Chapter 8

Growth of Entire Functions of Several Complex Variables

8.1 Introduction

If \( \nu : \mathbb{C}^2 \rightarrow \mathbb{R}^+ = [0, \infty[ \), be a real-valued function such that the following conditions hold:

(i) \( \nu(z + z') \leq \nu(z) + \nu(z') \quad \forall \, z, z' \in \mathbb{C}^2 \),

(ii) \( \nu(\lambda z) \leq |\lambda| \nu(z) \quad \forall \, \lambda \in \mathbb{C} \)

(iii) \( \nu(z) = 0 \iff z = 0 \), then \( \nu \) is a norm.

Let \( f(z_1, z_2) = \sum_{m,n=0}^{\infty} P_{m,n}(z_1, z_2) \), be the Taylor series expansion of \( f(z_1, z_2) \) in terms of homogeneous polynomials \( P_{m,n}(z_1, z_2) : \mathbb{C}^2 \to \mathbb{C} \), of degree \((m + n)\), then

\[
M(r_1, r_2) = \sup_{\nu(z) \leq r} |f(z_1, z_2)|, \quad t = 1, 2, \quad r = \max(r_1, r_2),
\]

is the maximum modulus of \( f(z_1, z_2) \), \( \forall \, r_1, r_2 \in \mathbb{R}^+ \) with respect to the norm \( \nu \).

Define

\[
C_{m,n} = \sup_{\nu(z) \leq 1} |P_{m,n}(z_1, z_2)|.
\]

The order, lower order and type of the function are defined respectively by

\[
\rho = \lim_{r_1, r_2 \to \infty} \sup_{\nu(z) \leq 1} \frac{\log \log M(r_1, r_2)}{\log(r_1, r_2)}
\]

\[
\lambda = \lim_{r_1, r_2 \to \infty} \inf_{\nu(z) \leq 1} \frac{\log \log M(r_1, r_2)}{\log(r_1, r_2)}
\]

\[
\rho = \lim_{r_1, r_2 \to \infty} \sup_{\nu(z) \leq 1} \frac{\log \log M(r_1, r_2)}{\log(r_1, r_2)}
\]
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\[ T = \lim_{r_1, r_2 \to \infty} \sup \frac{\log M(r_1, r_2)}{\inf (r_1^\rho + r_2^\rho)}. \]

Kumar et al. [49], proved the following results

\[ \rho = \limsup_{m+n \to \infty} \frac{\log[(m+n)\alpha_{m+n}]}{\log(C_{m,n})^{-1/m+n}}, \quad (8.1.1) \]

\[ e\rho T = \limsup_{m+n \to \infty} \frac{(m+n)\alpha_{m,n}}{(C_{m,n})^{-\rho/m+n}}, \quad (8.1.2) \]

where

\[ \alpha_{m,n} = \begin{cases} \frac{[m^n n^m]^{1/m+n}}{(m+n)} & \text{if } m,n \geq 1 \\ 0 & \text{if } m,n = 0. \end{cases} \]

Analogously, the lower order and lower type are defined by

\[ \lambda = \liminf_{m+n \to \infty} \frac{\log[(m+n)\alpha_{m,n}]}{\log(C_{m,n})^{-1/m+n}}, \quad (8.1.3) \]

\[ e\lambda T = \liminf_{m+n \to \infty} \frac{(m+n)\alpha_{m,n}}{(C_{m,n})^{-\rho/m+n}}. \quad (8.1.4) \]

In this chapter we generalized and improved the results of Srivastava and Kumar [86], Dalal [18]. Surprisingly, they have not mention the factor \( \alpha_{m,n} \) in their results. Here, we have defined the orders and types different from those of above. To reduce the mechanical labour we have considered only two variables, though the results can easily be extended to several complex variables.

### 8.2 Orders, Lower Orders And Homogeneous Polynomials Expansion of Entire Functions

In this section, we will study some relations between finite orders, nonzero lower orders and homogeneous polynomial expansion of entire functions of two complex variables.
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Theorem 8.2.1. \( f_i(z_1, z_2) = \sum_{m,n=0}^{\infty} P_{m,n}^{(i)}(z_1, z_2) \), where \( i = 1, \ldots, k \), be \( k \) entire functions of finite orders \( \rho_1, \rho_2, \ldots, \rho_k \), and nonzero lower orders \( \lambda_1, \lambda_2, \ldots, \lambda_k \), respectively. Then the function

\[
f_i(z_1, z_2) = \sum_{m,n=0}^{\infty} P_{m,n}(z_1, z_2),
\]

where

\[
C_{m,n} \sim \prod_{i=1}^{k} \left( C_{m,n}^{(i)} \right)^{m_i}, \quad m_i \text{ are constant}
\]
is an entire function such that

\[
\sum_{i=1}^{k-1} \frac{m_i}{\rho_i} \leq \left\{ \left( \frac{1}{\rho} - \frac{m_k}{\rho_k} \right), \left( \frac{1}{\lambda} - \frac{m_k}{\lambda_k} \right) \right\} \leq \sum_{i=1}^{k-1} \frac{m_i}{\lambda_i} \quad (8.2.1)
\]

