5 Design of 1 Dimensional Linear Phase FIR Filter with Orthogonal Polynomials

5.1 Introduction

To design linear phase FIR filters, various techniques have been discussed by several authors. Some of them, like Remez exchange algorithm [18] [6], are generally used for efficient implementation of linear phase FIR filters, but as the number of conditions grow the computational time increases exponentially.

Linear programming (LP), semi-infinite programming (SIP) and iterative re-weighted least square (IRLS) algorithms are generally used to design filters with constraints. Liang et al. [7] used LP technique to design some filters with constraints, like optimal Nyquist filters, partial response filters, etc. but this method consumes a great deal of CPU time. Potchinkov [8] applied SIP technique for constrained filters but the time requirements were still high. IRLS algorithms by Burrus et al [39] and many others have good computational efficiency. But they are bad when convergence is considered, because it is not guaranteed that they will converge.

Some techniques to design FIR filter are discussed by various authors, where either the filter has good magnitude response but poor phase characteristics or if they possess exact linear phase then magnitude characteristics are poor or they can not be controlled completely. Chan and Tsui [40] discusses an approach where the group delay is not constant and consequently the phase is not linear. Hanna [41] discusses a method where as bandwidth of the filter increases, the level of sidebands increases and also when the cutoff frequency goes far from the origin, pass band characteristic becomes less flat. These requirements motivate us to discuss a new technique to design an arbitrary magnitude response and exact linear phase FIR filter using a very simple technique.
5.2 Preliminaries

A sequence of orthogonal polynomials with weight $W(x)$ satisfies the relation

$$\left\langle f_i(x), f_j(x) \right\rangle = \int_{x_1}^{x_2} f_i(x)f_j(x)W(x)dx = 0 \quad i \neq j$$  \hspace{1cm} (5.1)

where, $f_i(x)$ and $f_j(x)$ are any two members of the orthogonal set, and $(\bullet, \bullet)$ represents the inner product.

In the present discussion, we consider only Legendre polynomials as an example of orthogonal polynomials. Since they are the simplest orthogonal polynomials among all – generally used – yet the procedure is general in nature and one can use any of the other orthogonal polynomials and consider their interval of orthogonality.

For Legendre polynomials the interval of orthogonality is $[-1, 1]$ and weight function is $1$. We can generate the Legendre polynomials by using Rodrigues’ Formula [42] given by

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n$$  \hspace{1cm} (5.2)

where, $n=1, 2, 3, \ldots$. The first few polynomials are defined below:

\begin{align*}
P_0(x) &= 1 \quad \text{(5.3)} \\
P_1(x) &= x \quad \text{(5.4)} \\
P_2(x) &= \frac{3x^2 - 1}{2} \quad \text{(5.5)} \\
P_3(x) &= \frac{5x^3 - 3x}{2} \quad \text{(5.6)} \\
&\vdots
\end{align*}

These all are orthogonal to each other over $[-1, 1]$; that is

$$\int_{-1}^{1} P_m(x)P_n(x)dx = 0 \quad \text{whenever } m \neq n$$  \hspace{1cm} (5.7)

The plot of the first few Legendre polynomials is shown in Figure 5.1.
5.3 Procedure

Suppose user needs to design a low pass filter (LPF), whose magnitude response, $H(\omega)$, is shown in Figure 5.2\(^1\) where,

$A_{\text{max}}$ represents the amplitude of the pass band of the filter,

$A_{\text{min}}$ represents the amplitude of the stop band of the filter,

$\omega_{\text{CH}}$ is the end of pass band and the start of the transition band, and

$\omega_{\text{CL}}$ is the end of transition band and start of the stop band.

$H(\omega)$ (Figure 5.2) is first converted to an object function (independent variable $x$), shown in Figure 5.3. The mapping from Figure 5.2 to 5.3 is done by previously defined transformation (repeated below)

$$x = x_0\cos(\omega/2) \quad -\pi < \omega < \pi \quad (5.8)$$

The object function is then approximated using a linear combination of several Legendre polynomials\(^2\). Note that we approximate the object

\(^1\)Though here an LPF is considered, any other ideal magnitude characteristic may be used. The procedure is general in nature.

