CHAPTER 3

SYSTEM OF QUASI-VARIATIONAL INEQUALITY PROBLEMS

3.1. INTRODUCTION

One of the efficient numerical techniques for solving variational inequality problems in Hilbert spaces is the projection method and its variant forms. Since the standard projection method strictly defined on the inner product property of Hilbert spaces, it can no longer be applied for variational inequality problems in Banach spaces. This fact motivates us to develop alternative method to study iterative algorithms for approximating the solutions of variational inequality problems in Banach spaces.

In 2004, Verma [126] studied the convergence analysis of a iterative algorithm for approximating the solution of system of variational inequality problems involving relaxed cocoercive mappings in Hilbert spaces. Since then, many authors developed and studied different iterative methods for various classes of variational inequality problems and systems of variational inequality problems involving relaxed cocoercive mappings in Hilbert spaces, see for example [60,61,98,100,101].

In this chapter, we consider a system of general quasi-variational inequality problems (in short, SGQVIP) in uniformly smooth Banach space. Further, using retraction method, we prove the existence of a unique solution for SGQVIP. Furthermore, a Mann-type partial implicit iterative algorithm is proposed for SGQVIP and discussed its convergence analysis. The iterative algorithms and results presented here improve the iterative algorithms and results given in [60,61,126].

The remaining part of this chapter is organized as follows:

In Section 3.2, we formulate SGQVIP in uniformly smooth Banach space and discuss some of its special cases. Further we show that SGQVIP is equivalent to a system of relations. Furthermore, we show some relations between relaxed $(\nu, r)$-cocoercive, expansive, strongly accretive and relaxed accretive mappings.

In Section 3.3, we prove the existence and uniqueness of solution of SGQVIP.
In Section 3.4, we suggest a Mann-type partial implicit for finding the approximate solution of SGQVIP and discuss its convergence analysis.

3.2. PRELIMINARIES AND FORMULATION OF PROBLEM

Throughout the chapter, we assume that $E$ is a real uniformly smooth Banach space with its dual space $E^*$.

Let $K : E \times E \to 2^E$ be a set-valued mapping such that for each $(x, y) \in E \times E$, $K(x, y)$ be a nonempty, closed and convex set in $E$. Let $g : E \to E$ be a single-valued mapping and $F, G : E \times E \to E$ be nonlinear mappings. We consider the problem of finding $x^*, y^* \in E$ with $g(x^*), g(y^*) \in K(x^*, y^*) \cap K(y^*, x^*) \neq \emptyset$ such that

$$
\begin{align*}
\langle \rho_1 F(y^*, x^*) + g(x^*) - g(y^*), J(z_1 - g(x^*)) \rangle & \geq 0, \quad \forall z_1 \in K(y^*, x^*), \\
\langle \rho_2 G(x^*, y^*) + g(y^*) - g(x^*), J(z_2 - g(y^*)) \rangle & \geq 0, \quad \forall z_2 \in K(x^*, y^*),
\end{align*}
$$

(3.2.1)

where $J : E \to E^*$ is a normalized duality mapping. Problem (3.2.1) is called a system of general quasi-variational inequality problems (SGQVIP).

Special cases of SGQVIP (3.2.1):

1. If $g \equiv I$, identity mapping and $E = H$, Hilbert space, then SGQVIP (3.2.1) reduces to a system of quasi-variational inequality problems of finding $x^*, y^* \in K(x^*, y^*) \cap K(y^*, x^*)(\neq \emptyset)$ such that

$$
\begin{align*}
\langle \rho_1 F(y^*, x^*) + x^* - y^*, z_1 - x^* \rangle & \geq 0, \quad \forall z_1 \in K(y^*, x^*), \\
\langle \rho_2 G(x^*, y^*) + y^* - x^*, z_2 - y^* \rangle & \geq 0, \quad \forall z_2 \in K(x^*, y^*),
\end{align*}
$$

which is the correct form of the system (1)-(2) studied by Noor and Huang [101]. We remark that to find the solution $(x^*, y^*)$ of system (1)-(2) [101] is not equivalent to find $(x^*, y^*)$ such that

$$
\begin{align*}
x^* & = P_{K(y^*, x^*)}[y^* - \rho_1 F(y^*, x^*)], \\
y^* & = P_{K(x^*, y^*)}[x^* - \rho_2 G(x^*, y^*)],
\end{align*}
$$

unless $x^*, y^* \in K(y^*, x^*) \cap K(x^*, y^*)(\neq \emptyset)$.