Proof. In can be easily seen (Lelong et al. [54, pp.9]) that necessary and sufficient condition for \( f(z_1, z_2) \) to represent an entire function of two complex variables \( z_1 \) and \( z_2 \) is

\[
\lim_{m,n \to \infty} \sup \left( C_{m,n} \right)^{1/m+n} = 0.
\]

Since \( f_i(z_1, z_2) \) are entire functions, so it leads to

\[
\lim_{m,n \to \infty} \sup \left( C_{m,n}^{(i)} \right)^{1/m+n} = 0, \quad \text{for } i = 1, \ldots, k.
\]

Also

\[
C_{m,n} \sim \prod_{i=1}^{k} \left( C_{m,n}^{(i)} \right)^{m_i},
\]

which gives

\[
\lim_{m,n \to \infty} \sup \left( \frac{C_{m,n}}{\alpha_{m,n}} \right)^{1/m+n} \leq \prod_{i=1}^{k} \lim_{m,n \to \infty} \left[ \left( \frac{C_{m,n}^{(i)}}{\alpha_{m,n}^{(i)}} \right)^{m_i} \right]^{1/m+n} \quad (8.2.2)
\]

Hence \( f(z_1, z_2) \) is an entire function.
Now applying (8.1.1) and (8.1.3) for functions $f_i(z_1, z_2)$, we get

$$\lim_{m+n \to \infty} \sup \frac{\log[(m + n)\alpha_{m,n}^{(i)}]}{\log(C_{m,n}^{(i)})^{-1/m+n}} = \frac{\rho_i}{\lambda_i},$$

$$\log(C_{m,n}^{(i)})^{m_i} < -m_i(m + n) \log((m + n)\alpha_{m,n}) \left(\frac{1}{\rho_i + \varepsilon}\right) \quad \text{for} \quad m + n > m_0 + n_0, \quad (8.2.3)$$

and

$$\log(C_{m,n}^{(i)})^{m_i} < \frac{-m_i(m + n) \log((m + n)\alpha_{m,n})}{(\lambda_i + \varepsilon)}, \quad (8.2.4)$$

for an infinite sequence of values of $m, n$. Taking $i = 1, \ldots, k$, in (8.2.3) and adding, we get

$$\log \prod_{i=1}^{k} (C_{m,n}^{(i)})^{m_i} < -(m + n) \log \left((m + n)\alpha_{m,n}\right) \sum_{i=1}^{k} \frac{m_i}{\rho_i + \varepsilon}. \quad \text{Using (8.2.2), we have}$$

$$\log C_{m,n} < -(m + n) \log \left((m + n)\alpha_{m,n}\right) \sum_{i=1}^{k} \frac{m_i}{\rho_i + \varepsilon}. \quad \text{or}$$

$$\limsup_{m+n \to \infty} \frac{\log[(m + n)\alpha_{m+n}]}{\log(C_{m,n})^{-1/m+n}} < \sum_{i=1}^{k} \left(\frac{m_i}{\rho_i}\right)^{-1} \quad \text{or}$$

$$\rho \leq \left(\sum_{i=1}^{k} \frac{m_i}{\rho_i}\right)^{-1} \quad \text{or}$$

$$\sum_{i=0}^{k-1} \frac{m_i}{\rho_i} \leq \frac{1}{\rho} - \frac{m_k}{\rho_k},$$

which proves a part of the left hand side of (8.2.1).

In the same way by taking $i = 1, \ldots, k - 1$ in (8.2.3) and $i = k$ in (8.2.4) and adding, we obtain
\[ \sum_{i=0}^{k-1} \frac{m_i}{\rho_i} \leq \frac{1}{\lambda} - \frac{m_k}{\lambda_k}, \]

which proves the second part of left hand side of (8.2.1).

Similarly The right hand side can be proved in the similar way, which completes the proof of the theorem.

### 8.3 Types And Lower Types

In this section we shall study the relation between types, lower types and homogeneous polynomial expansion of entire functions of two complex variables.

