \( p \) and \( q \), defined by

\[ B(p, q) = \begin{cases} \int_0^1 x^{p-1}(1-x)^{q-1} \, dx, & \text{Re}(p) > 0, \ \text{Re}(q) > 0, \\ \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}, & \text{Re}(p) < 0, \ \text{Re}(q) < 0, \ p, q \neq -1, -2, \ldots \end{cases} \]  

(1.2.2)

The Pochhammer Symbol

The Pochhammer symbol \((\lambda)_n\) is defined by

\[ (\lambda)_n = \begin{cases} 1, & \text{if } n = 0 \\ \lambda(\lambda+1)\cdots(\lambda+n-1), & \text{if } n = 1, 2, 3, \ldots \end{cases} \]  

(1.2.3)

In terms of Gamma function, we have

\[ (\lambda)_n = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)}, \ \lambda \neq 0, -1, -2, \ldots \]  

(1.2.4)

Further,

\[ (\lambda)_{m+n} = (\lambda)_m(\lambda+m)_n \]  

(1.2.5)
(\lambda)_n = \frac{(-1)^n}{(1-\lambda)_n}, \quad n = 1, 2, 3, \ldots, \quad \lambda \neq 0, \pm 1, \pm 2, \quad (1.2.6)

(\lambda)_{n-m} = \frac{(-1)^m(\lambda)_n}{(1-\lambda-n)_m}, \quad 0 \leq m \leq n. \quad (1.2.7)

For $\lambda = 1$, equation (1.2.7) reduces to

\[(n-m)! = \frac{(-1)^m n!}{(-n)_m}, \quad 0 \leq m \leq n.\]  \quad (1.2.8)

Another useful relation of Pochhammer's symbol $(\lambda)_n$ is included in Gauss's multiplication theorem:

\[(\lambda)_{mn} = (m)^m \prod_{j=1}^{m} \left(\frac{\lambda + j - 1}{m}\right)_n, \quad n = 0, 1, 2, \cdots \]  \quad (1.2.9)

where $m$ is positive integer.

For $m = 2$ the equation (1.2.9) reduces to Legendre's duplication formula

\[(\lambda)_{2n} = 2^{2n} \left(\frac{\lambda}{2}\right)_n \left(\frac{\lambda + 1}{2}\right)_n, \quad n = 0, 1, 2, \cdots \]  \quad (1.2.10)

In particular, one has

\[(2n)! = 2^{2n} \left(\frac{1}{2}\right)_n \quad n! \quad \text{and} \quad (2n+1)! = 2^{2n} \left(\frac{3}{2}\right)_n \quad n! \]  \quad (1.2.11)

Also, the binomial coefficient \(\binom{\lambda}{n}\) is defined by (see [5])

\[\binom{\lambda}{n} = \frac{(-1)^n(-\lambda)_n}{n!} \quad \ ]  \quad (1.2.12)

where

\[-\lambda)_n = (-\lambda)(-\lambda + 1)(-\lambda + 2)\cdots(-\lambda + n - 1),

\[-\lambda)_0 = 1. \quad \]  \quad (1.2.13)
Gaussian Hypergeometric Series

The hypergeometric series is given by

\[ \, _2F_1(a, b; c; z) = 1 + \frac{a \cdot b \cdot z}{1 \cdot c} + \frac{a(a+1) b(b+1)}{1 \cdot 2 \cdot c(c+1)} z^2 + ... \]

\[ = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}, \quad c \neq 0, -1, -2, ... \] \hspace{1cm} (1.2.14)

was introduced by a German mathematician Gauss, C.F. (1777-1855). Who in the year (1812) introduced this series into analysis and give the F-notation for it.

The special case \( a = c, b = 1 \) or \( b = c, a = 1 \) yields the elementary geometric series.

\[ \sum_{n=0}^{\infty} z^n = 1 + z + z^2 + ... + z^n + ... \] \hspace{1cm} (1.2.15)

Hence\((1.2.14)\) is called the hypergeometric series or more precisely, Gauss hypergeometric series.

In \((1.2.14)\), \((a)_n\) denotes the pochhammer’s symbol defined by \((1.2.3)\), \(z\) is real or complex variable, \(a, b\) and \(c\) are parameters which can take arbitrary real or complex values and \(c \neq 0, -1, -2, ... \)

If \(c\) is zero or negative integer, the series \((1.2.14)\) does not exist and hence the function \(_2F_1(a, b; c; z)\) is not defined unless one of the parameters \(a\) or \(b\) is negative integer such that \(-c < -a\) is also negative integer. If either of the parameters \(a\) or \(b\) is negative integer, then in this case, equation \((1.2.14)\) reduces to hypergeometric polynomials.

The hypergeometric series \((1.2.14)\), converges absolutely within the unit circle \(|z| < 1\), provided that \(Re(c - a - b) > 0\) for \(z = 1\) and \(Re(c - a - b) > -1\) for \(z = -1\).

**Generalized Hypergeometric Function**

The hypergeometric function \(_2F_1\) defined in equation \((1.2.14)\) can be generalized with notation \(_pF_q\) in an obvious way:

\[ \, _pF_q \left[ \begin{array}{c} \alpha_1, \alpha_2, \cdots, \alpha_p; \\ \beta_1, \beta_2, \cdots, \beta_q; \\ z \end{array} \right] = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_p)_n}{(\beta_1)_n \cdots (\beta_q)_n} \frac{z^n}{n!} \]
\[ \, _p F_q (\alpha_1, \cdots, \alpha_p; \beta_1, \cdots, \beta_q; z), \quad (1.2.16) \]

where \( p \) \& \( q \) are positive integers or zero. The numerator parameter \( \alpha_1, \cdots, \alpha_p \) and the denominator parameter \( \beta_1, \cdots, \beta_q \) take on complex values, provided that

\[ \beta_j \neq 0, -1, -2, \cdots, j = 1, 2, \cdots, q. \]

**Convergence of \( _p F_q \)**

The series \( _p F_q \) defined by (1.2.16)

(i) converges for all \( |z| < \infty \) if \( p \leq q \)

(ii) converges for \( |z| < 1 \) if \( p = q + 1 \) and

(iii) diverges for all \( z, z \neq 0 \) if \( p > q + 1 \).

Furthermore, if we set

\[ \omega = \text{Re} \left( \sum_{j=1}^{q} \beta_j - \sum_{j=1}^{p} \alpha_j \right) > 0, \]

the \( _p F_q \) series with \( p = q + 1 \) is

(i) Absolutely convergent for \( |z| = 1 \) if \( \text{Re}(\omega) > 0 \)

(ii) Conditionally convergent for \( |z| = 1, z \neq 1 \) if \( -1 < \text{Re}(\omega) < 0 \)

(iii) Divergent for \( |z| = 1 \) if \( \text{Re}(\omega) \leq -1 \).

An important special case when \( p = q = 1 \), the equation (1.2.16) reduces to the confluent hypergeometric series \( _1 F_1 \) named as Kummer’s series \([31]\), (see also Slater \([43]\)) and is given by

\[ _1 F_1 (\alpha; c; z) = \sum_{n=0}^{\infty} \frac{(\alpha)_n}{(c)_n} \frac{z^n}{n!}, \quad (1.2.17) \]

when \( p = 2 \) \& \( q = 1 \), equation (1.2.16) reduces to ordinary hypergeometric function \( _2 F_1 \) of second order given by (1.2.14).
Further, a symbol of type $\Delta(m, \lambda)$ stands for the set $m$ parameters

\[
\left( \frac{\lambda}{m} \right) \left( \frac{\lambda+1}{m} \right) \left( \frac{\lambda+2}{m} \right) \cdots \left( \frac{\lambda+m-1}{m} \right)
\]

Thus

\[
p_{m} \sum_{n=0}^{\infty} \left( \frac{a_{1} (a_{2})_{n} \cdots (a_{p})_{n}}{b_{1} (b_{2})_{n} \cdots (b_{q})_{n}} \left( \frac{\lambda}{m} \right)_{n} \left( \frac{\lambda+1}{m} \right)_{n} \cdots \left( \frac{\lambda+m-1}{m} \right)_{n} \right) \frac{z^{n}}{n!}.
\]

Also in terms of hypergeometric function, we have

\[
(1 - z)^{-a} = \sum_{n=0}^{\infty} \frac{(a)_{n} z^{n}}{n!} = \,_{1}F_{0} \left[ \frac{a}{-} ; z \right]
\]

\[
e^{z} = \sum_{n=0}^{\infty} \frac{z^{n}}{n!} = \,_{0}F_{0} \left[ \frac{-}{-} ; z \right]
\]

