6 Design of 2 Dimensional Linear Phase FIR Filter with Orthogonal Polynomials

6.1 Introduction

Due to rapid development in the field of image processing, sonar signal processing and other related fields, the design of 2 dimensional (2D) digital filters has become very important. 2D finite impulse response (FIR) filters are preferred over infinite impulse response (IIR) filters because of their inherent stability. These 2DFIR filters may have linear phase\(^1\).

Various techniques have been discussed in the literature to design 2DFIR digital filters. If we take into consideration least square (LS) error criterion, we observe that an overshoot of the frequency response at the pass band and the stop band edges occur due to Gibb's phenomenon [43]. Whereas, a minmax design approach results in an equiripple solution [44, 45].

Efficient 2DFIR filters can also be designed by using transformations. J. H. McClellen [46,47] has proposed a powerful technique, where 1DFIR filters are used with a transformation to design 2DFIR filters. This technique is very efficient and fast. This technique was implemented by [48]. But in this technique the transformation steps are very complicated and difficult to implement, and can be used to design 2D square filters only.

What we are presenting here is a new method to design 2DFIR filter with linear phase through the transformation and orthogonal polynomials. The whole procedure is simple and produce excellent results. Any type of filter can be designed by this approach.

\(^1\)It is intuitively clear that non linear phase will introduce a distortion.
6.2 Procedure

Because we are going to design an orthogonal polynomial based 2D FIR filter, therefore first we have to calculate 2D orthogonal polynomials.

We know that a sequence of orthogonal polynomials, $P_n(\rho)$ in cylindrical coordinates, must satisfy the relation

$$\langle P_i(\rho), P_j(\rho) \rangle = \int_0^{2\pi} \int_0^1 P_i(\rho)P_j(\rho)\rho d\rho d\phi = 0 \quad i \neq j$$

(6.1)

where,

$P_i(\rho)$ and $P_j(\rho)$ are any two members of the orthogonal set,

$\langle \bullet, \bullet \rangle$ represents the inner product, and

$\rho d\rho d\phi$ is the element of area.

The polynomials can be obtained by applying the well known Graham-Schmidt procedure [49] over the interval $[-1, 1]$ for $\rho$. Equation (6.2) represents some of the even polynomials.

$$P_0(\rho) = 1,$$

$$P_2(\rho) = 1 + 2\rho^2,$$

$$P_4(\rho) = 1 - 6\rho^2 + 6\rho^4,$$

$$P_6(\rho) = 1 - \frac{190}{31}\rho^2 + \frac{190}{31}\rho^4 + \frac{20}{31}\rho^6$$

$$\vdots$$

(6.2)

Figure 6.1 shows some of these polynomials.

The filters which we propose to design are circularly symmetric 2DFIR filters, half of the symmetric magnitude response of such a filter is shown in Figure 6.2, other half being mirror image. Since the design is circularly symmetric, we use the cylindrical co-ordinate system - $(\rho, \phi)$ - to represent the object function in 2D. $\rho$ is a mapping function and is related to Cartesian coordinates $(x, y)$ by

$$\rho^2 = x^2 + y^2 \quad -1 \leq \rho \leq 1$$

(6.3)

Since the filter is circularly symmetric, therefore, if we take the cross section of the 2D FIR filter perpendicular to $u - v$ plane passing through
Figure 6.1: First few 2D orthogonal polynomials calculated using Equation (6.1).

the origin, it will look like a 1D filter. Figure 6.3 shows the 1D equivalent magnitude response for the filter shown in Figure 6.2. In Figure 6.3

$A_{\text{max}}$ represents the amplitude of the pass band of the filter,

$A_{\text{min}}$ represents the amplitude of the stop band of the filter,

$\omega_{0H}$ is the end of pass band and start of the transition band, and

$\omega_{0L}$ is the end of transition band and start of the stop band.

