Chapter 4

fixed point theorem for four mappings in complete 2-metric spaces
This chapter contains the proof of a common fixed point theorem for four mappings in complete 2-metric spaces. This theorem is a version of many fixed point theorems in complete 2-metric spaces. This chapter is accepted for publication in International Journal of Applied Mathematics Analysis and Application.

Abstract: In the present chapter we prove a common fixed point theorem for four mappings in complete 2-metric spaces. This theorem is a version of many fixed point theorems in complete 2-metric spaces.

Introduction
The concept of 2-metric space is a natural generalization of the metric space. Initially, it has been investigated by S. Gähler [27, 28, 29] in 1960. After this, number of fixed point theorems have been proved by Many researchers like H.K. Pathak [10,11] and M.S. Khan [19, 20, 21, 22] for 2-metric spaces by introducing compatible mappings, which are more general than commuting and weakly commuting mappings. K. Iseki [15, 16, 17] is well-known in this literature which also includes J. Matkowski et.al. [14], M. Imdad et.al., P.P. Muthuy et.al. [24], S.V.R. Naidu and J.R. Prasad, et.al., Commutability of two mappings was weakened by Sessa with weakly commuting mappings. Vishal Gupta et al. [35, 36, 37] also prove some common fixed point theorems for a class of A-contraction on 2-metric space. Jungck and Rhoades defined the concepts of d-compatible and weakly compatible mappings as extensions of the concept of compatible mapping for single-valued mappings on metric spaces. Several authors used these concepts to prove some common fixed point theorems. Jungck extended the class of non-commuting mappings by compatible mappings and extended the class of non-commuting mappings by compatible mappings.

Main Result
Theorem 4.1: Let F, G, S and T be four self mappings of a 2-metric space \((X, d)\) satisfying the following conditions:

1. The pair \((F, S)\) and \((G, T)\) are weakly compatible.
2. \(F(X) \subseteq T(X)\) and \(G(X) \subseteq S(X)\) is closed subset of \(X\).
3. \(d(Fx, Gy, t) \leq \phi(\min\{d(Sx, Ty, t), d(Fx, Sx, t), d(Gy, Ty, t), d(Fx, Ty, t), d(Sx, Gy, t)\})\) where \(\phi\) is a contractive modulus.

Then the maps F, G, S and T have a unique common fixed point in X.
Proof: Let \( Y_n \) be a sequence in \( X \) such that \( Y_n = Fx_n = Tx_n \) and \( Y_{n+1} = Gx_{n+1} = Sx_{n+2} \) by (3)

\[
d(Y_{n+1}, Y_n, t) = d(Fx_n, Gx_{n+1}, t)
\]

\[
\leq \phi\left(\min \{d(Sx_n, T x_{n+1}, t), d(Fx_n, Sx_n, t), d(Gx_n, T x_n, t), d(Fx_n, T x_n, t), d(Sx_n, Gx_{n+1}, t)\}\right)
\]

\[
\leq \phi\left(\min \{d(Y_{n+1}, Y_n, t), d(Y_n, Y_{n-1}, t), d(Y_{n-1}, Y_{n-2}, t), d(Y_{n-2}, Y_{n-3}, t), \ldots \} \right)
\]

\[
\leq \phi\left(\min \{d(Y_n, Y_{n+1}, t), d(Y_{n+1}, Y_{n+2}, t) \} \right) \leq \phi\left[d(Y_n, Y_{n+1}, t)\right] \Rightarrow d(Y_n, Y_{n+1}, t) \leq \phi\left[d(Y_n, Y_{n+1}, t)\right]
\]

But \( \phi \) is a contractive modulus

\[
\Rightarrow \phi\left[d(Y_n, Y_{n+1}, t)\right] < d(Y_n, Y_{n+1}, t) \text{ this is possible only if } \lim_{n \to \infty} d(Y_n, Y_{n+1}, t) = 0.
\]

Now we show that \( Y_n \) is a Cauchy sequence in \( X \).