Some useful linear transformations of the hypergeometric function, known as Euler’s transformations, may be recalled here as follows:

\[
\,_{2}F_{1} \left[ \frac{a, b}{c} ; z \right] = (1 - z)^{-a} \,_{2}F_{1} \left[ \frac{a, c - b}{c} ; \frac{z}{z - 1} \right],
\]

\[
c \neq 0, -1, -2, \cdots, |arg(1 - z)| < \pi;
\]

\[
\,_{2}F_{1} \left[ \frac{a, b}{c} ; z \right] = (1 - z)^{-b} \,_{2}F_{1} \left[ \frac{c - a, b}{c} ; \frac{z}{z - 1} \right],
\]

\[
c \neq 0, -1, -2, \cdots, |arg(1 - z)| < \pi;
\]

\[
\,_{2}F_{1} \left[ \frac{a, b}{c} ; z \right] = (1 - z)^{c - a - b} \,_{2}F_{1} \left[ \frac{c - a, c - b}{c} ; z \right],
\]

\[
c \neq 0, -1, -2, \cdots, |arg(1 - z)| < \pi.
\]
1.3 Hypergeometric Function of Two and Several Variables

Appell Function

In 1880, Appell [1] considered the product of two Gauss hypergeometric functions,

\[ {}_2F_1(a, b; c; x) {}_2F_1(a', b'; c'; y) = \sum_{m,n=0}^{\infty} \frac{(a)_m(a')_n(b)_m(b')_n x^m y^n}{(c)_m(c')_n m! n!} \]  

(1.3.1)

This double series, in itself, yields nothing new, but if one or more of the three pairs of products

\[(a)_m(a')_n, (b)_m(b')_n, (c)_m(c')_n\]

be replaced by the corresponding expressions

\[(a)_{m+n}, (b)_{m+n}, (c)_{m+n}\]

then we are led four distinct possibilities of getting new functions. One such possibility, however, gives us the double series.

\[ \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_{m+n} x^m y^n}{(c)_{m+n} m! n!} \]

which is simply the Gaussian series for

\[ {}_2F_1(a, b; c; x + y), \]

since it is easily verified that \{cf. [48], p. 4\}

\[ \sum_{m,n=0}^{\infty} f(m + n) \frac{x^m y^n}{m! n!} = \sum_{N=0}^{\infty} f(N) \frac{(x + y)^n}{N!}, \]

(1.3.2)

or, more generally,

\[ \sum_{m_1, \ldots, m_n=0}^{\infty} f(m_1 + \ldots + m_n) \frac{x_1^{m_1} \ldots x_n^{m_n}}{m_1! \ldots m_n!} = \sum_{m=0}^{\infty} f(m) \frac{(x_1 + \ldots + x_n)^m}{m!}, \]

(1.3.3)
then the remaining four possibilities lead to the four Appell functions of two variables that are given below:

$$F_1[a, b, b'; c; x, y] = \sum_{m, n=0}^{\infty} \frac{(a)_{m+n}(b)_{m}(b')_{n}x^{m}y^{n}}{(c)_{m+n}m!n!} \left( \max\{|x|, |y|\} < 1 \right);$$  \hspace{1cm} (1.3.4)

$$F_2[a, b, b'; c, c'; x, y] = \sum_{m, n=0}^{\infty} \frac{(a)_{m+n}(b)_{m}(b')_{n}x^{m}y^{n}}{(c)_{m}(c')_{n}m!n!} \left( |x| + |y| < 1 \right);$$  \hspace{1cm} (1.3.5)

$$F_3[a, a', b, b'; c; x, y] = \sum_{m, n=0}^{\infty} \frac{(a)_{m}(a')_{n}(b)_{m}(b')_{n}x^{m}y^{n}}{(c)_{m+n}m!n!} \left( \max\{|x|, |y|\} < 1 \right);$$  \hspace{1cm} (1.3.6)

$$F_4[a, b; c, c'; x, y] = \sum_{m, n=0}^{\infty} \frac{(a)_{m+n}(b)_{m+n}x^{m}y^{n}}{(c)_{m}(c')_{n}m!n!} \left( \sqrt{|x|} + \sqrt{|y|} < 1 \right).$$  \hspace{1cm} (1.3.7)

The standard work on the theory of Appell series is the monograph by Appell and Kampé de Fériet [2]. See also (Slater [43] and Exton [15]; p.23 (28)) for a review of the subsequent work.

**Humbert Function**

In 1920, Humbert [17] has studied seven confluent form of the four Appell functions and is denoted these confluent hypergeometric functions of two variables by $\Phi_1, \Phi_2, \Phi_3, \Psi_1, \Psi_1, \Xi_1, \Xi_2$.

Here we list four Humbert functions which are used in our subsequent work as given below:

$$\Psi_1[\alpha, \beta; \gamma, \gamma'; x, y] = \sum_{m, n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta)_{m}x^{m}y^{n}}{(\gamma)_{m}(\gamma')_{n}m!n!} \left( |x| < 1, |y| < \infty \right);$$  \hspace{1cm} (1.3.8)

$$\Psi_2[\alpha; \gamma, \gamma'; x, y] = \sum_{m, n=0}^{\infty} \frac{(\alpha)_{m+n}x^{m}y^{n}}{(\gamma)_{m}(\gamma')_{n}m!n!} \left( |x| < \infty, |y| < \infty \right);$$  \hspace{1cm} (1.3.9)

$$\Xi_1[\alpha, \alpha', \beta; \gamma; x, y] = \sum_{m, n=0}^{\infty} \frac{(\alpha)_{m}(\alpha')_{n}(\beta)_{m}x^{m}y^{n}}{(\gamma)_{m+n}m!n!} \left( |x| < 1, |y| < \infty \right);$$  \hspace{1cm} (1.3.10)

$$\Xi_2[\alpha, \beta; \gamma; x, y] = \sum_{m, n=0}^{\infty} \frac{(\alpha)_{m}(\beta)_{m}x^{m}y^{n}}{(\gamma)_{m+n}m!n!} \left( |x| < 1, |y| < \infty \right).$$  \hspace{1cm} (1.3.11)
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Kampé de Fériet Function

Appell’s four double hypergeometric functions $F_1$, $F_2$, $F_3$ and $F_4$, were unified and generalized by Kampé de Fériet [18], (see also [2], p.150) who defined a general hypergeometric function of two variables. The notation introduced by Kampe’ de Feriet for his double hypergeometric function of superior order was subsequently abbreviated in 1941 by Burchnall and Chaundy [6]. We recall here definition of a more general double hypergeometric function (than the one defined by Kampe de Feriet) in a slightly modified notation [ see, for example, [54], p.423. Eq.(26) ]:

$$F_{p_q;k_l;m_n}^{p+s;k+l;0} \left[ \begin{array}{c} (a_p) ; (b_q) ; (c_k) ; \\ (\alpha_l) : (\beta_m) ; (\gamma_n) ; \\ x, y \end{array} \right] = \sum_{r,s=0}^{\infty} \prod_{j=1}^{p} (a_j)^r \prod_{j=1}^{q} (b_j)^s \prod_{j=1}^{k} (c_j)^r x^r y^s (p+s)!,$$  \hspace{1cm} (1.3.12)

where, for convergence

(i) $p + q < l + m + 1$, $p + k < l + n + 1$, $|x| < \infty$, $|y| < \infty$

(ii) $p + q = l + m + 1$, $p + k = l + n + 1$

$$|x|^{1/(p-l)} + |y|^{1/(q-l)} < 1$$

$$\max \{ |x|, |y| < 1 \text{ if } p \leq l \}.$$  \hspace{1cm} (1.3.13)

Although the double hypergeometric function defined by (1.3.12) reduces to the Kampe de Feriet function in the special case

$q = k \text{ and } m = n,$

yet it is usually referred to in the literature as Kampe de Ferièt function.

We shall also need here the definition of triple hypergeometric $F^{(3)} [x, y, z]$ (ef. Srivastava, H.M. [47], p. 428) defined as:

$$F^{(3)} [x, y, z] \equiv F^{(3)} \left[ \begin{array}{c} (a) :: (b) ; (b') ; (b'') : (d) ; (d') ; (d'') ; \\ (a) :: (g) ; (g') ; (g'') : (h) ; (h') ; (h'') ; \\ x, y, z \end{array} \right] = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \Lambda (m, n, p) \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!},$$  \hspace{1cm} (1.3.14)
where, for convenience

\[
\Lambda(m, n, p) = \frac{\prod_{j=1}^{A} (a_j)_{m+n+p} \prod_{j=1}^{B} (b_j)_{m+n} \prod_{j=1}^{B'} (b'_j)_{n+p} \prod_{j=1}^{B''} (b''_j)_{p+m}}{\prod_{j=1}^{E} (c_j)_{m+n+p} \prod_{j=1}^{G} (g_j)_{m+n} \prod_{j=1}^{G'} (g'_j)_{n+p} \prod_{j=1}^{G''} (g''_j)_{p+m}} \\
\times \frac{\prod_{j=1}^{C} (c_j)_m \prod_{j=1}^{C'} (c'_j)_n \prod_{j=1}^{C''} (c''_j)_p}{\prod_{j=1}^{H} (h_j)_m \prod_{j=1}^{H'} (h'_j)_n \prod_{j=1}^{H''} (h''_j)_p},
\] (1.3.15)

where \((a)\) abbreviates, the array of a parameters \(a_1, a_2, \ldots, a_A\), with similar interpretations for \((b), (b'), (b''), etc. The triple hypergeometric series in (1.3.15) converges absolutely when

\[
\begin{align*}
1 + E + G + G'' + H - A - B - B'' - C &\geq 0, \\
1 + E + G + G' + H' - A - B - B' - C &\geq 0, \\
1 + E + G' + G'' + H'' - A - B' - B'' - C'' &\geq 0,
\end{align*}
\] (1.3.16)

where the equalities hold true for suitable constrained values of \(|x|, |y| and |z|.

