Design of 1 Dimensional Linear Phase FIR Filter with Chebyshev Polynomials

3.1 Introduction

Due to large number of applications [28, 29], several approaches have been proposed for designing a Chebyshev FIR filter. To design a linear phase Chebyshev FIR filter one has to convert the filter design problem into an approximation problem [28, 29]. Imposing the linear phase restriction in the pass band it leads to a complex approximation problem. An approach discussed by Cuthbert [30] proposes a technique where real and imaginary parts of the filter frequency response are separately approximated to design a non linear phase FIR filter. This approach was further extended using Remez-exchange algorithm [31, 32] by Attikiouzel et al [33] for two real approximations. Linear programming was used by Delves et al [34] to approximate real and imaginary parts simultaneously. These papers in general discuss approximation methods, while the approach discussed in the present chapter gives exact design of FIR filter in Chebyshev sense with linear phase.

In this chapter, we are going to discuss a new approach to realize FIR filters using Chebyshev polynomials. Chebyshev polynomials play a vital role in antenna as well as in signal processing theory. The Dolph-Chebyshev distribution of currents feeding the elements of a linear array, comprising an antenna, gives a sharp main lobe and small side lobes all of which have the same power level [17], we use this concept in the present discussion.

This chapter presents a method by which we can design a filter with linear phase and, 

(i) A given pass band to stop band ratio,

(ii) A given pass band to stop band transition, and to some extent
(iii) The frequency band of the pass band.

### 3.2 Preliminaries

A linear equispaced antenna array with n elements, labeled from left to right gives rise to a far field of [17]

\[ |E| = |A_0 e^{i\psi} + A_1 e^{i\beta} + A_2 e^{i\gamma} + \ldots + A_{n-2} e^{i(\alpha-2)\psi} + A_{n-1} e^{i(\alpha-1)\psi}| \]  \hspace{1cm} (3.1)

where,

\[ \psi = \beta + \ln \phi + \gamma \]  \hspace{1cm} (3.2)

|E| is the magnitude of the far field, \( \beta = 2\pi/\lambda \), \( \lambda \) is the free space wavelength, \( d \) is the spacing between elements, \( \phi \) is the angle from the normal to the linear array, \( \gamma \) is the progressive phase shift from left to right, and \( A_0, A_1, A_2, \ldots \) are complex amplitudes which are proportional to the current amplitudes.

If we substitute \( z = e^{i\psi} \) and rewrite Equation (3.1), it leads to

\[ H(z) = A_0 + A_1 z + A_2 z^2 + \ldots + A_{n-2} z^{n-2} + z^{n-1} \]  \hspace{1cm} (3.3)

this equation represents an FIR filter. Where, \( H(z) \) represents the impulse response of the filter with \( z = e^{i\omega} \), and \( A_0, A_1, A_2, \ldots \) represent amplitudes at the corresponding frequencies.

The Chebyshev polynomials are given by

\[ T_m(x) = \begin{cases} \cos(m \cos^{-1} x) & 0 < |x| < 1 \\ \cosh(m \cosh^{-1} x) & |x| > 1 \end{cases} \]  \hspace{1cm} (3.4)

### 3.3 Procedure

Let us assume that the order of the filter, which we intend to design, is \( m \). The stepwise procedure to design the required FIR filter is as follows:

**Step 1**: As discussed earlier the pass-band to stop-band ratio is user dependent. Therefore, first we calculate the absolute value of attenuation,
defined by the user, in the stop band, \( b \) (refer Chapter 1),

\[
b = 10 \text{[attenuation in dB]} \quad (3.5)
\]

**Step 2**: We find the stop band, \( \omega_s \), and pass band, \( \omega_p \), frequencies following the steps discussed in [17]

\[
\omega_s = 2\cos^{-1}\left(\frac{1}{\cosh\left(1/m \cosh^{-1} b\right)}\right) \quad (3.6)
\]

\[
\omega_p = 2\cos^{-1}\left[\frac{\cosh\left((1/m) \cosh^{-1}(b/\sqrt{2})\right)}{\cosh\left(1/m \cosh^{-1} b\right)}\right] \quad (3.7)
\]

**Step 3**: The location of zeros on unit circle, \( \omega_m \), are calculated by the following equation, which is discussed in [17]

\[
\omega_m = 2\cos^{-1}\left(\frac{\cos(\omega_k)}{\cosh\left(1/m \cosh^{-1} b\right)}\right) \quad (3.8)
\]

where, \( \omega_k = (2k-1)\pi/2m \), and \( k = 1 \ldots m \).

**Step 4**: Applying the relation \( z_m = e^{j\omega_m} \) we can write frequency response, \( H(z) \), in \( z \) domain using Equation (3.3) as follows

\[
H(z) = (z - z_1)(z - z_2) \ldots (z - z_m) \quad (3.9)
\]

where, \( z_1, z_2, \ldots \) are location of zeros of the transfer function \( H(z) \). Replacing \( z \) by \( e^{j\omega} \) and \( z_m \)'s by \( e^{j\omega_m} \)'s in Equation (3.9) we get frequency response in frequency domain as follows

\[
H(\omega) = (e^{j\omega} - e^{j\omega_1})(e^{j\omega} - e^{j\omega_2}) \ldots (e^{j\omega} - e^{j\omega_m}) \quad (3.10)
\]

To get a vivid picture of the procedure discussed above, we design some actual filters in the following section.