If not \( \exists \varepsilon > 0 \) such that \( m < n < N, d(Y_n, Y_m, t) \geq \varepsilon \) but \( d(Y_{n-1}, Y_m, t) < \varepsilon \)

And, \( \varepsilon \leq d(Y_n, Y_{n+1}, t) = d(Fx_n, Gx_{n+1}, t) \)

\[
\leq \phi\left[\min \{d(Sx_n, T x_n, t), d(Fx_n, Sx_n, t), d(Gx_n, T x_n, t), d(Fx_n, T x_n, t), d(Sx_n, Gx_{n+1}, t)\}\right]
\]

\[
\leq \phi\left[\min \{d(Y_n, Y_{n+1}, t), d(Y_n, Y_{n-1}, t), d(Y_{n-1}, Y_{n-2}, t), d(Y_{n-2}, Y_{n-3}, t), \ldots \} \right)
\]

\[
\leq \phi\left[\min \{\varepsilon, \varepsilon, 0, \varepsilon, \varepsilon \} \right) \Rightarrow \varepsilon \leq \phi(\varepsilon)
\]

But \( \phi \) is a contractive modulus \( \Rightarrow \phi(\varepsilon) < \varepsilon \Rightarrow \varepsilon < \varepsilon \) which is a contradiction hence \( Y_n \) is a Cauchy sequence. Since \( X \) is complete \( \exists \) a point \( z \) in \( X \) such that

\[
\lim_{n \to \infty} Y_n = z \Rightarrow \lim_{n \to \infty} Gx_n = \lim_{n \to \infty} Sx_n = z = \lim_{n \to \infty} Fx_n = \lim_{n \to \infty} T x_n
\]

since \( F(X) \subseteq T(X) \) \( \exists \) a point \( \alpha \in X \) \( \alpha \in X \ s.t. \ z = T \alpha \) if \( z \neq Ga \) using (3), we get

\[
d(G\alpha, z, t) = d(G\alpha, Fx_n, t)
\]

\[
\leq \phi\left[\min \{d(Sx_n, T \alpha, t), d(Fx_n, Sx_n, t), d(G\alpha, T \alpha, t), d(Fx_n, T \alpha, t), d(Sx_n, G\alpha, t)\}\right]
\]

\[
\leq \phi\left[\min \{d(z, z, t), d(z, z, t), d(G\alpha, z, t), d(z, z, t), d(z, G\alpha, t)\} \leq \phi\left[d(G\alpha, z, t)\right]
\]

\[
\Rightarrow d(G\alpha, z, t) \leq \phi\left[d(G\alpha, z, t)\right]
\]

But \( \phi \) is a contractive modulus \( \Rightarrow \phi\left[d(G\alpha, z, t)\right] < d(G\alpha, z, t) \) which is a contradiction

So \( G\alpha = z \) i.e. \( G\alpha = z = T \alpha \)

\( \Rightarrow \alpha \) is a co-incidence point of \( G \) and \( T \) and \( (G, T) \) is weakly compatible.
\[ \Rightarrow GT\alpha = TG\alpha \Rightarrow Gz = Tz \] Now \( G(X) \subseteq S(X) \) \( \exists \) a point \( w \in X \) s.t. \( Sw = z \) if \( Fw \neq z \) using (3)

\[ d(Fw, z, t) = d(G\alpha, Fw, t) \]

\[ \leq \phi[\min\{d(Sw, T\alpha, t), d(Fw, Sw, t), d(G\alpha, T\alpha, t), d(Fw, T\alpha, t), d(Sw, G\alpha, t)\}] \]

\[ \leq \phi[\min\{d(z, z, t), d(Fw, z, t), d(z, z, t), d(Fw, z, t), d(z, z, t)\} \leq \phi[\min\{d(z, z, t), d(Fw, z, t), d(z, z, t), d(Fw, z, t), d(z, z, t)\} \leq \phi[\min\{d(z, z, t), d(Fw, z, t), d(z, z, t), d(Fw, z, t), d(z, z, t)\}] \leq \phi[\min\{d(z, z, t), d(Fw, z, t), d(z, z, t), d(Fw, z, t), d(z, z, t)\}] \leq \phi[\min\{d(z, z, t), d(Fw, z, t), d(z, z, t), d(Fw, z, t), d(z, z, t)\}] \]

\[ \Rightarrow d(Fw, z, t) \leq \phi[d(Fw, z, t)] \]

But \( \phi \) is a contractive modulus \( \Rightarrow \phi[d(Fw, z, t)] < d(Fz, z, t) \) which is a contradiction.