**Kampé de Fériet Function of n-Variables**

Karlsson, Per W. [19] was introduced functions of \(n\)-variables as finite sums of similar hypergeometric functions multiplied by elementary functions. One of these is defined by

\[
\begin{align*}
F^{1;2}_{1;1}: & \left[\begin{array}{c}
\alpha, \beta_1, \beta'_1, \ldots, \beta_n, \beta'_n, \\
\gamma, \delta, \ldots, \delta_n
\end{array}\right] \\
&= \sum_{j_1=0}^{\infty} \cdots \sum_{j_n=0}^{\infty} \prod_{k=1}^{n} \frac{(\alpha)_{j_1+\cdots+j_n} (\beta_k)_{j_1+j''_k} \beta''_k}{(\gamma)_{j_1+\cdots+j_n} (\delta_k)_{j_1+j''_k} j''_k!},
\end{align*}
\] (1.3.17)

for \(|z_k| < 1, k \in \{1, \cdots, n\}\), and by analytical continuation elsewhere; none of the quantities \(\gamma, \delta_1, \cdots, \delta_n\) may be zero or a negative integer.

**Lauricella Function of n Variables**

Lauricella [33], generalized the Appell’s double hypergeometric functions \(F_1, \cdots, F_4\) (cf. [14], p.224) to functions of \(n\) variables. These Lauricella functions, viz. \(F_A^{(n)}, F_B^{(n)},\)
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Two important confluent hypergeometric functions of two variables defined by (1.3.9).

\[ F_C^{(n)}(a, b_1, \cdots, b_n; c_1, \cdots, c_n; x_1, \cdots, x_n) = \sum_{m_1, \ldots, m_n = 0}^\infty \frac{(a)_{m_1+\cdots+m_n} (b_1)_{m_1} \cdots (b_n)_{m_n}}{(c_1)_{m_1} \cdots (c_n)_{m_n}} \frac{x_1^{m_1} \cdots x_n^{m_n}}{m_1! \cdots m_n!} |x_1| + \cdots + |x_n| < 1; \quad (1.3.18) \]

\[ F_B^{(n)}(a, b_1, \cdots, b_n; c; x_1, \cdots, x_n) = \sum_{m_1, \ldots, m_n = 0}^\infty \frac{(a)_{m_1} \cdots (a_n)_{m_n}}{(c)_{m_1+\cdots+m_n}} \frac{x_1^{m_1} \cdots x_n^{m_n}}{m_1! \cdots m_n!} \max \{|x_1|, \ldots, |x_n|\} < 1; \quad (1.3.19) \]

\[ F_C^{(n)}(a, b; c_1, \cdots, c_n; x_1, \cdots, x_n) = \sum_{m_1, \ldots, m_n = 0}^\infty \frac{(a)_{m_1} \cdots (a_n)_{m_n}}{(c_1)_{m_1} \cdots (c_n)_{m_n}} \frac{x_1^{m_1} \cdots x_n^{m_n}}{m_1! \cdots m_n!} \left\{ \sqrt{|x_1|}, \ldots, \sqrt{|x_n|} \right\} < 1; \quad (1.3.20) \]

\[ F_D^{(n)}(a, b_1, \cdots, b_n; c; x_1, \cdots, x_n) = \sum_{m_1, \ldots, m_n = 0}^\infty \frac{(a)_{m_1} \cdots (a_n)_{m_n}}{(c)_{m_1} \cdots (c_n)_{m_n}} \frac{x_1^{m_1} \cdots x_n^{m_n}}{m_1! \cdots m_n!} \max \{|x_1|, \ldots, |x_n|\} < 1. \quad (1.3.21) \]

Clearly, we have \( F_A^{(2)} = F_2, \ F_B^{(2)} = F_3, \ F_C^{(2)} = F_4 \) and \( F_D^{(2)} = F_1 \).

Lauricella [32] also gave several elementary properties of these functions. A summary of Lauricella’s work is given by Appell and Kampé de Fériet ([2], p.114-120), {see also Carlson [8], Carlitz and Srivastava [7] and Srivastava and Exton [50], [49], [51]}. \[ v \]

Confluent forms of Lauricella Function

Two important confluent hypergeometric functions of \( n \) variables are the functions \( \Phi_2^{(n)} \) and \( \Psi_2^{(n)} \) (see [53], p.62). But in our subsequent work we need only \( \Psi_2^{(n)} \), which is defined as

\[ \psi_2^{(n)}(a, c_1, \ldots, c_n; x_1, \ldots, x_n) = \sum_{m_1, \ldots, m_n = 0}^\infty \frac{(a)_{m_1+\cdots+m_n}}{(c_1)_{m_1} \cdots (c_n)_{m_n}} \frac{x_1^{m_1} \cdots x_n^{m_n}}{m_1! \cdots m_n!}. \quad (1.3.22) \]

Clearly, we have \( \Psi_2^{(2)} = \Psi_2 \), where \( \Psi_2 \) is Humbert confluent hypergeometric function of two variables defined by (1.3.9).
1.4 Integral Transforms

Integral transforms play an important role in various fields of applied mathematics and physics. The method of solution of problems arising in physics lie at the heart of the use of integral transform.

Let \( f(t) \) be a real or complex valued function of real variable \( t \), defined on interval \( a \leq t \leq b \), which belongs to a certain specified class of functions and let \( F(p,t) \) be a definite function of \( p \) and \( t \), where \( p \) is a complex quantity, whose domain is prescribed, then the integral equation

\[
\phi[f(t);p] = \int_a^b F(p,t)f(t) \, dt
\]

represents an integral transform \( \phi[f(t);p] \) of the function \( f(t) \) with respect to the function \( F(p,t) \). Where the class of functions to which \( f(t) \) belongs and the domain of \( p \) are so prescribed that the integral on the right exists. \( F(p,t) \) is called the kernel of the transform \( \phi[f(t),p] \).

If we can define an integral equation

\[
f(t) = \int_c^d F(t)\phi[f(t),p] \, dp
\]

then the equation (1.4.2) defines the inverse transform for the equation (1.4.1). By given different values to the function \( F(p,t) \), different integral transforms are defined by various authors like Fourier, Laplace, Hankel and Mellin as given below:

**Fourier Transform**

We call

\[
\mathcal{F}[f(x); \xi] = (2\pi)^{-1/2} \int_{-\infty}^{\infty} f(x)e^{i\xi x} \, dx
\]

the Fourier transform of \( f(x) \) and regard \( x \) as complex variable.

**Laplace Transform**

We call

\[
\mathcal{L}[f(t); p] = \int_0^{\infty} f(t)e^{-pt} \, dt
\]

the Laplace transform of \( f(t) \) and regard \( p \) as complex variable.
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**Hankel Transform**

We call

\[ \mathcal{H}_\nu[f(t); \xi] = \int_0^\infty f(t) t J_\nu(\xi t) \, dt \]  

(1.4.5)

the Hankel transform of \( f(t) \) and regard \( \xi \) as complex variable.

**Mellin transform**

We call

\[ \mathcal{M}[f(x); s] = \int_0^\infty f(x) x^{s-1} \, dx \]  

(1.4.6)

the Mellin transform of \( f(x) \) and regard \( s \) as complex variable.

**Gauss transform**

The Gauss transform of a function \( F(t) \) is defined by

\[ g_\alpha \{ F(t) \} = \left\{ \frac{1}{\sqrt{2\pi\alpha}} \int_{-\infty}^{+\infty} F(t) e^{-\frac{(x-t)^2}{2\alpha}} \, dt \right\} \]  

(1.4.7)

\( \alpha \) being parameter.

The most complete set of integral transforms are given in Erdélyi et al. [12, 13], Ditkin and Prudnikov [11] and Prudnikov et al. [37, 38].

**1.5 The Classical Orthogonal Polynomials**

Orthogonal polynomials constitute an important class of special functions in general and of hypergeometric function in particular. Some of the orthogonal polynomials and their connection with hypergeometric functions used in our work are as given as below:

**Legendre Polynomials**

The Legendre polynomials \( P_n(x) \) is defined by means of the generating relation.

\[ (1 - 2xt + t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x) t^n, \]  

(1.5.1)
in which \((1 - 2xt + t^2)^{-1/2}\) denotes the particular branch which tends to 1 as \(t \to 0\).

