CHAPTER I

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Boundary value problems for ordinary differential equations of second and higher order involving a parameter in the differential equation are considered in this thesis.

i) Determination of the range of parameter-values for which the solution exists (or does not exist) and

ii) the behaviour of the solution with respect to changing parameter values, changing boundary values or changing domain (called together data of the problem)

are the main aspects studied in this thesis.

1.1 Occurrence:

Such boundary value problems naturally occur in many branches of science, engineering and technology such as mechanics, heat conduction, fluid dynamics etc. They involve a physical parameter in the differential equation or boundary values or the domain, which govern the behaviour of the solution.

To quote a very simple example, the vertical projectile motion of a particle of mass ‘m’ thrown vertically upwards to cover a given distance ‘d’ in a given interval of time ‘T’ under a given resistance dependent on the speed of the particle, is governed by

\[ mu'' = -H(u') \]
\[ u(0) = 0, \quad u(T) = d. \]

Here the problem is governed by three physical parameters: the mass ‘m’ occurring in the differential equation, the distance ‘d’ occurring in the boundary
values and the time interval ‘T’ specifying the domain. These three together constitute the data of the problem.

Some sophisticated problems of this type such as steady state heat conduction in a wire or a fin, thermistor problem, heat transfer beneath oceanic rises etc. are given in chapter VI of the thesis, while discussing the applications of the theory.

1.2 Examples of existence or non-existence:

In general a prototype boundary value problem of this type is

\[ u'' = H(t, u, u', \alpha) \]  \hspace{1cm} (1.1)
\[ u(a) = \gamma_1, \; u(b) = \gamma_2 \]  \hspace{1cm} (1.2)

Such a problem involving a parameter \( \alpha \) may have a unique solution for certain values of \( \alpha \), more than one solutions for certain values of \( \alpha \) and no solution for some other values of \( \alpha \). So it is interesting to determine these values of the parameter. For instance the boundary value problem

\[ u'' = u^{2+} \alpha^2 \]
\[ u(0) = 0, \; u(1) = 0 \]

has a solution for \( \alpha = 1 \) but no solution for \( \alpha = \pi \).

1.2.1 For the eigen value problem:

\[ u'' + \lambda^2 u = 0 \]
\[ u(0) = 0, \; u(1) = 1 \]

the solution exists for all values of \( \lambda \) except the values \( \lambda = \pm n \pi, \; n = 1, 2, \ldots \).

More general version of this is a Sturm-Liouville problem.
\[ u'' + g(t) u' + [h(t) + \lambda \kappa(t)] u = 0, \quad a < t < b \]
\[-u(a) \cos \theta + u'(a) \sin \theta = 0 \]
\[u'(b) \cos \phi + u(b) \sin \phi = 0.\]

It is known that a solution exists only for a discrete set of eigen values \(\lambda_1, \ldots, \lambda_n, \ldots\), and there exists the smallest eigen value \(\lambda_1\) such that all eigen values are \(\geq \lambda_1\). These well studied problems are therefore excluded from this thesis.

1.2.2 Singular perturbation problems:

Deshpande and Kasture, [8], have considered a singular perturbation problem
\[ \varepsilon u'' + F(t, u, u') = 0 \]
\[u(a) = \gamma_1, \quad u(b) = \gamma_2,\]
where \(\varepsilon\) is a parameter multiplying the highest derivative. The set \(S\) of all values of the parameter \(\varepsilon\) in \((0, \infty)\) for which the solution exists is called the parameter set. If \(S \neq (0, \infty)\), then there are values of \(\varepsilon\) in \((0, \infty)\) which are not in \(S\). These values of \(\varepsilon\) give rise to problems inimical to singular perturbation theory (SPT), while the problem is favourable to SPT for all \(\varepsilon \in S\); provided \(S\) includes a right neighbourhood of zero. For our simple example in § 1.1, it is shown by Deshpande and Kasture, [8], that if \(F\) has a superquadratic growth, then \(S\) excludes a right neighbourhood of zero. Thus if \(H(u') = u^3\), then in the simple example quoted in § 1, a particle of arbitrary small mass \(m (= \varepsilon)\) cannot cover a distance 'd' in a given interval 'T' of time, however large its initial velocity may be.

Thus existence of a solution is not assured for every value of a parameter \(\alpha\) in the differential equation. A further study of the existence theory for parameter dependent boundary value problems is therefore essential.
1.3 Monotonicity:

Another important problem is the monotonicity of a solution $u(t, \alpha)$ with respect to $\alpha$ or with respect to the boundary values. The problem of monotonicity with respect to $\alpha$, has been studied for elliptic boundary value problems by Kasture [24].

In general the monotonicity with respect to $\alpha$ is not obvious even when the solution is explicitly known. For example for the boundary value problem

$$u'' = -\alpha^2 u$$
$$u(0) = 0, \quad u\left(\frac{\pi}{2}\right) = 1$$

the solution is given by

$$u(t, \alpha) = \csc \frac{\alpha \pi}{2} \sin \alpha t$$

We see that even for such a linear problem, it is not easy to decide the monotonicity of a solution with respect to $\alpha$.