### 3.4 Application

Let us design some Chebyshev low pass FIR filters with side band 40 dB down from the pass band; that is, \( b = 10^{40/20} = 100 \).
3.4.1 Design 1

Let us design a filter with order 6; that is, \( m = 6 \). From Equations (3.6) and (3.7) we calculate

\[
\omega_s = 1.5732, \\
\omega_p = 0.5622.
\]

The values of the \( \omega_m \)'s can be calculated by using Equation (3.8)

\[
\omega_1 = 1.64, \\
\omega_2 = 2.0958, \\
\omega_3 = 2.7739, \\
\omega_4 = 3.5093, \\
\omega_5 = 4.1874, \\
\omega_6 = 4.6431.
\]

From Equation (3.10) we can write \( H(\omega) \) as

\[
H(\omega) = (e^{j\omega} - e^{j1.64})(e^{j\omega} - e^{j2.0958})(e^{j\omega} - e^{j2.7739})(e^{j\omega} - e^{j3.5093})(e^{j\omega} - e^{j4.1874})(e^{j\omega} - e^{j4.6431})
\]

If we plot the filter characteristics; that is, \( |H(\omega)| \) versus frequency \( \omega \), the results will become clear. The dark continuous lines in Figures 3.1, 3.2, and 3.3 show the magnitude response, magnitude response in \( dB \), and phase response of above mentioned FIR filter, respectively. Figure 3.1 confirms that the location of \( \omega_s \) and \( \omega_p \) are at the points where we have calculated them. Figure 3.2 show that side bands are 40 dB down. From Figure 3.3 it is clear that the phase is linear.
Figure 3.1: Magnitude response of 6\textsuperscript{th} order Chebyshev low pass FIR filter.

Figure 3.2: Magnitude response in dB of 6\textsuperscript{th} order Chebyshev low pass FIR filter.
Figure 3.3: Phase response of 6\textsuperscript{th} order Chebyshev low pass FIR filter.

3.4.2 Design 2

Let us calculate the frequency response of 3\textsuperscript{rd} order filter by using the steps mentioned in previous section. The values of $\omega_s$ and $\omega_p$ for 3\textsuperscript{rd} order filter are

$$\omega_s = 2.4641,$$

$$\omega_p = 0.9136.$$ 

Zeros, $\omega_m$'s, for this filter are as follows

$$\omega_1 = 2.5578,$$

$$\omega_2 = 3.1416,$$

$$\omega_3 = 3.7254.$$ 

While the dark continuous lines in Figure 3.4 represents the magnitude response of 3\textsuperscript{rd} order FIR filter, Figure 3.5 shows the magnitude response in dB. The side bands are 40 dB down and the bandwidth of the filter is
increased when compared with Figure 3.2. The transition band is wider than the previous design, or in other words filter will pass some of the non required frequencies. Phase remains linear and can be verified easily.

![Graph showing magnitude response](image)

**Figure 3.4:** Magnitude response of 3rd order Chebyshev low pass FIR filter.

### 3.4.3 Design 3

Next we calculate the frequency responses of 24th order filter. Suppose we design the filter with side bands 40 dB down, the values of $\omega_s$ and $\omega_p$ for the 24th order filter are as follows

$$\omega_s = 0.4380,$$

$$\omega_p = 0.1558,$$

Values of $\omega_m$'s are as follows

$$\omega_1 = 0.4568, \omega_2 = 0.5861;$$

$$\omega_3 = 0.7831, \omega_4 = 1.0088;$$
Figure 3.5: Magnitude response in dB of 3\textsuperscript{rd} order Chebyshev low pass FIR filter.

\[ \omega_5 = 1.2478, \omega_6 = 1.4935; \]
\[ \omega_7 = 1.7432, \omega_8 = 1.9952; \]
\[ \omega_9 = 2.2488, \omega_{10} = 2.5033; \]
\[ \omega_{11} = 2.7584, \omega_{12} = 3.0138; \]
\[ \omega_{13} = 3.2694, \omega_{14} = 3.5248; \]
\[ \omega_{15} = 3.7799, \omega_{16} = 4.0344; \]
\[ \omega_{17} = 4.2879, \omega_{18} = 4.5400; \]
\[ \omega_{19} = 4.7896, \omega_{20} = 5.0354; \]
\[ \omega_{21} = 5.2743, \omega_{22} = 5.5001; \]
\[ \omega_{23} = 5.6970, \omega_{24} = 5.8264 \]
The dark continuous lines in Figure 3.6 represents the magnitude response of this order FIR filter and Figure 3.7 shows the magnitude response in dB. When Figure 3.6 is compared with Figures 3.1 and 3.4, we found that the present filter has narrow pass band and sharper transition band while side bands remains 40 dB down. Therefore, to design a narrow band filter we have to increase the order of the filter.