2. If $F = G = T$, SGQVIP (3.2.1) is equivalent to finding $x^*, y^* \in E$ with $g(x^*), g(y^*) \in K(x^*, y^*) \cap K(y^*, x^*)(\neq \emptyset)$ such that

$$
\begin{align*}
\langle \rho_1 T(y^*, x^*) + g(x^*) - g(y^*), J(z_1 - g(x^*)) \rangle & \geq 0, \quad \forall z_1 \in K(y^*, x^*), \\
\langle \rho_1 T(x^*, y^*) + g(y^*) - g(x^*), J(z_2 - g(y^*)) \rangle & \geq 0, \quad \forall z_2 \in K(x^*, y^*),
\end{align*}
$$

(3.2.2)
which appears to be new one.

3. If \( K(x^*, y^*) = K(y^*, x^*) = K \), a nonempty, closed and convex set in \( E \), then 

SGQVIP (3.2.1) reduces to the following system of variational inequality problems of finding \( x^*, y^* \in K \) such that

\[
\langle \rho_1 F(y^*, x^*) + g(x^*) - g(y^*), J(z - g(x^*)) \rangle \geq 0, \quad \forall z \in K,
\]

\[
\langle \rho_2 G(x^*, y^*) + g(y^*) - g(x^*), J(z - g(y^*)) \rangle \geq 0, \quad \forall z \in K.
\]

For appropriate and suitable choice of mappings \( F, G \) and the set-valued mapping \( K \), one can obtain a number of new and previously known problems from the SGQVIP (3.2.1) as special cases.

Next, we have the following technical lemma.

**Lemma 3.2.1.** \( (x^*, y^*) \in E \times E \) with \( g(x^*), g(y^*) \in K(x^*, y^*) \cap K(y^*, x^*) \), is a solution of SGQVIP (3.1) if and only if \( (x^*, y^*) \) satisfies the relations

\[
g(x^*) = P_{K(y^*, x^*)}[g(y^*) - \rho_1 F(y^*, x^*)],
\]

\[
g(y^*) = P_{K(x^*, y^*)}[g(x^*) - \rho_2 G(x^*, y^*)],
\]

where \( \rho_1, \rho_2 > 0 \) are constants.

Finally, we give some relations between relaxed \((\nu, r)\)-cocoercive, expansive, strongly accretive and relaxed accretive mappings.

**Lemma 3.2.2.** If the mapping \( T : E \to E \) is relaxed \((\gamma, r)\)-cocoercive then \( T \) is \( \left( \frac{2r - 1}{1 + 2\gamma} \right)^{1/2} \)-expansive for \( r > \frac{1}{2} \).

**Proof.** Since \( T \) is relaxed \((\gamma, r)\)-cocoercive mapping then there exist constants \( r, \gamma > 0 \) such that

\[
-\gamma\|Tx - Ty\|^2 + r\|x - y\|^2 \leq \langle Tx - Ty, J(x - y) \rangle
\]

\[
\leq \frac{1}{2}\left(\|Tx - Ty\|^2 + \|x - y\|^2 \right),
\]

where we have used inequality \( 2ab \leq a^2 + b^2 \) for real numbers \( a, b \).
Hence, we have

\[(1 + 2\gamma)\|Tx - Ty\|^2 \geq (2r - 1)\|x - y\|^2,\]

or

\[\|Tx - Ty\| \geq \left(\frac{2r - 1}{1 + 2\gamma}\right)^{1/2}\|x - y\|, \text{ for } r > \frac{1}{2}.\]

This completes the proof.

**Lemma 3.2.3.** Let the mapping \( T : E \to E \) be relaxed \((\gamma, r)\)-cocoercive and \(\beta\)-Lipschitz continuous.