1.4 Problems in this thesis:

We discuss the existence of a solution, continuous dependence on parameter and boundary values and monotonicity of a solution with respect to boundary values and parameter for the boundary value problem (1.1), (1.2). Further we have discussed the applicability of our results especially monotonicity to some real world problems.
1.5 Review of earlier results:

In chapter II we briefly review the earlier work regarding the existence of a solution, its continuous dependence on α or γ₁ or γ₂ and differentiability with respect to α, γ₁, γ₂, focusing the attention on those results which are used in this thesis.

To prove the existence using the method of topological transversality [17], we require bounds on a solution and its derivative. Therefore sufficient conditions for boundedness of solution and its derivative for boundary value problems of second or higher order, which do not involve a parameter, are briefly reviewed in this chapter. These results are extended to parameter dependent boundary value problems, subsequently.

The essential results included in chapter II are some results on differential inequalities [28], Nagumo condition [4], barrier strip condition [25], and Bernstein-Nagumo condition [17].

Theorems on continuous dependence and differentiability by Ehme, [12], Ingram, [22] and Gaines, [16], which are used in subsequent chapters are stated in chapter II. Further, for differentiability with respect to a parameter studied in chapter V, we used the results of Pontryagin, [27], for initial value problems, which are also stated there.

1.6 Parameter multiplying the highest derivative:

In chapter III we mainly discuss the existence of a solution of a boundary value problem in which a parameter ε multiplies the highest derivative. A second order boundary value problem of this type is

\[ \varepsilon \, u'' = H(t, u, u') \]
\[ u(a) = \gamma_1, \quad u(b) = \gamma_2 \]

We are interested in the parameter set S – the set of all values of ε for which a solution exists; with an emphasis on conditions under which \( S = (0, \infty) \).
The techniques that we adopt are the disconjugacy condition, Bailey et al [2], the topological transversality theorem, Granas et al, [17], shooting method and modified function technique. These results are extended to nth order boundary value problems using the topological method given in [1].

Existence theorems are developed in this chapter for those $H$ in equation (1.3), which satisfy a Lipschitz condition or differential inequalities with Nagumo condition or Bernstein-Nagumo condition with

$$uH(t, u, u') \geq 0.$$  

**1.6.1 Left or right injectivity:**

The parameter set $S$ is said to satisfy the right injectivity property, if it is non-empty [10], and

$$\varepsilon_0 \in S \Rightarrow \varepsilon \in S \text{ for } \forall \varepsilon > \varepsilon_0.$$  

Conditions under which this property holds for a boundary value problem of nth order, $n \geq 2$, are obtained in this chapter. Similarly the left injectivity can be defined as follows.

$S$ is said to be left injective, if it is non-empty and

$$\varepsilon_0 \in S \Rightarrow (0, \varepsilon_0] \subseteq S.$$  

**1.7 General parameter dependent boundary value problems:**

Study of chapter III is continued in chapter IV for theorems on existence of a solution for a general boundary value problem (1.1), (1.2) containing a parameter $\alpha$, using the topological transversality technique of Granas et al, [17], with some necessary changes. Such theorems are also developed for nth order boundary value problems.
The conditions under which the existence of a solution for $\alpha = \alpha_0$ implies existence of a solution for all $\alpha > \alpha_0$, called the property of right injectivity, are obtained in this chapter IV for second or higher order boundary value problems.

A special existence theory is developed for a quasilinear boundary value problem using Schauder's fixed point theorem and the method of Dorr, Parter and Shampine, [11]. Sufficient conditions required for this, namely, uniform boundedness of solution and its derivative, are given.

1.8 Differentiability and Monotonicity:

Chapter V is devoted for the discussion on the important aspect of a solution of a boundary value problem, that is the monotonicity with respect to boundary values or the parameter, using the results of Ehme, [12] and Ingram, [22].

This aspect of monotonicity is also studied for a quasi-linear boundary value problem, where differentiability is assumed.

The monotonicity results with respect to a boundary value $\gamma_1$ are obtained assuming some monotonicity conditions on $H$ in (1.1). The results are extended to nth order boundary value problems.

Existence and differentiability of a solution with respect to a parameter is proved for all $\alpha \in [\alpha_1, \alpha_2]$, by using Ingram's result [22].

Continuous dependence on parameter has been proved in this chapter without using the strong uniqueness condition of Ingram, [22]. In another theorem proved here, strong uniqueness condition is assumed for a variational boundary value problem. At the end of this chapter we prove a result about monotonicity with respect to the domain, without using differentiability, for a non-linear boundary value problem. Some numerical examples are given for illustration.
1.9 Applications:

Applications of our results to the real world problems are discussed in chapter VI. They include, the thermistor problem, Zhou & Westbrook, [32], Fowler et al, [15], and some problems of mechanics.

1.10 Concluding remarks:

Thus various aspects of the theory of boundary value problems for second or higher order ordinary differential equations involving a parameter in the differential equation and in boundary values are studied in this thesis. This is achieved by developing some new existence theorems, by obtaining some criteria for continuous dependence on parameters and differentiability to them and then by obtaining some criteria for monotonicity with respect to parameters. Applications of our results to some important realistic problems are also discussed in the last chapter. Since the parameters represent some physical quantities in many problems of science and technology, such studies are essential for a proper development of these branches on sound theoretical footings.