So \( Fw = z = Sw \) hence \( w \) is a co-incidence point of \( F \) and \( S \) and \( (F, S) \) is weakly compatible

\[ \Rightarrow FSw = SFw \Rightarrow Fz = Sz \] now if \( Fz \neq z \) using (3)

\[ d(Fz, z, t) = d(Fz, G\alpha, t) \leq \phi[\min\{d(Sz, T\alpha, t), d(Fz, Sz, t), d(G\alpha, T\alpha, t), d(Fz, T\alpha, t), d(Sz, G\alpha, t)\}] \]

\[ \leq \phi[\min\{d(Sz, z, t), d(Fz, Sz, t), d(z, z, t), d(z, z, t), d(Sz, z, t)\}] \text{ and } Fz = Sz \]

\[ \Rightarrow d(Fz, z, t) \leq \phi[d(Fz, z, t)] \] and \( \phi \) is a contractive modulus \( \Rightarrow \phi[d(Fz, z, t)] < d(Fz, z, t) \)

which is a contradiction. Hence \( Fz = Sz = z \) Now if \( Gz \neq z \) using (3)

\[ d(z, Gz, t) = d(Fz, Gz, t) \leq \phi[\min\{d(Sz, Tz, t), d(Fz, Sz, t), d(Gz, Tz, t), d(Fz, Tz, t), d(Sz, Gz, t)\}] \]

\[ d(z, Gz, t) \leq \phi[\min\{d(z, Tz, t), d(z, z, t), d(Gz, Tz, t), d(z, Tz, t), d(z, Gz, t)\}] \]

And \( Gz = Tz \Rightarrow d(z, Gz, t) \leq \phi[d(z, Gz, t)] \) and \( \phi \) is a contractive modulus therefore \( \phi[d(z, Gz, t)] < d(z, Gz, t) \) which is a contradiction.

So \( Gz = z = Tz \) hence we have \( Gz = Tz = Fz = Sz = z \) So \( F, S, T, G \) have a common fixed point in \( X \).

Now we prove uniqueness let there be another point say \( w \) s.t. \( w \neq z \) then by (3)

\[ d(Fz, Gw, t) \leq \phi[\min\{d(Sz, Tw, t), d(Fz, Sz, t), d(Gw, Tw, t), d(Fz, Tw, t), d(Sz, Gw, t)\}] \]

\[ d(z, w, t) \leq \phi[\min\{d(z, w, t), d(z, z, t), d(w, w, t), d(z, w, t), d(z, w, t)\}] \Rightarrow d(z, w, t) \leq \phi[d(z, w, t)] \]

and \( \phi \) is a contractive modulus \( \Rightarrow \phi[d(z, w, t)] < d(z, w, t) \) which is a contradiction \( z = w \) and hence the uniqueness.

**Corollary 4.1:** Let \( F, G, S \) and \( T \) be four self mappings of a 2-metric space \((X, d)\) satisfying the following conditions:
1. The pair \((F, S)\) and \((G, T)\) are weakly compatible.

2. \(\lim_{n \to \infty} Fx_n = \lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Gy_n = \lim_{n \to \infty} Ty_n = z\) for some \(z\) in \(X\).

3. \(d(Fx_n, Gy_n, t) \leq \phi [\min \{d(Sx, Ty, t), d(Fx_n, Sx_n, t), d(Gy_n, Ty_n, t), d(Fx_n, Ty_n, t), d(Sx_n, Gy_n, t)\}]\)

Where \(\phi\) is a contractive modulus. Then the pair \(F, G, S\) and \(T\) has a unique common fixed point in \(X\).

Proof: Since \(\lim_{n \to \infty} Fx_n = \lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Gy_n = \lim_{n \to \infty} Ty_n = z\) for some \(z\) in \(X\) since \(\lim_{n \to \infty} Ty_n = z\) \(\exists\) a point \(\alpha \in X\) s.t. \(z = T\alpha\) refers to proof of theorem 4.1 we have corollary 4.1.