(i) If \( \gamma/\beta^2 < r \) then \( T \) is \((r - \gamma\beta^2)\)-strongly accretive;

(ii) If \( r < \gamma/\beta^2 \) then \( T \) is \((r - \gamma\beta^2)\)-relaxed accretive.

**Proof.** Since \( T \) is relaxed \((\gamma, r)\)-cocoercive and \(\beta\)-Lipschitz continuous, we have

\[\langle Tx - Ty, J(x - y) \rangle \geq -\gamma\|Tx - Ty\|^2 + r\|x - y\|^2 \]

\[\geq -\gamma\beta^2\|x - y\|^2 + r\|x - y\|^2 \]

\[= (r - \gamma\beta^2)\|x - y\|^2.\]

This completes the proof.

**Remark 3.2.1.**

(i) In the spirit of Lemma 3.2.3(i), Theorem 3.1 [98], Theorem 3.1 [100,101], Theorem 2.1 [61], and Theorem 3.1 [60] are actually for variational inequality problems for strongly monotone mappings in Hilbert space.

(ii) It is easily observed from Lemma 3.2.3 that \( |\beta - \frac{1}{\gamma}| \geq \frac{\sqrt{4r\gamma - 1}}{2\gamma} \) and \( \gamma r > \frac{1}{4} \).

### 3.3. EXISTENCE AND UNIQUENESS OF SOLUTION

First, we define the following concepts.
Definition 3.3.1. Let $g : E \to E$ be a nonlinear mapping. A mapping $F : E \times E \to E$ is said to be:

(i) **$\alpha$-strongly accretive with respect to $g$ in the first argument** if there exists a constant $\alpha > 0$ such that

\[ \langle F(x_1, y_1) - F(x_2, y_2), J(g(x_1) - g(x_2)) \rangle \geq \alpha \|x_1 - x_2\|^2, \quad \forall x_1, x_2, y_1, y_2 \in E; \]

(ii) **$\beta$-Lipschitz continuous in the first argument** if there exists a constant $\beta > 0$ such that

\[ \|F(x_1, y_1) - F(x_2, y_2)\| \leq \beta \|x_1 - x_2\|, \quad \forall x_1, x_2, y_1, y_2 \in E. \]

We remark that if $g$ is $\mu$-Lipschitz continuous, then $\alpha \leq \mu \beta$.

Now, we prove the existence and uniqueness of solution for SGQVIP(3.2.1).

Theorem 3.3.1. Let $E$ be a uniformly smooth Banach space with $\rho_E(t) \leq ct^2$ for some $c > 0$; $K$ be a nonempty, closed and convex set in $E$ and let the mapping $g : E \to E$ be $\sigma$-strongly accretive and $\mu$-Lipschitz continuous. Let $F, G : K \times K \to E$ be two mappings such that $F$ be $\alpha_1$-strongly accretive with respect to $g$ and $\beta_1$-Lipschitz continuous in the first argument and $G$ be $\alpha_2$-strongly accretive with respect to $g$ and $\beta_2$-Lipschitz continuous in the first argument. Suppose that there is a constant $\lambda > 0$ such that

\[ \|P_K(x_1, y_1)(x) - P_K(x_2, y_2)(x)\| \leq \lambda \|x_1 - x_2\|, \quad \forall (x_1, y_1), (x_2, y_2) \in E \times E, x \in E, \tag{3.3.1} \]

and $\rho_1, \rho_2 > 0$ satisfy the following conditions:

\[ k_1 := \theta + \theta_2 < 1; \quad k_2 := \theta + \theta_1 < 1 \quad \text{and} \quad \theta + \sqrt{\theta_1} < 1, \tag{3.3.2} \]

where $\theta := \sqrt{(1 - 2\sigma + 64c\mu^2)}$; $\theta_1 := \sqrt{(\mu^2 - 2\rho_1\alpha_1 + 64c\rho_1^2\beta_1^2) + \lambda}$;

\[ \theta_2 := \sqrt{(\mu^2 - 2\rho_2\alpha_2 + 64c\rho_2^2\beta_2^2) + \lambda}. \]