Thus equation (1.5.1) gives the Legendre polynomials in the form

\[
P_n(x) = \sum_{k=0}^{[n/2]} \frac{(-1)^k \left(\frac{1}{2}\right)_{n-k} (2x)^{n-2k}}{k!(n-2k)!},
\]

form which it follows that \(P_n(x)\) is a polynomial of degree precisely \(n\) in \(x\).

Also, the relation (1.5.1) gives

\[
P_n(-x) = (-1)^n P_n(x)
\]

which shows that \(P_n(x)\) is an odd function of \(x\) for \(n\) odd, an even function of \(x\) for \(n\) even.

The hypergeometric form of the relation (1.5.2), can be written as

\[
P_n(x) = {}_2F_1 \left[ \begin{array}{c} -n, n + 1; \\ 1 - x \\ \frac{1}{2} \end{array} \right],
\]

or equivalently

\[
P_n(x) = (-1)^n {}_2F_1 \left[ \begin{array}{c} -n, n + 1; \\ 1 + x \\ \frac{1}{2} \end{array} \right].
\]

**Hermite Polynomials**

Hermite Polynomials \(H_n(x)\) is defined by means of generating relation

\[
\exp(2xt - t^2) = \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!},
\]

valid for all finite \(x\) and \(t\) and one can easily obtain

\[
H_n(x) = \sum_{k=0}^{[n/2]} (-1)^k \frac{n! (2x)^{n-2k}}{k! (n-2k)!}.
\]

and the hypergeometric form of (1.5.6) can be written as

\[
H_n(x) = (2x)^n {}_2F_0 \left[ \begin{array}{c} -\frac{n}{2}, -\frac{n}{2} + \frac{1}{2}; \\ -1 \end{array} \right].
\]
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Laguerre Polynomials

The generalized Laguerre Polynomial $L_n^{(\alpha)}(x)$ is defined by means of generating relation.

$$\sum_{n=0}^{\infty} L_n^{(\alpha)}(x)t^n = (1 - t)^{-(\alpha+1)} \exp\left(\frac{xt}{t-1}\right). \quad (1.5.8)$$

For $\alpha = 0$, the above equation (1.5.8) yield the generating function for simple Laguerre Polynomial $L_n(x)$.

$$L_n(x) = L_n^{(0)}(x) = {}_1F_1[-n; 1; x]. \quad (1.5.9)$$

A series representation of $L_n^{(\alpha)}(x)$ for non negative integers $n$, is given by

$$L_n^{(\alpha)}(x) = \sum_{k=0}^{n} \frac{(-1)^k(1 + \alpha)_n x^k}{k! (n - k)!(1 + \alpha)_k}. \quad (1.5.10)$$

The hypergeometric form of generalized Laguerre polynomials is defined by (see [40], p.200),

$$L_n^{(\alpha)}(x) = \frac{(1 + \alpha)_n}{n!} {}_1F_1[-n; \alpha + 1; x]. \quad (1.5.11)$$

Gegenbauer Polynomials

The Gegenbauer polynomials, denoted by $C_n^{(\nu)}(x)$, is a generalization of the Legendre polynomial and is defined by the generating relation

$$(1 - 2xt + t^2)^{-\nu} = \sum_{n=0}^{\infty} C_n^{(\nu)}(x)t^n. \quad (1.5.12)$$

Thus equation (1.5.12), gives the Gegenbauer polynomials in the form

$$C_n^{(\nu)}(x) = \sum_{k=0}^{\lfloor\nu\rfloor} \frac{(-1)^n(\nu)_n(2x)^{n-2k}}{k!(n - 2k)!}. \quad (1.5.12)$$

Also, the hypergeometric form of Gegenbauer polynomials is as given below

$$C_n^{(\nu)}(x) = \frac{(2\nu)_n}{n!} {}_2F_1\left[ -n, 2\nu + n; \nu + \frac{1}{2}; \frac{1 - x}{2} \right]. \quad (1.5.13)$$
Generalized Rice Polynomials

Rice, S.O. [41] investigated a polynomial set,

\[ H_n(\xi, p, \nu) = \binom{n + 1}{n} \binom{-n, n + 1, \xi; \nu}{1, p; \nu}, \quad (1.5.14) \]

In 1964, Khandekar [22] generalized Rice polynomial as

\[ H_n^{(\alpha, \beta)}(\xi, p; x) = \binom{n + \alpha}{n} \binom{-n, \alpha + \beta + n + 1, \xi; x}{\alpha + 1, p; x}, \quad (1.5.15) \]

where \( (\alpha) > -1, \ Re(\beta) > -1. \)

In 1972, Khan, I.A. (see [20] and [21]) introduced a generalization of Rice polynomial defined by (1.5.15) in the form

\[ f_n^{(\alpha, \beta)}[(a_p), (b_q); x] = \frac{(1 + \alpha)n}{n!} p+2F_{q+1} \left[ \begin{array}{c} -n, \alpha + \beta + n + 1, (a_p); x \\ \alpha + 1, (b_q); x \end{array} \right], \quad (1.5.16) \]

Clearly, we note that

\[ f_n^{(\alpha, \beta)}(\xi, p; x) = H_n^{(\alpha, \beta)}(\xi, p; x), \quad (1.5.17) \]

and

\[ P_n^{(\alpha, \beta)}(x) = H_n^{(\alpha, \beta)}[\xi, \xi; (1 - x)/2]. \quad (1.5.18) \]

Bedient Polynomials

Bedient [3] polynomials, denoted by \( R_n(\beta, \gamma; x) \) and \( G_n(\alpha, \beta; x) \), are defined by means of the generating relations

\[ \sum_{n=0}^{\infty} R_n(\beta, \gamma; x)t^n = (1 - 2xt)^{-\beta} 2F_1 \left[ {\beta, \gamma - \beta; \frac{-t^2}{1 - 2xt}} \right] \quad (1.5.19) \]

\[ \sum_{n=0}^{\infty} G_n(\alpha, \beta; x)t^n = 2F_1 \left[ {\alpha, \beta; 2xt - t^2} \right]. \quad (1.5.20) \]
Also $R_n(\beta, \gamma; x)$ and $G_n(\alpha, \beta; x)$ can be represented in the following Hypergeometric forms

\[
R_n(\beta, \gamma; x) = \frac{(\beta)_n (2x)^n}{n!} \frac{\gamma, 1 - \beta - n; 1}{x^2} \binom{3}{2} F_2 \left[ \frac{-n, \gamma - \frac{1}{2}, \gamma - \beta;}{1 - \beta - n, 1 - \beta - n; x^2} \right].
\] (1.5.21)

\[
G_n(\alpha, \beta; x) = \frac{(\alpha)_n (\beta)_n (2x)^n}{n! (\alpha + \beta)_n} \frac{1 - \alpha - n, 1 - \beta - n; 1}{x^2} \binom{3}{2} F_2 \left[ \frac{-n, -n + \frac{1}{2}, 1 - \alpha - \beta - n;}{1 - \alpha - n, 1 - \beta - n; x^2} \right].
\] (1.5.22)

The Rodrigues formulae for $R_n(\beta, \gamma; x)$ and $G_n(\alpha, \beta; x)$ are as given below

\[
R_n(\beta, \gamma; x) = \frac{(-1)^n (\gamma - \beta)_n}{2^n n! (\gamma)_n} D^n \left\{ 3 F_2 \left[ \frac{-n, \beta, 1 - \gamma - n;}{1 - \gamma + \beta - n, \frac{1}{2}; x^2} \right] \right\}.
\] (1.5.23)

\[
G_n(\alpha, \beta; x) = \frac{(-1)^n}{2^n n!} D^n \left\{ 3 F_2 \left[ \frac{-n, \alpha, \beta;}{\frac{1}{2}, \alpha + \beta; x^2} \right] \right\}.
\] (1.5.24)

### 1.6 Operator Representations

In a recent paper in 2008, Khan, M.A. and Shukla, A.K. [30] obtained binomial and trinomial operator representations of certain polynomials. Using their technique in our work we have obtained certain results of binomial and trinomial operator representation type for various polynomials by using their Rodrigues formula. Here we need the following results of [30]:

\[
(D_x + D_y)^n \{ f(x) g(y) \} = \sum_{r=0}^{n} \binom{n}{r} D_x^{n-r} f(x) D_y^r g(y).
\] (1.6.1)

\[
(D_x + D_y + D_z)^n \{ f(x) g(y) h(z) \} = \sum_{r=0}^{n} \sum_{s=0}^{n-r} \binom{n}{r} (-1)^{r+s} r! s! \times D_x^{n-r-s} f(x) D_y^r g(y) D_z^s h(z).
\] (1.6.2)
and

\[(D_x D_y + D_x D_z + D_y D_z)^n \{f(x)g(y)h(z)\} = \sum_{r=0}^{n} \sum_{s=0}^{n-r} \frac{(-n)^{r+s} (-1)^{r+s}}{r!s!} \times D_x^{n-s} f(x) D_y^{n-r} g(y) D_z^{r+s} h(z). \quad (1.6.3)\]

where \(D_x = \frac{\partial}{\partial x}, \ D_y = \frac{\partial}{\partial y} \) and \(D_z = \frac{\partial}{\partial z}. \) Also in deriving the operational representations of polynomials, use has been made of the fact that

\[D_x^n \{x^\lambda\} = \lambda(\lambda - 1) \cdots (\lambda - n + 1) = \frac{\Gamma(\lambda + 1)}{\Gamma(\lambda - n + 1)} x^{\lambda-n}, \quad (n = 0, 1, 2, \cdots) \quad (1.6.4)\]

to general form

\[D_x^\mu \{x^\lambda\} = \frac{\Gamma(\lambda + 1)}{\Gamma(\lambda - \mu + 1)} x^{\lambda-\mu}, \quad D \equiv \frac{d}{dx}, \quad (1.6.5)\]

where \(\lambda \) and \(\mu, \ \lambda \geq \mu\) are arbitrary real numbers.