The relationship between the 1D frequency axis, $\omega$-axis, of Figure 6.3
and 2D frequency axis, $u, v$-axis, of Figure 6.2 is given by

$$\omega^2 = u^2 + v^2 \quad -\pi \leq \rho \leq \pi$$ \hspace{1cm} (6.4)

The transformation used for conversion from object function – $\rho$ domain – to frequency characteristics – $\omega$ domain – is carried out by

$$\rho = \rho_0 \cos \left( \frac{\omega}{2} \right) \quad -\pi \leq \omega \leq \pi$$ \hspace{1cm} (6.5)

Because the filter is circularly symmetric, therefore, the object function will also be circularly symmetric around the $z$-axis. Figure 6.4 shows this object function in 3D and its corresponding 1D representation in Figure 6.5.
Figure 6.2: 3D desired filter response.

Figure 6.3: 1D representation of a low pass 2D filter characteristics.
Figure 6.4: 3D object function for filter characteristics of Figure 6.2, showing slight non linearity in the transition region.

Figure 6.5: 1D object function, to be approximated using Legendre polynomials.
To transform the 2D filter function, in terms of $u - v$, to 2D object function, in terms of $x - y$, we use

\[
    u = 2\cos^{-1}(x/\rho_0) \quad \pi \leq |u|, \rho_0 < |x|
\]

\[
    v = 2\cos^{-1}(y/\rho_0) \quad \pi \leq |v|, \rho_0 < |y|
\]

(6.6)

\[f(x, y) \xrightarrow{T} H(u, v)\]

where, $\rho_0$ is the maximum value of $\rho$ and $T$ is the transformation of Equation (6.6), it is also the maximum value of $x$ and $y$, Equation (6.3).

We approximate the object function using a linear combination of several orthogonal polynomials $^2$, calculated above. Note that approximation of the object function will be done using even polynomials only; that is, $P_0(\rho), P_2(\rho), \ldots$, because the object function and frequency response both are symmetric in nature, and polynomials with even power will give us the required symmetrical characteristics.

Let us denote the object function of Figure 6.4 by $f(\rho)$ (recall Equation (6.3)), which can be written as

\[
f(\rho) = \sum_{n=0}^{\infty} a_{2n}P_{2n}(\rho)
\]

(6.7)

where, $a_0, a_2, \ldots$ are coefficients to be multiplied with orthogonal polynomials to get the required characteristics, shown in Figure 6.4. To calculate these coefficients, $a_0, a_2, \ldots$, we use $^3$

\[
a_i = \frac{\int_0^1 f(\rho)P_i(\rho)\rho d\rho}{\int_0^1 P_i(\rho)P_i(\rho)\rho d\rho}
\]

(6.8)

where, $i = 0, 2, 4, \ldots$

We calculate the approximate object function $f_a(\rho)$ by

\[
f_a(\rho) = \sum_{i=0}^{N} a_i P_i(\rho)
\]

(6.9)

$^2$Any function can be represented as a linear combination of several orthogonal polynomials, refer Appendix A.

$^3$Refer Appendix A.
Note that \( f_\delta(\rho) \) tends to \( f(\rho) \) when we use infinite number of terms to approximate the object function. Therefore, as we increase number of terms in \( f_\delta(\rho) \) we get better approximation of \( f(\rho) \).

This approximate object function is then transformed from \((x, y)\) or \(\rho\) domain to frequency domain, or \((u, v)\) or \(\omega\) domain, by using Equation (6.5).

The zeros of \( f_\delta(\rho) \) can be calculated by using any standard routine, and zeros of \( f_\delta(\omega) \) are obtained by using

\[
\rho_i \mapsto \omega_i
\]

Here \( \rho_i \)'s are the zeros of \( f_\delta(\rho) \), and \( \rho \) is transformation of Equation (6.5).

## 6.3 Application and Discussion

Suppose we want to design a 2DFIR filter with transfer function defined as

\[
|H(\omega)| = \begin{cases} 
1000 & 0 \leq \omega < 1.5908 \\
\text{transition band} & 1.5908 \leq \omega < 2.0944 \\
10 & 2.0944 \leq \omega < \pi
\end{cases} \tag{6.10}
\]

Comparing these characteristics with Figure 6.2, we note that

\[
A_{max} = 1000, \\
A_{min} = 10.
\]

First we find the corresponding object function by using the transformation given in Equation (6.5). The object function comes out to be

\[
f(\rho) = \begin{cases} 
10 & 0 \leq \rho < 0.5 \\
\text{transition band} & 0.5 \leq \rho < 0.7 \\
1000 & 0.7 \leq \rho < 1.0
\end{cases} \tag{6.11}
\]

Let us design this filter.
6.3.1 Design 1

To calculate the value of $f_a(\rho)$ (Equation (6.9)), we first calculate the values of $a_i$'s using Equation (6.8). As mentioned earlier, when we increase the number of terms to approximate our object function the approximation becomes better and better. Therefore, it is up to the designer where he wants to stop.