Then SGQVIP (3.2.1) has a unique solution.
Proof. For \( (x, y) \in E \times E \), define a mapping \( Q : E \times E \to E \times E \) by
\[
Q(x, y) = (T(x, y), S(x, y)), \quad \forall (x, y) \in E \times E,
\]
where \( T, S : E \times E \to E \) are defined by
\[
T(x, y) = x - g(x) + P_{K(y, x)}[g(y) - \rho_1 F(y, x)],
\]
and
\[
S(x, y) = y - g(y) + P_{K(x, y)}[g(x) - \rho_2 G(x, y)],
\]
where \( \rho_1, \rho_2 > 0 \) are some constants.

For any \( (x_1, y_1), (x_2, y_2) \in E \times E \), it follows that from (3.3.1) and (3.3.4) that
\[
\|T(x_1, y_1) - T(x_2, y_2)\| \leq \|x_1 - x_2 - (g(x_1) - g(x_2))\| + \|P_{K(y_1, x_1)}[g(y_1) - \rho_1 F(y_1, x_1)] - P_{K(y_2, x_2)}[g(y_2) - \rho_1 F(y_2, x_2)]\|
\]
\[
\leq \|x_1 - x_2 - (g(x_1) - g(x_2))\| + \|P_{K(y_1, x_1)}[g(y_1) - \rho_1 F(y_1, x_1)] - P_{K(y_2, x_2)}[g(y_1) - \rho_1 F(y_1, x_1)]\|
\]
\[
+ \|P_{K(y_2, x_2)}[g(y_1) - \rho_1 F(y_1, x_1)] - P_{K(y_2, x_2)}[g(y_2) - \rho_1 F(y_2, x_2)]\|
\]
\[
\leq \|x_1 - x_2 - (g(x_1) - g(x_2))\| + \lambda \|y_1 - y_2\| + \|g(y_1) - g(y_2) - \rho_1(F(y_1, x_1) - F(y_2, x_2))\|. \tag{3.3.6}
\]
and similarly,
\[
\|S(x_1, y_1) - S(x_2, y_2)\| \leq \|y_1 - y_2 - (g(y_1) - g(y_2))\| + \lambda \|x_1 - x_2\| + \|g(x_1) - g(x_2) - \rho_2(G(x_1, y_1) - G(x_2, y_2))\|. \tag{3.3.7}
\]

Since \( g \) is \( \sigma \)-strongly accretive and \( \mu \)-Lipschitz continuous, by using Lemma 1.2.2, we have
\[
\|x_1 - x_2 - (g(x_1) - g(x_2))\|^2 \leq \|x_1 - x_2\|^2 - 2\langle g(x_1) - g(x_2), J(x_1 - x_2 - (g(x_1) - g(x_2))) \rangle
\]
\[
\leq \|x_1 - x_2\|^2 - 2\langle g(x_1) - g(x_2), J(x_1 - x_2) \rangle
\]
\[
-2\langle g(x_1) - g(x_2), J(x_1 - x_2 - (g(x_1) - g(x_2))) - J(x_1 - x_2) \rangle
\]
\[
\leq \|x_1 - x_2\|^2 - 2\sigma \|x_1 - x_2\|^2 + 64c \mu^2 \|x_1 - x_2\|^2
\]
\[
\leq (1 - 2\sigma + 64c \mu^2) \|x_1 - x_2\|^2. \tag{3.3.8}
\]
Similarly, we have
\[
\|y_1 - y_2 - (g(y_1) - g(y_2))\|^2 \leq (1 - 2\sigma + 64c\mu^2)\|y_1 - y_2\|^2. \tag{3.3.9}
\]

