Appendix

The sum of Legendre polynomials multiplied by suitable coefficients approximates the ideal polynomial in the least mean square error sense.

Proof:

Suppose a polynomial \( f(x) \) is approximated by \( f_n(x) \) and its representation is

\[
f_n(x) = \sum_{n=0}^{N} a_{2n} P_{2n}(x)
\]

The mean squared error (MSE) polynomial between the original polynomial and approximated polynomial is given by

\[
E(x) = \{ f(x) - f_n(x) \}^2
\]

or,

\[
E(x) = \left[ f(x) - \sum_{n=0}^{N} a_{2n} P_{2n}(x) \right]^2 \tag{A.1}
\]

Note that \( E(x) \) is either positive or zero hence the minimum value of its integral is zero. We define

\[
e(a_0, a_2, \ldots) = \int_{-1}^{1} E(x) dx = \int_{-1}^{1} \left[ f(x) - \sum_{n=0}^{N} a_{2n} P_{2n}(x) \right]^2 dx \tag{A.2}
\]

We need to find the coefficients \( a_0, a_2, \ldots \) from the above equation so that this integral is minimized. The general way of solving this problem is well known, and is described below

\[
\frac{\partial}{\partial a_i} \int_{-1}^{1} \left[ f(x) - \sum_{n=0}^{N} a_{2n} P_{2n}(x) \right]^2 dx = 0 \quad i = 1 \ldots N \tag{A.3}
\]

or
\[ \frac{\partial}{\partial a_i} \int_{-1}^{1} \left[ (f(x))^2 + \left( \sum_{n=0}^{N} a_{2n} P_{2n}(x) \right)^2 \right] \, dx = 0 \quad i = 1 \ldots N \quad (A.4) \]

which is

\[ \int_{-1}^{1} \left[ 0 + \frac{\partial}{\partial a_i} \left( \sum_{n=0}^{N} a_{2n} P_{2n}(x) \right)^2 \right] \, dx = -2 \frac{\partial}{\partial a_i} f(x) \left( \sum_{n=0}^{N} a_{2n} P_{2n}(x) \right) \bigg|_{-1}^{1} = 0 \quad i = 1 \ldots N \quad (A.5) \]

spelt out this is

\[ \int_{-1}^{1} \left[ \frac{\partial}{\partial a_i} \left( a_0^2 P_0^2 + 2a_0a_2 P_0 P_2 + \ldots + a_i^2 P_i^2 + 2a_i a_{i+2} P_i P_{i+2} + 2a_i a_{i+4} P_i P_{i+4} + \ldots \right) \right] \, dx \]

\[ -2 \int_{-1}^{1} f(x)a_i P_i \, dx = 0 \quad (A.6) \]

The orthogonality property of the Legendre polynomials reduces this integral to

\[ 2 \int_{-1}^{1} a_i P_i^2 \, dx - 2 \int_{-1}^{1} f(x)P_i \, dx = 0 \quad i = 1 \ldots N \quad (A.7) \]

\[ a_i = \frac{\int_{-1}^{1} f(x)P_i \, dx}{\int_{-1}^{1} P_i^2 \, dx} \quad i = 1 \ldots N \quad (A.8) \]