ACKNOWLEDGEMENT

All praise is to Allah alone, the almighty, the beneficent and the omnipotent who showed me the path and blessed with the patience and strength to embark upon his work, without his blessing nothing could be done. I bow my head before Him.

I feel emotionally charged, however delighted to say that I was, indeed, fortunate enough to become disciple of highly sincere, a dedicated teacher as well as a researcher Dr. Kaleem Raza Kazmi, Associate Professor, Department of Mathematics, Aligarh Muslim University, Aligarh under whose valuable guidance, inspiring attitude, constant encouragement and kind supervision, I am academically shaped as now and this work could have been accomplished. At this moment, it is too difficult to express in words (what I whole heartedly want) my most profound regards, heartiest gratefulness and high admirations to him.

With pleasure, I am especially thanks to Prof. Mumtaz Ahmad Khan, Chairman, Department of Applied Mathematics, Aligarh Muslim University, Aligarh, for providing me various departmental facilities and creating a congenial atmosphere to carry out this work. His deep knowledge and enthusiasm for the subject was a continuous source of inspiration to me and no words can truly convey my gratitude.

I am also great thankful to Prof. Afzal Beg, Chairman Department of Mathematics, Aligarh Muslim University, Aligarh, for his inspiration and encouragement which provided the continuing impetus for this work.

I express my deep sense of gratitude to Dr. (Mrs.) Subuhi Khan, Associate Professor, Department of Mathematics, Aligarh Muslim University, Aligarh for her worthwhile suggestions, continuous encouragement during the hard days of this work. I owe my profound gratitude for her.

I am very thankful to Prof. Syed Mohammad Amin, Department of Urdu, Aligarh Muslim University, Aligarh for his encouragement and moral support.

I wish to express my thanks to my colleagues, seniors and friends of department especially to Dr. Faizan Ahmad Khan, Dr. Mohd. Iqbal Bhat, Dr. Suhel Ahmad Khan, Dr. Naeem Ahmad, Dr. Javid Ali, Shuja Haider, Mohd. Furkan, Mohd. Ghayasuddin, Kashif Ali and Sanjeev Gupta for their continuous encouragement,
productive discussions, moral support and influenced the development during the
course of this work.

At this juncture, I do not want to fail in remembering to all my friends Feeroz,
Arif Husain, Zameer Ahmad, Mohammad Humayun Khan, Kamran Sabeel, Kaleem
Ali Rizvi, Mohammad Mehdi and Nafees Ahmad, who through their deep affection,
constant encouragement, loving advise, moral support and as a mentor shaped my
life to this stage. Acknowledging this, I express my sincere gratitude to their for all
help. I also wish to extend my gratefulness to my friends who encouraged and cared
for me all the time.

At this moment, I specially remember my parents and their devotion for me at
every step of my life and all my relatives whom I missed a lot during the course of
the work. I express my profound love to my loving brothers Mohammad Shabab
Anwer, Mohammad Shoeb Anwer and to my sisters Asia Khatoon, Rabia Khatoon
for their most cordial and helpful role in my academic endeavour. Their countless
blessings, deep love and affection always worked as a hidden power behind every
thing I did.

Financial support in the from of University Grant Commission (U.G.C.) an-
nounced in latter UGC No. 19-33/2006(CU) dated 1.2.2007, New Delhi, India needs
also to be acknowledged.

Last but not the least, I thank to the staff of Department of Applied Math-
ematics, Aligarh Muslim University, Aligarh, especially office staff for their nice
co-operation attitude.

All the above cited acknowledgements belong to those who some how played a
pivotal role along with this work. I can not forget the dandy strives of my unde-
scribed mates who supported me morally and mentally.

October 19, 2011

(Mohammad Shahzad)
PREFACE

The present thesis entitled “Solvability of some classes of nonlinear variational inequalities” is an outcome of the studies made by the author at the Department of Applied Mathematics and Department of Mathematics, Aligarh Muslim University, Aligarh, India during the last four years.

Theory of variational inequalities was initiated independently by G. Stampacchia [121] and G. Fichera [47] in the early 1960’s. Since then variational inequalities have been extended and generalized in several directions using novel and innovative techniques both for their own sake and for applications. It has been shown that variational inequality theory provides the natural, direct, unified and efficient framework for the general treatment of a wide class of unrelated linear and nonlinear problems arising in fluid flow through porous media, elasticity, transformation, economics, optimization, structural analysis, applied and engineering sciences, see for example [4-8,11,12, 26-28,30,40,41,48-51,55,88,104,109].