\(^2\)The proof that any object function can be approximated by a linear combination of
function with even Legendre polynomial terms only; that is, $P_0, P_2, P_4, \ldots$, since the characteristics of Figure 5.3 are symmetric.

Let us call the object function of Figure 5.3 as $f(x)$, and it is given by

$$f(x) = \sum_{n=0}^{\infty} a_{2n} P_{2n}(x) \quad (5.9)$$

where, $a_0, a_2, a_4, \ldots$ are coefficients which have to be multiplied with the corresponding Legendre polynomials to get the required characteristics, shown in Figure 5.3.

![Figure 5.2: Desired characteristics of low pass filter.](image)

Let us discuss the procedure to design the filter step by step.

**Step 1**: To calculate the coefficients $(a_0, a_1, a_2, \ldots)$ we obtain a set of equations by multiplying Equation (5.9) by $P_0(x), P_2(x), \ldots$ one by one and integrating over the interval $[-1, 1]$ \footnote{See Appendix A}. this leads to following array of equations

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See Appendix A
Figure 5.3: Object function, for filter characteristics of Figure 5.2, to be approximated using Legendre polynomials.

\[ 2 \int_0^1 f(x)P_0(x)dx = 2 \int_0^1 \left( \sum_{n=0}^{\infty} a_{2n} P_{2n}(x) \right) P_0(x)dx \quad (5.10) \]

\[ 2 \int_0^1 f(x)P_2(x)dx = 2 \int_0^1 \left( \sum_{n=0}^{\infty} a_{2n} P_{2n}(x) \right) P_2(x)dx \quad (5.11) \]

\[ 2 \int_0^1 f(x)P_4(x)dx = 2 \int_0^1 \left( \sum_{n=0}^{\infty} a_{2n} P_{2n}(x) \right) P_4(x)dx \quad (5.12) \]

\[ \vdots \]

The number of equations which we have to solve depends on how many coefficients we desire, or in other words how many Legendre polynomials we are going to use for the approximation of our object function, this point will be clear soon.

Using the orthogonality property Equations (5.10), (5.11), \ldots reduce to

\[ \int_0^1 f(x)P_0(x)dx = \int_0^1 a_0(P_0(x)P_0(x))^2dx \quad (5.13) \]
\[ \int_{0}^{1} f(x)P_2(x)dx = \int_{0}^{1} a_2(P_2(x)P_2(x))^2dx \quad (5.14) \]

From the above equations, it is clear that the coefficients \(a_0, a_2, \ldots\) are calculated directly by using

\[ a_i = \frac{\int_{0}^{1} f(x)P_i(x)dx}{\int_{0}^{1} P_i(x)P_i(x)dx} \quad (5.15) \]

where, \(i = 0, 2, 4, \ldots\)

**Step 2:** The coefficients are replaced in Equation (5.9), and the resulting approximate polynomial \(f_a(x)\) is calculated. Note that \(f_a(x)\) has a limited number of terms, up to \(N\), when compared with \(f(x)\) Equation (5.9); that is,

\[ f_a(x) = \sum_{n=0}^{N} a_{2n}P_{2n}(x) \quad (5.16) \]

**Step 3:** To transform from polynomial (or \(x\)) domain to frequency (or \(\omega\)) domain we use the transformation, Equation (5.8). We choose \(x_0\) to be 1, albeit one can choose any value.