Let us approximate the object function using 8 orthogonal polynomials. We calculate the coefficients using Equation (6.8). The approximate object function is

$$f_a(\rho) = 750 P_0 - 562.5 P_2 - 468.75 P_4 + 59.04 P_6 + 263.6712 P_8$$
$$+ 306.152 P_{10} + 66.65048 P_{12} - 198.4407 P_{14}$$

(6.12)

We apply Equations (6.3) and (6.6) on the approximated object function and get the filter in frequency domain. The magnitude response of the filter is shown in Figure 6.6, its cross section in Figure 6.7 and the magnitude response in dB is shown in Figure 6.8. When we look at Figures 6.6 and 6.7, we find that the pass band of the filter has some ripples. Figure 6.8 shows that first side band is approximately 35dB down.

In the next classification we consider the same problem while using more number of orthogonal polynomial terms to approximate the object function.

6.3.2 Design 2

In the present case we use 15 orthogonal polynomial terms to approximate the object function. This object function is then transformed using the transformation of Equation (6.5). The resulting 2DFIR is shown in Figure 6.9, the cross section in Figure 6.10 and the magnitude response in dB is shown in Figure 6.11, the first side band is approximately 45dB down.

Comparing Figures 6.7 and 6.10, it is clear that as we increase the number of terms to calculate object function, which in turn is used to construct the required filter, the sharpness of the transition band of the filter increases and ripples in the pass band decreases. The filter shown in Figure 6.11 has lower side bands than the filter represented in Figure 6.8.
6.3.3 Design 3

We can easily design high pass, bandpass, band reject, and other type of filters using the same approach. We only have to calculate the coefficients required to design the filter.

Let us design a high pass filter with characteristics as below

\[
|H(\omega)| = \begin{cases} 
10 & 0 \leq \omega < 2.0944 \\
\text{transition band} & 2.0944 \leq \omega < 2.5322 \\
1000 & 2.5322 \leq \omega < \pi
\end{cases}
\]  \hspace{1cm} (6.13)

The corresponding object function is

\[
f(\rho) = \begin{cases} 
1000 & 0 \leq \rho < 0.3 \\
\text{transition band} & 0.3 \leq \rho < 0.5 \\
10 & 0.5 \leq \rho < 1.0
\end{cases}
\]  \hspace{1cm} (6.14)

Suppose this object function is approximated using 15 orthogonal polynomial terms. The resulting high pass filter is shown in Figure 6.12, and
Figure 6.7: Cross section of the magnitude response of low pass FIR filter when object function is approximated using 8 orthogonal polynomial terms.
Figure 6.8: Magnitude response in dB of low pass FIR filter when object function is approximated using 8 orthogonal polynomial terms.

Figure 6.9: Magnitude response of low pass FIR filter when object function is approximated using 15 orthogonal polynomial terms.
Figure 6.10: Cross section of the magnitude response of low pass FIR filter when object function is approximated using 15 orthogonal polynomial terms.
Figure 6.11: Magnitude response in dB of low pass FIR filter when object function is approximated using 15 orthogonal polynomial terms.

The cross section of this filter is represented in Figure 6.13. From the figures it is clear that the filter has flat pass band and if we pass an image through it the filter will be able to cut off non required frequencies (we will pass some images through this filter in the next section). Phase remains linear and can easily be verified.