### 1.7 Fractional Integrals and Derivatives Operators

This section deals with the definition and basic properties of various operators of fractional integration and fractional differentiation of arbitrary order. Among the various operators studied, it involves the following fractional integral and fractional derivative operators used in our work are as given below:

**Riemann-Liouville fractional integral of order \(\mu\)**

\[I^\mu \{f(x)\} = \frac{1}{\Gamma(\mu)} \int_0^x (x-t)^{\mu-1} f(t)dt, \quad Re(\mu) > 0 \quad (1.7.1)\]

**Riemann-Liouville left sided fractional integral of order \(\alpha\)**

Let \(f(x) \in L(b, c), \ \alpha \in C \) and \(Re(\alpha) > 0, \) then

\[bI_x^\alpha \{f(x)\} = \frac{1}{\Gamma(\alpha)} \int_b^x (x-t)^{\alpha-1} f(t)dt, \quad x > b. \quad (1.7.2)\]
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Riemann-Liouville right sided fractional integral of order $\alpha$

Let $f(x) \in L(b, c), \alpha \in C$ and $\text{Re}(\alpha) > 0$, then

$$x^\alpha I_c\{f(x)\} = \frac{1}{\Gamma(\alpha)} \int_x^c (t-x)^{\alpha-1} f(t)dt, \; x < c. \quad (1.7.3)$$

The Weyl integral of $f(x)$ of order $\alpha$, denoted by $xW_\infty^\alpha$, is defined by

$$xW_\infty^\alpha \{f(x)\} = \frac{1}{\Gamma(\alpha)} \int_x^\infty (t-x)^{\alpha-1} f(t)dt, \; -\infty < x < \infty \quad (1.7.4)$$

where $\alpha \in C$ and $\text{Re}(\alpha) > 0$.

Erdely-Kober operator of first kind

We have

$$I[\alpha, \eta; f(x)] = \frac{x^{\alpha-\eta}}{\Gamma(\alpha)} \int_0^x t^\eta (x-t)^{\alpha-1} f(t)dt, \quad (1.7.5)$$

where $\alpha, \eta \in C$ and $\text{Re}(\alpha) > 0$.

Erdely-Kober operator of second kind

We have

$$I[\alpha, \eta; f(x)] = \frac{x^\eta}{\Gamma(\alpha)} \int_x^\infty t^{-\alpha} (t-x)^{\alpha-1} f(t)dt, \quad (1.7.6)$$

where $\alpha, \eta \in C$ and $\text{Re}(\alpha) > 0$.

Saigo integral operator of first kind

We have

$$I_{0+}^{\alpha, \beta, \eta} \{f(x)\} = \frac{x^{\alpha-\beta}}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} \binom{\alpha + \beta, -\eta}{\alpha; 1 - \frac{t}{x}} f(t)dt, \; \text{Re}(\alpha) > 0 \quad (1.7.7)$$

Saigo integral operator of second kind

We have

$$I_{0-}^{\alpha, \beta, \eta} \{f(x)\} = \frac{1}{\Gamma(\alpha)} \int_x^\infty (t-x)^{\alpha-1} t^{-\alpha} \binom{\alpha + \beta, -\eta}{\alpha; 1 - \frac{x}{t}} f(t)dt, \; \text{Re}(\alpha) > 0 \quad (1.7.8)$$
The left sided Riemann-Liouville fractional derivative of order $\alpha$

Let $\alpha \in \mathbb{C}$, $\text{Re}(\alpha) \geq 0$ and $n = [\text{Re}(\alpha)] + 1$ then

$$b D^n_x \{\phi(x)\} = \frac{1}{\Gamma(n - \alpha)} \left( \frac{d}{dx} \right)^n \int_b^x \frac{\phi(t)dt}{(x - t)^{\alpha-n+1}}, \ x > b \quad (1.7.9)$$

The right sided Riemann-Liouville fractional derivative of order $\alpha$

Let $\alpha \in \mathbb{C}$, $\text{Re}(\alpha) \geq 0$ and $n = [\text{Re}(\alpha)] + 1$ then

$$x D^n_c \{\phi(x)\} = (-1)^n \frac{1}{\Gamma(n - \alpha)} \left( \frac{d}{dx} \right)^n \int_x^c \frac{\phi(t)dt}{(t - x)^{\alpha-n+1}}, \ x < c \quad (1.7.10)$$

The Weyl fractional derivative of $f(x)$ of order $\alpha$, denoted by $x D^n_{\infty}$, is defined by

$$x D^n_{\infty} \{f(x)\} = (-1)^m \left( \frac{d}{dx} \right)^m \frac{1}{\Gamma(m - \alpha)} \int_{-\infty}^x \frac{f(t)dt}{(t - x)^{1+\alpha-m}}, \quad (1.7.11)$$

where $-\infty < x < \infty$, $\alpha \in \mathbb{C}$, $m - 1 \leq \alpha < m$ and $m \in \mathbb{N}$.

### 1.8 The Modified Hermite Polynomials of One and Two Variables

Khan, M.A., Khan, A.H. and Ahmad, N. [25] introduced the modified Hermite polynomials of one variable, denoted by $H_n(x; a)$, is defined by means of the generating relation

$$a^{2x^2-t^2} = \sum_{n=0}^{\infty} \frac{H_n(x; a)t^n}{n!}, \quad a > 0, \ a \neq 1 \quad (1.8.1)$$

It follows from (1.8.1) that

$$H_n(x; a) = \sum_{k=0}^{[\frac{n}{2}]} \frac{(-1)^k n!(2x)^{n-2k}(\log a)^{n-k}}{k!(n-2k)!}. \quad (1.8.2)$$

For $a = e$, (1.8.2) reduces to Hermite polynomials $H_n(x)$.

It may be remarked that $H_n(x; a)$ is an even function of $x$ for even $n$, an odd function of $x$ for odd $n$. 


Also the modified Hermite polynomials of two variables introduced by Khan, M.A., Khan, A.H. and Ahmad, N., denoted by $H_n(x, y; a)$ and is defined by means of the generating relation:

$$a^{2xt-(y+1)t^2} = \sum_{n=0}^{\infty} \frac{H_n(x, y; a)t^n}{n!}, \quad a > 0, \ a \neq 1 \quad (1.8.3)$$

It follows from (1.8.3) that

$$H_n(x, y; a) = \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n!H_{n-2r}(x; a)(-y)^r(\log a)^r}{r!(n-2r)!} \quad (1.8.4)$$

where $H_n(x; a)$ stands for the modified Hermite polynomial of one variable [25].

The definition (1.8.4) is equivalent to the following explicit representation of $H_n(x, y; a)$

$$H_n(x, y; a) = \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{s=0}^{\lfloor \frac{n-r}{2} \rfloor} (-n)_{2r+2s}(2x)^{n-2r-2s}(-y)^r(-1)^s(\log a)^{n-r-s} \left( \frac{1}{r!s!} \right). \quad (1.8.5)$$

In terms of double hypergeometric function, modified Hermite polynomials of two variables can be written as

$$H_n(x, y; a) = (2x\log a)^n F_{2;0;0}^{2;0;0} \left[ -\frac{a}{2}, -\frac{a}{2} + \frac{1}{2}; -; -; - \frac{y}{x^2 \log a}, -\frac{1}{y^2 \log a} \right] \quad (1.8.6)$$

For $a = e$, (1.8.4), (1.8.5) and (1.8.6) reduces to Hermite polynomials of two variables $H_n(x, y)$ due to Khan, M.A. and Abukhummash, G.S. [23].