**Step 4:** The zeros of \(f_a(x)\) are calculated using any standard routine and the zeros of \(f_a(\omega)\) are obtained by using

\[ x_i \xrightarrow{T} \omega_i \]

here \(x_i\)s are the zeros of \(f_a(x)\) and \(T\) is the transformation. The zeros of \(H(z)\) are \(z_i = \exp(j\omega_i)\), where \(i = 1, 2, 3, \ldots\)

**Step 5:** Using these zeros we calculate the transfer function in \(z\) domain; that is,

\[ H(z) = (z - z_1)(z - z_2)(z - z_3) \ldots \quad (5.17) \]

and replace \(z = e^{j\omega}\) to find \(H(\omega)\).
5.4 Application and Discussion

Let us design a filter with following values (refer Figure 5.2)

\[
A_{\text{max}} = 1000, \\
A_{\text{min}} = 0, \\
\omega_{\text{OL}} = 2.3186, \\
\omega_{\text{OH}} = 2.0007, \\
\omega_{\text{dB}} = 2.0944.
\]

Then, the "object function" characteristics are calculated to be

\[
f(x) = \begin{cases} 
0 & \text{if } 0 \leq x < 0.4 \\ 
7142.86(x - 0.4) & \text{if } 0.4 \leq x < 0.54 \\ 
1000 & \text{if } 0.54 \leq x < 1.0 
\end{cases} \tag{5.18}
\]

Let us consider some specific cases for above mentioned values.

5.4.1 Design 1

As we keep increasing the number of terms to approximate the object function we get better and better results. It is up to the designer, where he would like to stop. For example, when ten even Legendre polynomial terms are used to approximate the object function then

\[
f_a(x) = a_0 P_0 + a_2 P_2 + a_4 P_4 + a_6 P_6 + \ldots + a_{18} P_{18} \tag{5.19}
\]

where, \( f_a(x) \) is the approximation to \( f(x) \).

Using Equation (5.15) we calculate the coefficients \( a_0 \) to \( a_{18} \), and

\[
f_a(x) = 530 + 909.685 P_2 - 586.385 P_4 - 4.6429 P_6 + 401.108 P_8 + \ldots \tag{5.20}
\]

The approximated, together with ideal, object function is shown in Figure 5.4. It is clear from the figure that approximated object function follows
the user defined characteristics, but not very closely (in Design 2 we try to improve this).

Using the transformation described in Step 3, we transform $f_a(x)$ to frequency domain. Step 4 will give the position of the eighteen zeros in the frequency domain.

Following Step 5 we can write $H(z)$ as

$$H(z) = \prod_{i=1}^{18} (z - z_i) \tag{5.21}$$

where, $z_i$'s are $e^{j\omega}$; $i = 1, 2, \ldots 18$. This $H(z)$ gives the magnitude response shown in Figure 5.5 and is following the required filter characteristics, also shown in the figure. Magnitude response in dB is shown in Figure 5.6. If we look at it we found that the pass band is very linear and stop bands are approximately 25dB down from the pass band.

![Figure 5.4: Approximation of object function using first 10 Legendre polynomials terms, $P_0$ to $P_{18}$ (the object function here is not showing the non linearities for simplicity).](image)

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Figure 5.5: Magnitude response of the low pass FIR filter corresponding to the object function shown in Figure 5.4.

Figure 5.6: Magnitude response in dB of the low pass FIR filter corresponding to the object function shown in Figure 5.4.
5.4.2 Design 2

Let us approximate our object function of Equation (5.18) using 17 even Legendre polynomial terms. We get the magnitude response and the magnitude response in dB as shown in Figures 5.7 and 5.8, respectively. The object function and thus magnitude response are following the required characteristics more closely than in Design 1 (Figure 5.5), which shows that as we increase the number of terms to approximate our object function we get better approximation of magnitude response. The dB magnitude response (Figure 5.8) shows that the first side band is approximately 35 dB down; that is, there is an improvement of 10 dB over previous design. Figure 5.9 shows the phase response for the present design, the phase is exactly linear.

![Graph showing magnitude response with dB scale]

Figure 5.7: Magnitude response of low pass FIR filter when object function is approximated using 17 Legendre polynomial terms, $P_0$ to $P_{32}$.

5.4.3 Design 3

Following the above procedure we can easily design a high pass filter also. Let us consider the following values for the filter to be designed

$$A_{max} = 1000,$$
Figure 5.8: Magnitude response in dB of low pass FIR filter when object function is approximated using 17 Legendre polynomial terms, $P_0$ to $P_{32}$.