Since $F$ is $\alpha_1$-strongly accretive with respect to $g$ in the first argument and $\beta_1$-Lipschitz continuous in the first argument, and $G$ is $\alpha_2$-strongly accretive with respect to $g$ in the first argument and $\beta_2$-Lipschitz continuous in the first argument, we have
\[
\|g(y_1) - g(y_2) - \rho_1(F(y_1, x_1) - F(y_2, x_2))\|^2 \\
\leq \|g(y_1) - g(y_2)\|^2 - 2\rho_1\langle F(y_1, x_1) - F(y_2, x_2), J(g(y_1) - g(y_2)) \rangle \\
- 2\rho_1\langle F(y_1, x_1) - F(y_2, x_2), J(g(y_1) - g(y_2)) - \rho_1(F(y_1, x_1) - F(y_2, x_2)) \rangle - J(g(y_1) - g(y_2)) \\
\leq \mu^2\|y_1 - y_2\|^2 - 2\rho_1\alpha_1\|y_1 - y_2\|^2 + 64c\rho_1^2\beta_1^2\|y_1 - y_2\|^2 \\
\leq (\mu^2 - 2\rho_1\alpha_1 + 64c\rho_1^2\beta_1^2)\|y_1 - y_2\|^2,
\]
which implies
\[
\|g(y_1) - g(y_2) - \rho_1(F(y_1, x_1) - F(y_2, x_2))\| \leq \sqrt{(\mu^2 - 2\rho_1\alpha_1 + 64c\rho_1^2\beta_1^2)}\|y_1 - y_2\|. \tag{3.3.10}
\]

Similarly, we have
\[
\|g(x_1) - g(x_2) - \rho_2(G(x_1, y_1) - G(x_2, y_2))\| \leq \sqrt{(\mu^2 - 2\rho_2\alpha_2 + 64c\rho_2^2\beta_2^2)}\|x_1 - x_2\|. \tag{3.3.11}
\]

From (3.3.6), (3.3.8) and (3.3.10), we have
\[
\|T(x_1, y_1) - T(x_2, y_2)\| \\
\leq \sqrt{(1 - 2\sigma + 64c\mu^2)}\|x_1 - x_2\| + \left(\sqrt{(\mu^2 - 2\rho_1\alpha_1 + 64c\rho_1^2\beta_1^2) + \lambda}\right)\|y_1 - y_2\|. \tag{3.3.12}
\]

Also, from (3.3.7), (3.3.9) and (3.3.11), we have
\[
\|S(x_1, y_1) - S(x_2, y_2)\| \\
\leq \sqrt{(1 - 2\sigma + 64c\mu^2)}\|y_1 - y_2\| + \left(\sqrt{(\mu^2 - 2\rho_2\alpha_2 + 64c\rho_2^2\beta_2^2) + \lambda}\right)\|x_1 - x_2\|. \tag{3.3.13}
\]
From (3.3.12) and (3.3.13), we have

\[
\|T(x_1, y_1) - T(x_2, y_2)\| + \|S(x_1, y_1) - S(x_2, y_2)\| \\
\leq k_1\|x_1 - x_2\| + k_2\|y_1 - y_2\| \\
\leq \max\{k_1, k_2\}\left(\|x_1 - x_2\| + \|y_1 - y_2\|\right),
\]

(3.3.14)

where

\[
k_1 := \theta + \theta_2, \\
k_2 := \theta + \theta_1,
\]

(3.3.15)

\[
\theta := \sqrt{(1 - 2\sigma + 64c_2^2)}, \quad \theta_1 := \sqrt{(\mu^2 - 2\rho_1\alpha_1 + 64c_1^2\beta_1^2)} + \lambda, \\
\theta_2 := \sqrt{(\mu^2 - 2\rho_2\alpha_2 + 64c_2^2\beta_2^2)} + \lambda.
\]

(3.3.16)

Now, define the norm \(\| \cdot \|_*\) on \(E \times E\) by

\[
\|(x, y)\|_* = \|x\| + \|y\|, \forall (x, y) \in E \times E.
\]

(3.3.17)

We can easily observe that \((E \times E, \| \cdot \|_*)\) is a Banach space. Hence, it follows from (3.3.3), (3.3.14) and (3.3.17) that

\[
\|Q(x_1, y_1) - Q(x_2, y_2)\|_* \leq \max\{k_1, k_2\}\|(x_1, y_1) - (x_2, y_2)\|_*. 
\]