One of the thrust areas in the theory of variational inequalities to study the existence theory and to develop iterative methods which provide efficient iterative algorithms for solving new classes of variational inequalities. In recent past, a number of iterative methods has been developed for various classes of variational inequalities, see for example [1,2,13,18-24,32-36,38,39,42-46,53,56-63,67-73,90-102,106, 110,122-130,132,137-140].

The main objective of this thesis is to study the existence of solutions and to discuss the convergence analysis of iterative algorithms for some new classes of variational inequalities, system of variational inequalities and variational inclusions. This work generalizes, improves and unifies the concepts, techniques and results given by many authors for the various classes of variational inequalities and variational inclusions, see for example [38,42-44,60,61,68,70,83,99,106,122,126,132,137,139].

The thesis consists of six chapters.

In Chapter 1, we review the notations, definitions and results which are used in the presentation of the work of the subsequent chapters. Further, we give brief introduction of some classes of variational inequalities, variational inclusions, system of variational inequalities and system of variational inclusions.
In Chapter 2, we give the concepts of partially relaxed strongly mixed monotone and regularized partially relaxed strongly $\theta$-pseudomonotone mappings, which are extensions of the concepts given by Xia and Ding [132], Noor [99] and Kazmi et al. [71]. Further, we use the auxiliary principle technique to suggest a two step iterative algorithm for approximating the solution of regularized (nonconvex) generalized mixed variational inequality problem. We prove that the convergence of the iterative algorithm requires only the continuity, partially relaxed strongly mixed monotonicity and partially relaxed strongly $\theta$-pseudomonotonicity.

In Chapter 3, we consider a system of general quasi-variational inequality problems (in short, SGQVIP) in uniformly smooth Banach space. Further, using retraction method, we prove the existence of a unique solution for SGQVIP. Furthermore, a Mann-type partial implicit iterative algorithm is proposed for SGQVIP and discussed its convergence analysis.

In Chapter 4, we consider a class of accretive mappings called generalized $H(\cdot, \cdot)$-accretive mappings, an natural generalization of accretive (monotone) mappings studied in [42-44,83,122,137,139], in Banach spaces. We prove that the proximal mapping of the generalized $H(\cdot, \cdot)$-accretive mapping is single-valued and Lipschitz continuous. Further, we consider a system of generalized variational inclusions involving generalized $H(\cdot, \cdot)$-accretive mappings in real $q$-uniformly smooth Banach spaces. Using proximal mapping method, we prove the existence and uniqueness of solution and suggest an iterative algorithm for the system of generalized variational inclusions. Furthermore, we discuss the convergence criteria of the iterative algorithm under some suitable conditions.

In Chapter 5, we introduce a class of proximal mappings for $H(\cdot, \cdot)$-accretive mappings in $q$-uniformly smooth Banach space, which is a natural and important generalization of classes of proximal mappings given by Sun et al. [122] and Zou and Huang [139]. We study some properties of this new class of proximal mappings. Further, we consider a system of implicit variational inclusions (in short, SIVI). Using proximal mapping technique, we prove that SIVI is equivalent to a system of implicit Wiener-Hopf equations (in short, SIWHE) in $q$-uniformly smooth Banach spaces. Further, using this equivalence and Nadler’s technique [87], we suggest an iterative algorithm for SIWHE. Furthermore, we prove the existence of solution of SIVI through the convergence analysis of the iterative algorithm for SIWHE.
In Chapter 6, we give the notion of $M$-$\eta$-proximal mapping for a nonconvex, proper, lower semicontinuous and subdifferentiable functional on Banach space and prove its existence and Lipschitz continuity in the setting of reflexive Banach space. Further, we consider a new system of generalized implicit variational-like inclusions in the setting of Banach spaces. Using the concept of $M$-$\eta$-proximal mapping, we establish an equivalence between the system of generalized implicit variational-like inclusions and a new system of implicit equations in the setting of reflexive Banach spaces. Using this equivalence, we prove the existence of solution of the system of generalized implicit variational-like inclusions in the setting of uniformly smooth Banach spaces. Further, we construct an iterative algorithm for the system of implicit variational-like inclusions. Furthermore, we discuss its convergence criteria in the setting of uniformly smooth Banach spaces.

In the end, we give a comprehensive list of references of books, monographs, edited volumes and research papers used in carrying out this research work.

Five papers based on the research work contained in this thesis have been communicated for publication in national/international journals. Three of them which contain the work of Chapter 2, Chapter 3 and Chapter 4 have been published in journals entitled “Bulletin of Korean Mathematical Society”, “Thai Journal of Mathematics” and “Applied Mathematics and Computation”, respectively.
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