It may be remarked that $H_n(x, y; a)$ is an even function of $x$ for even $n$, an odd function of $x$ for odd $n$.

$$H_n(-x, y; a) = (-1)^n H_n(x, y; a)$$
1.9 New Two Variables Analogue of Modified Hermite Polynomials

The new two variables analogue of modified Hermite polynomials of two variables defined and studied by Khan, M.A., Khan, A.H. and Ahmad, N., denoted by $\widetilde{H}_n(x, y; a)$, is defined as follows:

$$\widetilde{H}_n(x, y; a) = \sum_{r=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{(-1)^r n! \ H_{n-2r}(x; a) x^{2r} y^{n-2r} (\log a)^r}{r! (n-2r)!}, \quad a > 0, \ a \neq 1$$ (1.9.1)

where $H_n(x; a)$ is the modified Hermite polynomial of one variable [25].

The definition (1.9.1) is equivalent to the following explicit representation of $\widetilde{H}_n(x, y; a)$:

$$\widetilde{H}_n(x, y; a) = (2xy \log a)^n \sum_{r=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \sum_{s=0}^{\left\lfloor \frac{n}{2} - r \right\rfloor} (-n)_{2r+2s} \left( -\frac{1}{x^2 \log a} \right)^s \left( -\frac{1}{y^2 \log a} \right)^r 2^{2r+2s} \ s! \ r!$$ (1.9.2)

In terms of double hypergeometric function, modified Hermite polynomials of two variables can be written as

$$\widetilde{H}_n(x, y; a) = (2xy \log a)^n \ _2F_2^{0,0;0;0} \left[ \begin{array}{c} -\frac{n}{2}, -\frac{n}{2} + \frac{1}{2} ; -; -; \ \frac{1}{x^2 \log a}, \frac{1}{y^2 \log a} \end{array} \right]$$ (1.9.3)

1.10 New First Kind Analogue of Hermite Polynomials of Three and $m$-Variables

Here we define the first kind analogue of Hermite polynomials of three and $m$-variables whose two variable analogue seems more natural than that of Hermite polynomials of two variables defined and studied by Khan, M.A. and Abukhammash, G.S. [23]. First kind Hermite polynomials of three variables $H_n(x, y, z)$ is defined as follows:

$$H_n(x, y, z) = \sum_{r=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{(-1)^r n! \ H_{n-2r}(x, y) (xy)^{2r} z^{n-2r}}{r! (n-2r)!},$$ (1.10.1)
where $H_n(x, y)$ is Hermite polynomial of two variables [26]. The definition (1.10.1) can also be written as

$$H_n(x, y, z) = \sum_{r=0}^{n} [\frac{n}{2}] [\frac{n-r}{2}] (-1)^{r+s} \frac{n!}{r!} \frac{H_{n-2r-2s}(x)}{s!} \frac{x^{2r+2s}}{(n-2r-2s)!} \frac{y^{n-2s}}{z^{n-2r}}.$$  (1.10.2)

where $H_n(x)$ is the well-known Hermite polynomial of one variable.

The definition (1.10.2) is equivalent to the following explicit representation of $H_n(x, y, z)$:

$$H_n(x, y, z) = (2xyz)^n \sum_{k=0}^{\frac{n}{2}} \sum_{r=0}^{\frac{n}{2}-k} \sum_{s=0}^{\frac{n}{2}-k-r} \frac{(-n)_{2k+2r+2s}}{2^{2k+2r+2s} k! r! s!} \left( -\frac{1}{x^2} \right)^{k} \left( -\frac{1}{y^2} \right)^{r} \left( -\frac{1}{z^2} \right)^{s}.$$

(1.10.3)

In terms of triple hypergeometric function, Hermite polynomials of three variables can be written as

$$H_n(x, y, z) = (2xyz)^n \sum_{r_1=0}^{\frac{n}{2}} \ldots \sum_{r_{m-1}=0}^{\frac{n}{2}-r_1-r_2-\ldots-r_{m-2}} \frac{(-n)_{r_1+2r_2+\ldots+r_{m-1}}}{r_1! r_2! \ldots r_{m-1}!} \times (-1)^{r_1+r_2+\ldots+r_{m-1}} x_1^{r_1} x_2^{r_2} \ldots x_{m-1}^{r_{m-1}} x_m^{n-2r_1-2r_2-\ldots-2r_{m-1}}.$$  (1.10.4)

where for right hand side of (1.10.4), it may be recalled that the definition of a general triple hypergeometric function (cf. Srivastava, H.M. and Manocha, H.L. [12], p. 428).

Also, the $m$-variable analogue of first kind of Hermite polynomials is denoted by $H_n(x_1, x_2, \ldots, x_m)$ and is defined as follows:

$$H_n(x_1, x_2, \ldots, x_m) = \sum_{r_1=0}^{\frac{n}{2}} \ldots \sum_{r_{m-1}=0}^{\frac{n}{2}-r_1-r_2-\ldots-r_{m-2}} \frac{(-n)_{r_1+2r_2+\ldots+r_{m-1}}}{r_1! r_2! \ldots r_{m-1}!} \times (-1)^{r_1+r_2+\ldots+r_{m-1}} x_1^{r_1} x_2^{r_2} \ldots x_{m-1}^{r_{m-1}} x_m^{n-2r_1-2r_2-\ldots-2r_{m-1}} \times H_{n-2r_1-2r_2-\ldots-2r_{m-1}}(x_1).$$  (1.10.5)

where $H_n(x)$ is well-known Hermite polynomial of one variable. The definition (1.10.5) can also be written as
In this section we define a new second kind analogue of Hermite polynomials of three variables where for right hand side of (1.10.8), it may be recalled that the definition of Kampe de Feriet function of two variables defined and studied by Khan, M.A., Khan, A.H. and Ahmad, N. [26].

The definition (1.10.5) is equivalent to the following explicit representation of $H_n(x, x_2, \cdots, x_m)$:

$$
H_n(x_1, x_2, \cdots, x_m) = \sum_{r_1=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \sum_{r_2=0}^{\left\lfloor \frac{n}{2} - r_1 \right\rfloor} \cdots \left( \begin{array}{c} (-n)_{2r_1+2r_2+\cdots+2r_{m-2}} \frac{r_1!r_2!\cdots r_{m-2}!}{2r_1+2r_2+\cdots+2r_{m-2}} \\
\end{array} \right) 
\times \left( -1 \right)^{r_1+r_2+\cdots+r_{m-2}} (x_1 x_2)^{2r_1+2r_2+\cdots+2r_{m-2}} x^n_{m-2} 2r_1 x_{m-2}^{2r_2} \cdots x_m^{2r_{m-2}} 
\times H_{n-2r_1-2r_2-\cdots-2r_{m-2}}(x_1, x_2),
$$

(1.10.6)

where $H_n(x_1, x_2)$ is the Hermite polynomials of two variables defined and studied by Khan, M.A., Khan, A.H. and Ahmad, N. [26].

According to the definition (1.3.17), in terms of hypergeometric function, Hermite polynomials of $m$-variables can be written as

$$
H_n(x_1, x_2, \cdots, x_m) = \left\{ 2(x_1 x_2 \cdots x_m) \right\}^n 
\times \frac{\prod_{r=0}^{\left\lfloor \frac{n}{2} \right\rfloor} (\frac{1}{x_1^n}, \frac{1}{x_2^n}, \cdots, \frac{1}{x_m^n})}{2^{r_1+r_2+\cdots+r_{m-2}} (x_1 x_2)^{2r_1+2r_2+\cdots+2r_{m-2}} x^{n-2r_1} x_{m-2}^{2r_2} \cdots x_m^{2r_{m-2}}},
$$

(1.10.7)

(1.10.8)

where for right hand side of (1.10.8), it may be recalled that the definition of Kampe de Feriet function of $m$-variables [19].

### 1.11 New Second Kind Analogue of Hermite Polynomials of Three and $m$-Variables

In this section we define a new second kind analogue of Hermite polynomials of three variables whose two variable analogue seems more natural than that of Hermite polynomials of two variables defined and studied by Khan, M.A. and Abukhhammash, G.S. [23]. Second kind Hermite polynomials of three variable $\tilde{H}_n(x, y, z)$ is defined as follows:

$$
\tilde{H}_n(x, y, z) = \sum_{r=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{(-n)_{2r} \cdot (-1)^r \tilde{H}_{n-2r}(x, y) z^{n-2r} [(x^2 + y^2)(1 - z^2) + z^2]}{r!},
$$

(1.11.1)
where $\widetilde{H}_n(x, y)$ is Hermite polynomial of two variables [26]. The definition (1.11.1) can also be written as

$$\widetilde{H}_n(x, y, z) = \sum_{r=0}^{\frac{n}{2}} \sum_{s=0}^{\frac{n}{2}-r} \frac{(-1)^{r+s}}{r! s! (n-2r-2s)!} H_{n-2r-2s}(xy) x^{2r} y^{2s} z^{n-2r-2s},$$  

(1.11.2)

where $H_n(x)$ is the well-known Hermite polynomial of one variable.