(3.3.18)

Since \(\max\{k_1, k_2\} < 1\) by condition (3.3.2), it follows from (3.3.18) that \(Q\) is contraction mapping. Hence, by Banach contraction principle (Theorem 1.2.16), there exists a unique \((x, y) \in E \times E\) such that \(Q(x, y) = (x, y)\), which implies that

\[
g(x) = P_{K(y,x)}[g(y) - \rho_1 F(y, x)], \\
g(y) = P_{K(x,y)}[g(x) - \rho_2 G(x, y)].
\]

It follows from Lemma 3.2.1, that \((x, y)\) is the unique solution of SGQVIP (3.2.1). This completes the proof.
3.4. ITERATIVE ALGORITHM AND CONVERGENCE ANALYSIS

In this section, we suggest the fixed-point formulation for SGQVIP (3.2.1), see Lemma 3.2.1, and Theorem 3.3.1 are very important from the numerical approximation point of view and help to suggest the following iterative algorithm for SGQVIP (3.2.1).

Mann-type partially implicit iterative algorithm (in short, MTPIIA) 3.4.1.

For given \((x_0, y_0) \in E \times E\), compute an approximate solution \((x_n, y_n)\) given by iterative schemes:

\[
\begin{align*}
x_{n+1} &= (1 - a_n)x_n + a_n[x_n - g(x_n) + P_{K(y_n,x_n)}(g(y_n) - \rho_1 F(y_n, x_n))], \\
y_{n+1} &= (1 - b_n)x_{n+1} + b_n[y_{n+1} - g(y_{n+1}) + P_{K(x_{n+1},y_n)}(g(x_{n+1}) - \rho_2 G(x_{n+1}, y_n))],
\end{align*}
\]

where \(a_n, b_n > 0\) are constants and \(a_n, b_n \in (0, 1]\), \(n \geq 0\) with \(\sum_{n=0}^{\infty} a_n = \infty\) and \(\lim_{n \to \infty} b_n = 1\).

Some special cases of MTPIIA 3.4.1:

1. If \(E \equiv H\) and \(g \equiv I\), MTPIIA 3.4.1 reduces to the following iterative algorithm. For given \((x_0, y_0) \in H \times H\), compute an approximate solution \((x_n, y_n)\) given by

\[
\begin{align*}
x_{n+1} &= (1 - a_n)x_n + a_n P_{K(y_n,x_n)}(y_n - \rho_1 F(y_n, x_n)), \\
y_{n+1} &= (1 - b_n)x_{n+1} + b_n P_{K(x_{n+1},y_n)}(x_{n+1} - \rho_2 G(x_{n+1}, y_n)),
\end{align*}
\]

which is suggested by Noor and Huang [101].

2. If \(K(x, y) \equiv K(y, x) \equiv K\), the nonempty, closed and convex set in \(E\) and \(g \equiv I\), MTPIIA 3.4.1 reduces to the following iterative algorithm. For given \((x_0, y_0) \in E \times E\), compute an approximate solution \((x_n, y_n)\) given by

\[
\begin{align*}
x_{n+1} &= (1 - a_n)x_n + a_n P_{K}(y_n - \rho_1 F(y_n, x_n)), \\
y_{n+1} &= (1 - b_n)x_{n+1} + b_n P_{K}(x_{n+1} - \rho_2 G(x_{n+1}, y_n)),
\end{align*}
\]

where \(a_n, b_n \in [0, 1]\) \(\forall n \geq 0\).
3. If $a_n \equiv b_n \equiv 1$; $K(y, x) = K(x, y) \equiv K$ and $g \equiv I$, MTPIIA 3.4.1 reduces to the following iterative algorithm. For given $(x_0, y_0) \in E \times E$, compute an approximate solution $(x_n, y_n)$ given by

$$x_{n+1} = P_K(y_n - \rho_1 F(y_n, x_n)),$$

$$y_{n+1} = P_K(x_{n+1} - \rho_2 G(x_{n+1}, y_n)).$$

Finally, we discuss the convergence criteria for MTPIIA 3.4.1.