![Magnitude response graph](image)

Figure 3.6: Magnitude response of 24th order Chebyshev low pass FIR filter.

### 3.5 Discussion

It is clear from the Figures 3.1, 3.4 and 3.6 and Figures 3.2, 3.5 and 3.7 that the width of the pass band decreases as the order of filter increases, and the transition band becomes steeper, or in other words, it follows brick-wall or ideal characteristics of a filter more closely.

Figure 3.3 shows the phase response of the 6th order FIR filter designed above, which clearly shows its linear nature. The phase responses of 3rd and 24th order filters is also linear and can be verified easily by following the same steps.
Figure 3.7: Magnitude response in dB of 24\textsuperscript{th} order Chebyshev low pass FIR filter.

By using the procedure discussed, we can not control the bandwidth of the resulting filter with predefined order. In the next section we modify the design procedure to overcome this problem.

### 3.6 Modified Chebyshev Filter

We introduce a new parameter in the Chebyshev polynomial to have some control over the bandwidth, when \( m \) and \( b \) are already defined. In the original Chebyshev polynomial (Equation (3.4)) we multiply a new parameter \( \alpha \) with parameter \( x \) (\( \alpha \) controls the bandwidth of the filter). Thus, Equation (3.4) becomes

\[
T_n(ax) = \begin{cases} 
\cos(m \cos^{-1} ax) & 0 < |x| < 1 \\
\cosh(m \cosh^{-1} ax) & |x| > 1
\end{cases} \tag{3.11}
\]

When we multiply \( x \) with \( \alpha \), it results in a change in \( \omega_s \) only, while \( \omega_p \) remains the same. The reason is because \( \alpha \) is present in both numerator
as well as in denominator terms (see Equations (3.6) and (3.7)).

Therefore, new stop band frequency, \( \omega_{s-new} \), is

\[
\omega_{s-new} = 2 \cos^{-1} \left[ 1/\alpha \left( \cosh(1/m \cosh^{-1} b) \right) \right] \tag{3.12}
\]

and,

\[
\omega_{p-new} = \omega_p \tag{3.13}
\]

We can calculate the location of zeros for modified Chebyshev polynomials by

\[
\omega_{m-new} = 2 \cos^{-1} \left[ \cos(\omega_k)/ \left( \alpha(\cosh(1/m \cosh^{-1} b)) \right) \right] \tag{3.14}
\]

where, \( \omega_k=(2k-1)\pi/2m \), and \( k=0 \ldots m \). We can write \( H(\omega) \) as

\[
H(\omega) = \prod_{m-new=0}^{m} (e^{j\omega} - e^{j\omega_{m-new}}) \tag{3.15}
\]

### 3.7 Application

Plotting the magnitude response of Design 1 – with the new parameter \( \alpha \) taken into consideration – we get the magnitude response and magnitude response in dB as shown in Figures 3.1 and 3.2, respectively ("dash followed by dot" for values \( \alpha > 1 \) and "dots" for values \( \alpha < 1 \), respectively). It is evident from figures that the bandwidth of our filter is increased in case of \( \alpha > 1 \) and the stop band is further down.

Figure 3.3 shows that our modified Chebyshev FIR filter has linear phase characteristics, further it confirms that the filter retains its linear phase even after introducing the new parameter \( \alpha \). Similarly Figures 3.4 and 3.5, Figures 3.6 and 3.7 show the magnitude response and magnitude response in dB for 3rd and 24th order FIR filter having side bands 40 dB down, respectively. For lower order filter \( \alpha \) makes very small difference, this can be verified by looking at Figures 3.4 and 3.5.

It is noteworthy that when \( \alpha \) has value less than 1, see Figures 3.1 and 3.6, the magnitude of the stop band is approximately equal to the pass band. Thus we should not use filters designed using value of \( \alpha < 1 \).
3.8 Discussion

Figure 3.8 shows the magnitude response in dB for various values of \( \alpha \) for 32\textsuperscript{nd} order low pass FIR filter. It is clear from the figure that as we increase the value of our new parameter \( \alpha \), the bandwidth of our filter increases and the level of sidebands goes further down. One should note that the value of \( \alpha \) less than 1 will give us a poor design, where pass band and stop band are approximately at the same level, and thus should not be used. The value of \( \alpha \) is obtained by trial to obtain the desired bandwidth. A filter was designed and synthesized based on this technique to control burst noise, and it produced good results.

![Magnitude responses of 32\textsuperscript{nd} order Chebyshev low pass FIR filter for various values of \( \alpha \).](image)

3.9 Conclusion

We can say that the FIR filter design technique discussed in the present chapter can easily be implemented for given sideband specifications and bandwidth. The new parameter \( \alpha \) further pulls the side band levels down
and increases the bandwidth, while maintaining the phase linear. By this
technique we can design band pass, band reject, multiband and other filters
using the standard frequency transformation techniques.

We extend this concept to two dimensions, thus making use of such
filters for image processing applications as well, in the next chapter.