The definition (1.11.2) is equivalent to the following explicit representation of $\widetilde{H}_n(x, y, z)$:

$$\widetilde{H}_n(x, y, z) = (2xyz)^n \sum_{k=0}^{\frac{n}{2}} \sum_{r=0}^{\frac{n}{2}-k} \sum_{s=0}^{\frac{n}{2}-k-r} \frac{(-n)_{2k+2r+2s}}{2^{2k+2r+2s} k! r! s!} \left(\frac{1}{x^2 y^2} \right)^s \left(\frac{1}{y^2 z^2} \right)^r \left(\frac{1}{z^2 x^2} \right)^r \right)$$  

(1.11.3)

In terms of triple hypergeometric function, Hermite polynomials of three variables can be written as

$$\widetilde{H}_n(x, y, z) = (2xyz)^n F^{(3)} \left[ -\frac{n}{2}, -\frac{n}{2} + \frac{1}{2}, \ldots; -1, -1; -1; -1; -1; \right.$$

$$\left.-1; -1; -1; -1; -1; -1; -1; -1; -1; -1; \right]$$

(1.11.4)

where for right hand side of (1.11.4), it may be recalled that the definition of a triple hypergeometric function [ see, for example, Srivastava and Panda [54], p.423. Eq.(26) ].

Also, the $m$-variable analogue of second kind of Hermite Polynomials is denoted by $\widetilde{H}_n(x_1, x_2, \ldots, x_m)$ and is defined as follows:

$$\widetilde{H}_n(x_1, x_2, \ldots, x_m) = \sum_{r_1=0}^{\frac{n}{2}} \sum_{r_2=0}^{\frac{n}{2}-r_1} \cdots \sum_{r_m=0}^{\frac{n}{2}-r_1-r_2-\cdots-r_{m-1}} \frac{(-n)_{2r_1+2r_2+\cdots+2r_m}}{r_1! r_2! \cdots r_m!} \left[ -1 \right. \cdots \right.$$

$$\left. \cdots -1; \right]$$

$$\times (-1)^{r_1+r_2+\cdots+r_m} x_1^{2r_1} x_2^{2r_2} \cdots x_m^{2r_m} H_{n-2r_1-2r_2-\cdots-2r_m}(x_1, x_2, \ldots, x_m),$$

(1.11.5)
where \( H_n(x_1) \) is well-known Hermite polynomial of one variable. The definition (1.11.5) can also be written as

\[
\tilde{H}_n(x_1, x_2, \ldots, x_m) = \sum_{r_1=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \sum_{r_2=0}^{\left\lfloor \frac{n-r_1}{2} \right\rfloor} \cdots \sum_{r_m=0}^{\left\lfloor \frac{n-r_1-\cdots-r_{m-1}}{2} \right\rfloor} \frac{(-n)_{2r_1+2r_2+\cdots+2r_m}}{r_1! r_2! \cdots r_m!} \times \left(1 - (x_3x_4 \cdots x_m)^2\right)^{r_1+r_2+\cdots+r_m} x_1^{2r_1} x_2^{2r_2} x_3^{n-2r_1-2r_2-\cdots-2r_m} \cdots x_m^{n-2r_1-2r_2-\cdots-2r_m} \times x_m^{n-2r_1-2r_2-\cdots-2r_m} \tilde{H}_{n-2r_1-2r_2-\cdots-2r_m}(x_1, x_2),
\]

(11.16)

where \( \tilde{H}_n(x_1, x_2) \) is the Hermite polynomials of two variables defined and studied by Khan, M.A., Khan, A.H. and Ahmad, N. [26].

The definition (1.11.5) is equivalent to the following explicit representation of \( \tilde{H}_n(x_1, x_2, \ldots, x_m) \):

\[
\tilde{H}_n(x_1, x_2, \ldots, x_m) = \sum_{r_1=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \sum_{r_2=0}^{\left\lfloor \frac{n-r_1}{2} \right\rfloor} \cdots \sum_{r_m=0}^{\left\lfloor \frac{n-r_1-\cdots-r_{m-1}}{2} \right\rfloor} \frac{(-n)_{2r_1+2r_2+\cdots+2r_m}}{2^{2r_1+2r_2+\cdots+2r_m}} \times \frac{1}{r_1!r_2!\cdots r_m!} \left(1 - (x_3x_4 \cdots x_m)^2\right)^{r_1} \left(1 \cdot 1 \cdots \frac{1}{x_m^{2}}\right)^{r_2} \cdots \left(1 \cdot 1 \cdots \frac{1}{x_1^{2}}\frac{1}{x_2^{2}}\cdots \frac{1}{x_m^{2}}\right)^{r_m}.
\]

(11.17)

According to the definition (1.3.17), in terms of hypergeometric function, Hermite polynomials of \( m \)-variables can be written as

\[
\tilde{H}_n(x_1, x_2, \ldots, x_m) = \left\{2(x_1x_2 \cdots x_m)^{n/2}\right\} F_{0;0}^{1;0} \left[ -\frac{n}{2}, -\frac{n}{2} + \frac{1}{2}; -; \cdots; -; -; -; \right]
\]

\[
- \left(\frac{1}{x_2^2} \frac{1}{x_3^2} \cdots \frac{1}{x_m^2}\right), - \left(\frac{1}{x_1^2} \frac{1}{x_2^2} \cdots \frac{1}{x_m^2}\right), \cdots, - \left(\frac{1}{x_1^2} \frac{1}{x_2^2} \cdots \frac{1}{x_{m-1}^2}\right),
\]

(11.18)

where for right hand side of (11.18), it may be recalled that the definition of Kampe de Feriet function of \( m \)-variables [19].
1.12 A Study of Hermite Polynomials of Several Variables by Burchnall’s Method

The present section deals with an extension of certain results obtained by Burchnall, J.L. for Hermite polynomials to similar results for Hermite polynomials of several variables.

Feldheim and Watson [16, 56] proved the formula

\[ H_m(x)H_n(x) = m!n! \sum_{r=0}^{\min(m,n)} \frac{2^r H_{m+n-2r}(x)}{(m-r)!(n-r)!r!}. \]  \hspace{1cm} (1.12.1)

where \( H_n(x) \) is Hermite’s polynomial, defined by

\[ H_n(x) = e^{x^2} (-D)^n e^{-x^2} = (-1)^n(D - 2x)^n, \quad D \equiv \frac{d}{dx}. \]  \hspace{1cm} (1.12.2)

Here Feldheim [16] employs the orthogonal properties of the polynomials, while Watson [56], starting from the generating function, changes the order of the summation in a multiple series. Burchnall, J.L. [5] gave a proof depending directly on the definition (1.12.2). For this Burchnall, J.L. obtained the following results:

\[ (-1)^n(D - 2x)^n y = \sum_{r=0}^{n} (-1)^n \left( \begin{array}{c} n \\ r \end{array} \right) H_{n-r}(x) D^r y, \]  \hspace{1cm} (1.12.3)

where \( y \) is any sufficiently differentiable function of \( x \).

\[ H_{n+1}(x) = 2xH_n(x) - H'_n(x), \]  \hspace{1cm} (1.12.4)

\[ H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x), \]  \hspace{1cm} (1.12.5)

and

\[ H'_n(x) = 2nH_{n-1}(x). \]  \hspace{1cm} (1.12.6)

The formulae (1.12.4), (1.12.5) and (1.12.6) are well known. Also by setting \( y = H_m(x) \) in (1.12.3) with \( n \leq m \), he obtained

\[ H_{m+n}(x) = m!n! \sum_{r=0}^{\min(m,n)} \frac{(-1)^r 2^r}{(m-r)!(n-r)!r!} H_{m-r}(x)H_{n-r}(x). \]  \hspace{1cm} (1.12.7)
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The formulae (1.12.1) and (1.12.7) may be regarded as inverse to one another and (1.12.1) may now be derived either by inverting (1.12.7) or by employing this latter formula on the right hand side of (1.12.1), thus obtaining a verification. It is, however, possible to prove (1.12.1) directly by employing the theory of adjoint operators.

By regarding Leibnitz’s theorem
\[ D^n(vw) = \sum_{r=0}^{n} \binom{n}{r} (D^r v)(D^{n-r} w), \]  
(1.12.8)
as expressing the equivalence of certain operations on the function \( w \), then, taking the adjoint of these operations, he obtained
\[ vD^n w = \sum_{r=0}^{n} (-1)^r \binom{n}{r} D^{n-r} (wD^r v). \]  
(1.12.9)
By taking \( v = H_m(x) \), \( w = 1 \) \((m \geq n)\), in (1.12.9) and using (1.12.2) and (1.12.6) then
\[ (-1)^n H_m(x)H_n(x) = (-1)^n \sum_{r=0}^{n} \binom{n}{r} \frac{2^r m!}{(m-r)!} H_{m+n-2r}(x). \]  
(1.12.10)
which is (1.12.1).