**Theorem 3.4.1.** Let the mappings $E, F, G, g$ be same as in Theorem 3.3.1 and let conditions (3.3.1) and (3.3.2) of Theorem 3.3.1 hold. Then approximate solution $(x_n, y_n)$ generated by MTPIIA 3.4.1, converges strongly to the unique solution $(x, y)$ of SGQVIP(3.2.1).

**Proof.** It follows from Theorem 3.3.1 that SGQVIP (3.2.1) has the unique solution $(x, y) \in E$. Hence, by Lemma 3.2.1, we have

$$x = (1 - a_n)x + a_n[x - g(x) + P_{K(y, x)}(g(y) - \rho_1 F(y, x))], \quad (3.4.3)$$

$$y = (1 - b_n)y + b_n[y - g(y) + P_{K(x, y)}(g(x) - \rho_2 G(x, y))]. \quad (3.4.4)$$

From (3.3.1), (3.4.1) and (3.4.3), we have

$$\|x_{n+1} - x\| \leq (1 - a_n)\|x_n - x\| + a_n\|x_n - x - (g(x_n) - g(x))\|$$

$$+ a_n\|P_{K(y_n, x_n)}(g(y_n) - \rho_1 F(y_n, x_n)) - P_{K(y, x)}(g(y) - \rho_1 F(y, x))\|$$

$$\leq (1 - a_n)\|x_n - x\| + a_n\|x_n - x - (g(x_n) - g(x))\|$$

$$+ a_n\|P_{K(y_n, x_n)}(g(y_n) - \rho_1 F(y_n, x_n)) - P_{K(y, x)}(g(y) - \rho_1 F(y, x))\|$$

$$+ a_n\|P_{K(y_n, x_n)}(g(y_n) - \rho_1 F(y_n, x_n)) - P_{K(y_n, x_n)}(g(y_n) - \rho_1 F(y_n, x_n))\|$$

$$\leq (1 - a_n)\|x_n - x\| + a_n\|x_n - x - (g(x_n) - g(x))\|$$

$$+ a_n\|g(y_n) - g(y) - \rho_1 F(y_n, x_n) - F(y, x)\| + a_n\lambda\|y_n - y\|. \quad (3.4.5)$$

Since $g$ is $\sigma$-strongly accretive and $\mu$-Lipschitz continuous, using Lemma 1.2.2, we have

$$\|x_n - x - (g(x_n) - g(x))\|^2 \leq (1 - 2\sigma + 64\mu^2)\|x_n - x\|^2. \quad (3.4.6)$$
Also, $F$ is $\alpha_1$-strongly accretive with respect to $g$ in the first argument and $\beta_1$-Lipschitz continuous in the first argument, we have

$$
\|g(y_n) - g(y) - \rho_1(F(y_n, x_n) - F(y, x))\| \leq (\mu^2 - 2\rho_1\alpha_1 + 64\rho_1^2\beta_1^2)\|y_n - y\|^2.
$$

(3.4.7)

From (3.4.5), (3.4.6) and (3.4.7), it follows that

$$
\|x_{n+1} - x\|
\leq (1 - a_n)\|x_n - x\| + a_n\sqrt{(1 - 2\sigma + 64\rho^2\mu^2)}\|x_n - x\|
+ a_n\sqrt{(\mu^2 - 2\rho_1\alpha_1 + 64\rho_1^2\beta_1^2)}\|y_n - y\|
+ a_n\lambda\|y_n - y\|
\leq (1 - a_n)\|x_n - x\| + a_n\lambda\|y_n - y\|
= (1 - a_n\theta)\|x_n - x\| + a_n\theta\|y_n - y\|.
$$

(3.4.8)

Now, from (3.3.1), (3.4.2) and (3.4.4), we have

$$
\|y_{n+1} - y\|
\leq (1 - b_n)\|x_{n+1} - x\| + (1 - b_n)\|y - x\| + b_n\|y_{n+1} - y - (g(y_{n+1}) - g(y))\|
+ b_n\|g(x_{n+1}) - g(x) - \rho_2(G(x_{n+1}, y_n) - G(x, y))\|
+ b_n\lambda\|x_{n+1} - x\|
$$