**Theorem 8.3.1.** \( f_i(z_1, z_2) = \sum_{m,n=0}^{\infty} P_{m,n}^{(i)}(z_1, z_2), \) where \( i = 1, \ldots, k, \) be \( k \) entire functions of finite nonzero orders \( \rho_1, \rho_2, \ldots, \rho_k, \) and types \( T_1, T_2, \ldots, T_k, \) and lower type \( t_1, t_2, \ldots, t_k \) respectively \((0 < t_i < T_i < \infty) \) \( (i = 1, \ldots, k) \). Then the function

\[ f(z_1, z_2) = \sum_{m,n=0}^{\infty} P_{m,n}(z_1, z_2), \]

is an entire function such that

\[ \prod_{i=1}^{k-1} (\rho_i t_i)^{m_i/\rho_i} \leq \left\{ \frac{(\rho t)^{1/\rho}}{(\rho_k T_k)^{m_k/\rho_k}} \right\} \leq \prod_{i=1}^{k-1} (\rho_i T_i)^{m_i/\rho_i}, \quad (8.3.1) \]

where \( \rho, T \) and \( t \) are order, type and lower type of \( f(z_1, z_2) \) respectively.

**Proof.** Using (8.1.2) and (8.1.4) for functions \( f_i(z_1, z_2) \), we have

\[ (C_{m,n}^{(i)})^{m_i} < \left[ \frac{[e \rho_i (T_i + \varepsilon)]^{m+n}}{((m+n)(\alpha_{m,n}))^{m+n}} \right]^{m_i/\rho_i}, \quad \text{for } m+n > m_0 + n_0, \quad (8.3.2) \]

and

\[ (C_{m,n}^{(i)})^{m_i} < \left[ \frac{[e \rho_i (t_i + \varepsilon)]^{m+n}}{((m+n)(\alpha_{m,n}))^{m+n}} \right]^{m_i/\rho_i}, \quad (8.3.3) \]

for an infinite sequence of values of \( m, n \). Taking \( i = 1, \ldots, k \) in (8.3.2) and multiplying, we have
\[
\prod_{i=1}^{k} (C_{m,n}^{(i)})^{m_i} < \prod_{i=1}^{k} \left[ \frac{\left[ e^{\rho_i(T_i + \varepsilon)} \right]^{m+n}}{((m+n)(\alpha_{m,n}))^{m+n}} \right]^{m_i/\rho_i}.
\]

Using (8.2.2), we get

\[
C_{m,n} < \left\{ \left[ \frac{e^{\rho_k(T_k + \varepsilon)}}{(m+n)\alpha_{m,n}} \right]^{m+n} \right\}^{m_k/\rho_k} \prod_{i=1}^{k-1} \left\{ \left[ \frac{e^{\rho_i(T_i + \varepsilon)}}{(m+n)\alpha_{m,n}} \right]^{m+n} \right\}^{m_i/\rho_i}
\]
or

\[
(C_{m,n})^{\rho/m+n} < \left\{ \left[ \frac{e^{\rho_k(T_k + \varepsilon)}}{(m+n)\alpha_{m,n}} \right]^\rho \right\}^{m_k/\rho_k} \prod_{i=1}^{k-1} \left\{ \left[ \frac{e^{\rho_i(T_i + \varepsilon)}}{(m+n)\alpha_{m,n}} \right]^\rho \right\}^{m_i/\rho_i}.
\]

Using the fact that if any \(k\), \(L\)’s out of \(L, L_1, \ldots, L_k\), are equal to one, then all the \((k+1)\), \(L\)’s are equal to one and

\[
\frac{1}{\rho} = \sum_{i=1}^{k-1} \frac{m_i}{\rho_i}, \quad L = \frac{\rho}{\lambda}, \quad L_i = \frac{\rho_i}{\lambda_i}.
\]

We get

\[
\frac{(m+n)\alpha_{m,n}}{(C_{m,n})^{-\rho/m+n}} < e \left\{ \left[ \rho_k(T_k + \varepsilon) \right]^{\rho} \right\}^{m_k/\rho_k} \prod_{i=1}^{k-1} \left\{ \left[ \rho_i(T_i + \varepsilon) \right]^{\rho} \right\}^{m_i/\rho_i}.
\]

In view of (8.1.2), we get

\[
\frac{(\rho T)^{1/\rho}}{(\rho_k T_k)^{m_k/\rho_k}} \leq \prod_{i=1}^{k-1} (\rho_i T_i)^{m_i/\rho_i}.
\] \hspace{1cm} (8.3.4)

Now taking \(i = 1, \ldots, k-1\) in (8.3.2) and \(i = k\) in (8.3.3) and multiplying, we get

\[
\frac{(\rho t)^{1/\rho}}{(\rho_k t_k)^{m_k/\rho_k}} \leq \prod_{i=1}^{k-1} (\rho_i T_i)^{m_i/\rho_i}.
\] \hspace{1cm} (8.3.5)

(8.3.4) and (8.3.5) together proves the left hand side of (8.3.1). Similarly, the right hand side can be proved. Hence the proof of Theorem 8.3.1 is complete.