Similar results hold for the polynomials
\[ \varphi_n(x) = (-1)^n e^{\frac{1}{2}x^2}D^n e^{-\frac{1}{2}x^2} = (-1)^n(D - x)^n 1, \quad D \equiv \frac{d}{dx}, \]  
(1.12.11)
the formulae corresponding to (1.12.3), (1.12.7) and (1.12.1) being
\[ (-1)^n(D - x)^n y = \sum_{r=0}^{n} (-1)^r \binom{n}{r} \varphi_{n-r}(x)D^r y. \]  
(1.12.12)
\[ \varphi_{m+n}(x) = m!n! \sum_{r=0}^{\min(m,n)} \frac{(-1)^r}{(m-r)!(n-r)!r!} \varphi_{m-r}(x)\varphi_{n-r}(x). \]  
(1.12.13)
\[ \varphi_m(x)\varphi_n(x) = m!n! \sum_{r=0}^{\min(m,n)} \frac{\varphi_{m+n-2r}(x)}{(m-r)!(n-r)!r!}. \]  
(1.12.14)
By interchanging $r$ with $n-r$ and set $e^{-\frac{1}{2}x^2}y$ for $y$, he obtained

$$H_n(x) = \sum_{r=0}^{n} \binom{n}{r} \varphi_r(x) \varphi_{n-r}(x). \quad (1.12.15)$$

Finally he obtained the following identities between operators immediately deducible from (1.12.3) and (1.12.12)

$$D^n y = \sum_{r=0}^{n} (-1)^r \binom{n}{r} H_r(x)(D+2x)^{n-r}y. \quad (1.12.16)$$

$$D^n y = \sum_{r=0}^{n} (-1)^r \binom{n}{r} \varphi_r(x)(D+x)^{n-r}y. \quad (1.12.17)$$

with the adjoint identities

$$D^n y = \sum_{r=0}^{n} \binom{n}{r} (D-2x)^{n-r}H_r(x)y. \quad (1.12.18)$$

$$D^n y = \sum_{r=0}^{n} \binom{n}{r} (D-x)^{n-r}\varphi_r(x)y. \quad (1.12.19)$$

Results similar to (1.12.2), (1.12.3), (1.12.7), (1.12.10), (1.12.11), (1.12.12), (1.12.13), (1.12.14), (1.12.15), (1.12.16), (1.12.17), (1.12.18) and (1.12.19) have been obtained for Hermite polynomials of several variables using the above technique by Burchnall, J.L.

### 1.13 Bedient Polynomials of Several Variables

Khan, M.A., Khan, A.H. and Ahmad, N. studied the Bedient polynomials and extend it in two, three and $m$-variables. They defined the Bedient polynomials of two variables as given below:

$$R_n(\beta, \gamma, \lambda, \mu; x, y) = \frac{(\beta + \lambda)_n (2xy)^n}{n!} \times \sum_{r=0}^{\left[\frac{n}{2}\right]} \sum_{s=0}^{\left[\frac{n-r}{2}\right]} \frac{(-n)_{2r+2s} (\gamma - \beta)_r (\mu - \lambda)_s}{2^{2r+2s} (1 - \beta - \lambda - n)_{r+s} (\gamma)_r (\mu)_s r! s!} x^{2r} y^{2s}. \quad (1.13.1)$$

$$G_n(\alpha, \beta, \lambda, \mu; x, y) = \frac{(\alpha + \lambda)_n (\beta + \mu)_n (2xy)^n}{(\alpha + \beta + \lambda + \mu)_n n!}$$
In terms of double hypergeometric function, Bedient polynomials of two variables can be written as

\[
R_n(\beta, \gamma, \lambda, \mu; x, y) = \frac{(\beta + \lambda)_n (2xy)^n}{n!} \times \, \, F^{2:0}_{1:2:0} \left[ \begin{array}{c} -\frac{n}{2}, -\frac{n}{2} + \frac{1}{2} : \gamma - \beta, \mu - \lambda; -; \\ 1 - \beta - \lambda - n : \gamma, \mu; -; \frac{1}{x^2}, \frac{1}{y^2} \end{array} \right]. \tag{1.13.3}
\]

\[
G_n(\alpha, \beta, \lambda, \mu; x, y) = \frac{(\alpha + \lambda)_n (\beta + \mu)_n (2xy)^n}{(\alpha + \beta + \lambda + \mu)_n n!} \times \, \, F^{3:0:0}_{2:0:0} \left[ \begin{array}{c} -\frac{n}{2}, -\frac{n}{2} + \frac{1}{2}, 1 - \alpha - \beta - \lambda - \mu - n : -; -; \\ 1 - \alpha - \lambda - n, 1 - \beta - \mu - n : -; -; \frac{1}{x^2}, \frac{1}{y^2} \end{array} \right]. \tag{1.13.4}
\]

Now, the Bedient polynomials of three variables, defined and studied by Khan, M.A., Khan, A.H. and Ahmad, N., is as follows:

\[
R_n(\beta, \gamma, \lambda, \mu, \delta, \nu; x, y, z) = \frac{(\beta + \lambda + \delta)_n (2xyz)^n}{n!} \sum_{p=0}^{[\frac{\nu}{2}]} \sum_{q=0}^{[\frac{\mu - q}{2}]} \sum_{r=0}^{(-n)_{2p+2q+2r}} \frac{(-n)_{2p+2q+2r}}{p! q! r!} \times \frac{(\gamma - \beta)_p (\mu - \lambda)_q (\nu - \delta)_r}{2^{2p+2q+2r}(1 - \beta - \lambda - \delta - n)_{p+q+r} (\gamma)_p (\mu)_q (\nu)_r x^{2p} y^{2q} z^{2r}}. \tag{1.13.5}
\]

In terms of triple hypergeometric function, Bedient polynomials of three variables can be written as

\[
R_n(\beta, \gamma, \lambda, \mu, \delta, \nu; x, y, z) = \frac{(\beta + \lambda + \delta)_n (2xyz)^n}{n!} \times \, \, F^{(3)} \left[ \begin{array}{c} -\frac{n}{2}, -\frac{n}{2} + \frac{1}{2} : \gamma - \beta, \mu - \lambda, \nu - \delta; -; -; \\ 1 - \beta - \lambda - \delta - n : \gamma, \mu, \nu; -; -; \frac{1}{x^2}, \frac{1}{y^2}, \frac{1}{z^2} \end{array} \right]. \tag{1.13.6}
\]

Also, they defined the Bedient polynomials of \(m\)-variables as follows:

\[
R_n(\alpha_1, \beta_1, \alpha_2, \beta_2, \ldots, \alpha_m, \beta_m; x_1, x_2, \ldots, x_m) = \frac{(\alpha_1 + \alpha_2 + \cdots + \alpha_m)_n}{n!}
\]
\[
\times \left\{2(x_1 x_2 \cdots x_m)\right\}^n \sum_{r_1=0}^{[{n/2}]} \sum_{r_2=0}^{[n-r_1]} \cdots \sum_{r_m=0}^{[n-r_1-r_2-\cdots-r_{m-1}]} \frac{(n)_{2r_1+2r_2+\cdots+2r_m}}{2^{2r_1+2r_2+\cdots+2r_m} r_1!r_2! \cdots r_m!} \\
\times \frac{(\beta_1 - \alpha_1) r_1 (\beta_2 - \alpha_2) r_2 \cdots (\beta_m - \alpha_m) r_m}{(1 - \alpha_1 - \alpha_2 - \cdots - \alpha_m - n) r_1 + r_2 + \cdots + r_m} (\beta_1)_{r_1} (\beta_2)_{r_2} \cdots (\beta_m)_{r_m} x_1^{2r_1} x_2^{2r_2} \cdots x_m^{2r_m}.
\]

\[
(1.13.7)
\]

According to the definition (1.3.17), in terms of hypergeometric function, Bedient polynomials of \(m\)-variables can be written as

\[
R_n(\alpha_1, \beta_1, \alpha_2, \beta_2, \cdots, \alpha_m, \beta_m; x_1, x_2, \cdots, x_m) = \frac{(\alpha_1 + \alpha_2 + \cdots + \alpha_m)_n}{n!}
\times \left\{2(x_1 x_2 \cdots x_m)\right\}^n \, _2F_1^{2:1} \left[ \begin{array}{c}
-\frac{n}{2}, -\frac{n}{2} + \frac{1}{2}; \\
1 - \alpha_1 - \alpha_2 - \cdots - \alpha_m - n; \\
\beta_m - \alpha_m;
\end{array} \right]
\begin{array}{c}
1 - \alpha_1 - \alpha_2 - \cdots - \alpha_m - n; \\
\beta_1; \\
\beta_2; \\
\cdots;
\end{array}
\begin{array}{c}
\frac{1}{x_1^2}; \\
\frac{1}{x_2^2}; \\
\cdots;
\end{array}
\left[ \begin{array}{c}
\frac{1}{x_m^2}.
\end{array} \right] 
\]

\[
(1.13.8)
\]

where for right hand side of (1.13.8), it may be recalled that the definition of Kampe de Feriet function of \(m\)-variables [19].