6.3.4 Design 4

Let us design a band pass filter with frequency domain characteristics

\[
|H(\omega)| = \begin{cases} 
1000 & 1.4455 \leq \omega < 2.6362 \\
\text{transition bands} & 1.1096 \leq \omega < 1.4455 \text{ and } 2.6362 \leq \omega < 2.8405 \\
10 & 2.8405 \leq \omega < \pi \text{ and } 0 \leq \omega < 1.1096 
\end{cases}
\]  \hspace{1cm} (6.15)

The corresponding object function is
Figure 6.12: Magnitude response of high pass FIR filter when object function is approximated using 15 orthogonal polynomial terms.

\[
f(\rho) = \begin{cases} 
1000 & 0.25 \leq \rho < 0.75 \\
\text{transition bands} & 0.15 \leq \rho < 0.25 \text{ and } 0.75 \leq \rho < 0.85 \\
10 & 0.0 \leq \rho < 0.15 \text{ and } 0.85 \leq \rho < 1.0
\end{cases} \quad (6.16)
\]

Using the procedure discussed and approximating the object function of Equation (6.16) with 15 polynomial terms we get the required filter characteristics, which are shown in Figure 6.14. The magnitude response in dB is shown in Figure 6.15, the pass band of filter is very flat and side bands are approximately 25 dB down.

### 6.4 Results

The results produced by the algorithm, proposed in this chapter, give promising results. By increasing the number of terms to approximate the object function we can design better filters; that is, filters with sharp cutoff, lower side band and less ripples in the pass band. This technique can
Figure 6.13: Cross section of magnitude response of high pass FIR filter when object function is approximated using 15 orthogonal polynomial terms.
Figure 6.14: Cross section of magnitude response of band pass FIR filter when object function is approximated using 15 orthogonal polynomial terms.
Figure 6.15: Full view of magnitude response in dB of band pass FIR filter when object function is approximated using 15 orthogonal polynomial terms.
easily be used to design low pass, high pass, band pass, band reject, and other type of filters.

If we pass some images through these filters, we can estimate the quality of our filter. Passing an image through a high pass filter means that the details of the image are kept, while the large scale gradients are removed, while in the case of low pass filter opposite of this happens.

Figure 6.16 shows an image which when passed through a high pass filter, designed using 15 orthogonal polynomial terms, becomes as shown in Figure 6.17, which depicts that only high frequency components (changes in the different gray levels of the image) are retained. When this image is passed through a low pass filter the resulting image is shown in Figure 6.18, which removes high frequency components (small details of image) of the image and only blurred details of the image are retained.

Similarly, an image with more details than the previous one is shown in Figure 6.19, its high pass filtered output is shown in Figure 6.20 and low pass in Figure 6.21, respectively.

A daily life image with a fair combination of changes is shown in Figure 6.22, its high pass filtered result is shown in Figure 6.23 and low pass filtered output is the Figure 6.24.

One more image is shown in Figure 6.25 its high pass and low pass filtered outputs are shown in Figures 6.26 and 6.27, respectively. We have shown various type of images to show that the filter designed can be used for any type of image. The filters used above are designed using 15 orthogonal polynomial terms to approximate the corresponding object function.

Figure 6.16: Image-1.
Figure 6.17: Image-1 passed through high pass filter designed using 15 orthogonal polynomial terms.

Figure 6.18: Image-1 passed through low pass filter designed using 15 orthogonal polynomial terms.
6.5 Conclusion

A simple new approach has been discussed here for designing 2D linear phase FIR filters. The cut off characteristics and ripples in the pass band can be controlled by using the method proposed in this chapter. As we increase the number of terms, the resulting filter becomes more flat in pass band and the transition region approximates the defined value. We can design any type of filter; that is, band pass, band reject, etc., using the present technique.
Figure 6.20: Image-2 passed through high pass filter designed using 15 orthogonal polynomial terms.
Figure 6.21: Image-2 passed through low pass filter designed using 15 orthogonal polynomial terms.
Figure 6.23: Image-3 passed through high pass filter designed using 15 orthogonal polynomial terms.
Figure 6.24: Image-3 passed through low pass filter designed using 15 orthogonal polynomial terms.
Figure 6.25: Image-4.
Figure 6.26: Image-4 passed through high pass filter designed using 15 orthogonal polynomial terms.
Figure 6.27: Image-4 passed through low pass filter designed using 15 orthogonal polynomial terms.