(3.4.9)

Since $g$ is $\sigma$-strongly accretive and $\mu$-Lipschitz continuous, using Lemma 1.2.2, we have

$$
\|y_n - y - (g(y_n), g(y))\|^2 \leq (1 - 2\sigma + 64\rho^2\mu^2)\|y_n - y\|^2.
$$

(3.4.10)

Since $G$ is $\alpha_2$-strongly accretive with respect to $g$ in the first argument and $\beta_2$-Lipschitz continuous in the first argument, we have

$$
\|g(x_{n+1}) - g(x) - \rho_2(G(x_{n+1}, y_n) - G(x, y))\|^2 \leq (\mu^2 - 2\rho_2\alpha_2 + 64\rho_2^2\beta_2^2)\|x_{n+1} - x\|^2.
$$

(3.4.11)

From (3.4.6), (3.4.9), (3.4.10) and (3.4.11) it follows that
\[\|y_{n+1} - y\| \leq (1-b_n)\|x_{n+1} - x\| + (1-b_n)\|y-x\| + b_n\sqrt{(1-2\sigma + 64c\mu^2)}\|y_{n+1} - y\| + b_n\sqrt{\mu^2 - 2\rho_2\alpha_2 + 64c\rho_2^2\beta_2^2}\|x_{n+1} - x\| + b_n\lambda\|x_{n+1} - x\| \leq \left(1-b_n\left(1-\sqrt{\mu^2 - 2\rho_2\alpha_2 + 64c\rho_2^2\beta_2^2} + \lambda\right)\right)\|x_{n+1} - x\| + b_n\sqrt{(1-2\sigma + 64c\mu^2)}\|y_{n+1} - y\| + (1-b_n)\|y-x\| = (1-b_n(1-\theta_2))\|x_{n+1} - x\| + b_n\theta\|y_{n+1} - y\| + (1-b_n)\|y-x\| \leq \|x_{n+1} - x\| + \theta\|y_{n+1} - y\| + (1-b_n)\|y-x\|, \quad (3.4.12)\]

where \(\theta_2 := \sqrt{\mu^2 - 2\rho_2\alpha_2 + 64c\rho_2^2\beta_2^2} + \lambda \leq 1; b_n \in (0,1].\)

From (3.4.12), we have
\[
\|y_{n+1} - y\| \leq \frac{1}{(1-\theta)}\left(\|x_{n+1} - x\| + (1-b_n)\|y-x\|\right). \quad (3.4.13)
\]

Combining (3.4.8) and (3.4.13), we have
\[
\|x_{n+1} - x\| \leq (1-a_n(1-\theta))\|x_n - x\| + a_n\theta_1\left\{\frac{1}{(1-\theta)}\left(\|x_n - x\| + (1-b_{n-1})\|y-x\|\right)\right\} \leq \left[1-a_n\left(1-\left[\theta + \frac{\theta_1}{1-\theta}\right]\right)\right]\|x_n - x\| + a_n\frac{\theta_1(1-b_{n-1})}{1-\theta}\|y-x\|. \quad (3.4.14)
\]

By condition (3.3.2) it follows that \(1 - \left(\theta + \theta_1\right)\) is in \((0,1],\) and \(\sum_{n=0}^{\infty} a_n\left(1 - \left(\theta + \frac{\theta_1}{1-\theta}\right)\right) = \infty,\) and \(\sum_{n=0}^{\infty} a_n\frac{\theta_1(1-b_{n-1})}{1-\theta}\) is in \((0,1],\) and then by Lemma 1.2.5, it follows that \(\lim_{n \to \infty} \|x_n - x\| = 0.\) Further, the result \(\lim_{n \to \infty} \|y_n - y\| = 0\) follows from (3.4.13) and condition \(\lim_{n \to \infty} b_n = 1.\) This completes the proof.

**Remark 3.4.1.** The proof of theorems presented in this chapter for SGQVIP(3.2.1) under the assumption of relaxed cocoercivity on mappings \(F\) and \(G\) need further research effort.