Figure 5.9: Phase response of low pass FIR filter when object function is approximated using 17 Legendre polynomial terms, $P_0$ to $P_{32}$. 
\[ A_{\min} = 0, \]
\[ \omega_{OH} = 2.3186, \]
\[ \omega_{OL} = 2.0007, \]
\[ \omega_{MB} = 2.0944. \]

Figures 5.10 and 5.11 show the magnitude response, and magnitude response in dB, respectively, of the high pass filter. The high pass filter is derived from the object function which is approximated using 21 Legendre polynomials.

Figure 5.10: Magnitude response of high pass FIR filter when object function is approximated using 21 Legendre polynomial terms, \( P_0 \) to \( P_{40} \).

It is clear from the Figures 5.10 and 5.11 that the pass band of the resulting high pass filter is very flat, phase of this filter is absolutely linear, and the realized filter closely follows the restrictions; that is, cutoff frequency and pass band frequency.
Figure 5.11: Magnitude response in dB of high pass FIR filter when object function is approximated using 21 Legendre polynomial terms, \( P_0 \) to \( P_{40} \).

### 5.4.4 Design 4

Next we design a band pass filter with the transition band, stop band, and pass band characteristics as follows

\[
A_{\text{max}} = 1000,
\]

\[
A_{\text{min}} = 0,
\]

\[
\omega_{OL1} = 1.5908,
\]

\[
\omega_{OH1} = 2.0944,
\]

\[
\omega_{OL2} = 2.7389,
\]

\[
\omega_{OH2} = 2.5322.
\]

Note that the filter has unequal transition bands.

The magnitude response, and magnitude response in dB are calculated using 21 Legendre polynomial terms to approximate the corresponding...
Figure 5.12: Magnitude response of band pass FIR filter when object function is approximated using 21 Legendre polynomial terms, $P_0$ to $P_{40}$.

Figure 5.13: Magnitude response in dB of band pass FIR filter when object function is approximated using 21 Legendre polynomial terms, $P_0$ to $P_{40}$. 

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object function and are shown in Figures 5.12 and 5.13, respectively. Unequal side bands are distinguishable in Figure 5.12. The pass band is not as flat as we need but if we increase the number of terms, used to approximate object function, it will become better. The phase response remains linear and can be verified easily.

5.5 Results

We can conclude that as we increase the number of terms in our polynomial, the approximation of the object function becomes more and more like the desired object function (see Figures 5.5 and 5.7).

![Figure 5.14: Magnitude response in dB of low pass FIR filter when object function is approximated using 21 Legendre polynomial terms, $P_0$ to $P_{20}$.](image)

Figures 5.6, 5.8, and 5.14 show a steady improvement in all round performance in the magnitude responses. It is evident from the figures that the sharpness of the transition band, side band level and ripple in the pass band are all improved. We can also easily deduce from the figures that the filter characteristics are in accordance with the defined values of $\omega_{OL}$, $\omega_{OH}$ and $\omega_{3dB}$.
Figures 5.11 and 5.13 make it clear that the method proposed can easily be implemented to design FIR filters of another kind; that is, high pass, bandpass, as well as band reject, multiband, and any other type, all we have to do is write down the corresponding object function characteristics and follow the procedure discussed.

Important Note: If we approximate the “ideal brick wall characteristics”; that is, with an abrupt change at a particular point in place of a gradual change, the results show serious limitations because of Gibbs type of phenomenon. In the case of an abrupt change the approximation gives us a flat pass band but it turns out that the stop band does not reduce below a certain value irrespective of how ever many terms we use.

5.6 Conclusion

An alternative simple approach has been presented to design the linear phase FIR filters, whose characteristics are modeled using orthogonal polynomials. The orthogonal polynomials give us good frequency domain characteristics in terms of sharp cutoff, low stop band, low ripple in the pass band and linear phase. These may be improved to any desired level by increasing the number of terms used to approximate the object function. Further, there is no restriction on the type of filter to be designed.