Summary of the dissertation

The present dissertation deals with the study of fixed point theory in 2-metric space, which is used to show the existence and uniqueness of the solution of differential equation, integral equation and many other branches of mathematics and sciences.

Chapter 1 contains a brief introduction about fixed point theory and use of fixed point in various fields. Thus this chapter contains definition and examples of fixed point, types of fixed point theory such as discrete fixed point theory, metric fixed point theory etc. It also includes history of fixed point in 2-metric spaces, various papers established by many authors in 2-metric space.

Chapter 2 contains two common fixed theorems for four mappings in complete 2-metric spaces. Main result in this chapter is published in Applied Science Research, 2012, 3(5):2807-2814, this journal is published by Pelagia Research Library which is abstracted and indexed in Scopus, Google scholar, Indian Science abstracts, index Copernicus, EBSCO, SCIRUS and many more databases.

Chapter 3 includes a unique common fixed point theorem for two pairs of weakly compatible maps in a complete 2-metric space. We use weakly compatibility and contractive modulus to prove the main result. The theorem in this chapter is published in International Journal of Mathematics Archive 3(10), 2012 and this journal is indexed in Copernicus, Google scholar and many more.

Chapter 4 contains the proof of a common fixed point theorem for four mappings in complete 2-metric spaces. This chapter is accepted for publication in International Journal of Applied Mathematics Analysis and Application.

Chapter 5 Contains references
1. Introduction and Preliminaries

A fixed point of a function is a point that is mapped to itself by the function. It is also known as an invariant point. Let X be a set, A and B two nonempty subsets of X and $f: A \rightarrow B$ be a map. Then $x$ called a fixed point of $f$. When does a point $x \in A$ such that $f(x) = x$. A fixed point is not a root of the equation $g(x) = 0$ if it is a solution of the equation $g(x) = x$.

Geometrically, the fixed points of a function $g(x)$ are the point of intersection of the curve $y = g(x)$ and the line $y = x$. For example, if $f$ is defined on the real numbers by $f(x) = x^2 - 3x + 4$ then 2 is a fixed point of $f$, because $f(2) = 2$.

The theory of fixed point has been applied to show the existence and uniqueness of the solution of differential equation, integral equation and many other branches of mathematics. Fixed point theorems are very important in fields of economic theory, steady state temperature, flow of fluids, chemical equations. Fixed point theory of ordered banach spaces provide us exact or approximate solution of boundary value problems.

Fixed Point Theory is divided into three major areas:

1. Topological Fixed Point Theory

2. Metric Fixed Point Theory

3. Discrete Fixed Point Theory topology induced by 2-metric space is called 2-metric topology, which is generated by the set of all open spheres with two centers. Many authors used the topology in many applications; for example, El Naschie used this sort of the topology in physical applications.

One of important generalization is probabilistic metric space (PM space) which was introduced by Austrian mathematician K. Menger in 1942, using the notion of distribution functions in place of non-negative real numbers. Then, in 1959, A. Sklar and B. Schweizer obtained some fundamental results in probabilistic metric spaces. In 1972, H. Sehgal and A. T. Bharucha-Reid introduced contraction mapping on probabilistic metric space as an extension of famous Banach contraction principle to study a fixed point under the special $t$-norm. Also, H. Sherwood showed
that, for a very large class of triangular norms, it is possible to construct complete Menger probabilistic metric space together with the contractive mappings which do not have fixed point. The compatibility idea was introduced by G. Jungek[5, 6, 7].

Metric space was introduced by French mathematician M. Fréchet in 1906 and since then there corresponds several generalizations of metric space in the literature. A metric space is a set $X$ together with a function $d : X \times X \rightarrow \mathbb{R}$ satisfying the properties:

1. $d(x, y) \geq 0$ and $d(x, y) = 0 \iff x = y$,
2. $d(x, y) = d(y, x)$,
3. $d(x, y) + d(y, z) \geq d(x, z)$.

For example the function $d : \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$d(x, y) = |x - y|$$

for all $x, y \in \mathbb{R}$, is a metric on $\mathbb{R}$ of all real numbers. This is also known as usual metric on $\mathbb{R}$.

In 1922 Banach published his fixed point theorem also known as Banach's Contraction Principle uses the concept of Lipschitz mappings. A lipschitzian mapping with a lipschitz constant $k$ less than 1, i.e. $k < 1$, is called contraction. He gives the idea that $T$ has a unique fixed point, which can be reached from any starting value, $z$ in $X$. fixed point theory was extended to multivalued mappings in 1941 with the fixed point theorems given by S. Nadler (1969) and R. Fraser (1969). The lipschitz condition is a purely metric condition as it makes sense for functions from one metric space to another.

The concept of 2-metric space is a natural generalization of the metric space. A 2-metric space is a set $X$ with a function $d : X \times X \times X \rightarrow [0, \infty)$ satisfying the following conditions:

(1) For two distinct points $x, y \in X$, there exists a point $z \in X$ such that $d(x, y, z) \neq 0$,
(2) $d(x, y, z) = 0$ if at least two of $x, y, z$ are equal,
(3) $d(x, y, z) = d(x, z, y) = d(y, z, x)$,
(4) $d(x, y, z) \leq d(x, y, u) + d(x, u, z) + d(u, y, z)$ for all $x, y, z, u \in X$.

A frequent example of 2-metric is $(\mathbb{R}^2, p)$, where $p(x, y, z)$ is area of the triangle formed by joining the three points $x, y, z \in \mathbb{R}^2$.
Initially, it has been investigated by S. Gähler[17,18] in 1960s. It has been shown by Gahler that a 2-metric is a continuous function of any one of its three arguments but it need not be continuous in two arguments. If it is continuous in two arguments, then it is continuous in all three arguments. A 2-metric which is continuous in all of its arguments will be called continuous. Gahler claimed that a 2-metric is a generalization of the usual notion of a metric, but different authors proved that there is no relation between these two functions. For instance Ha et al [8] show that a 2-metric need not be a continuous function of its variables, whereas an ordinary metric is, further there is no easy relationship between results obtained in the two settings, in particular the contraction mapping theorem in metric spaces and in 2-metric spaces are unrelated. A 2-metric is a function \( d(x, y, z) \) symmetric under permutations, satisfying the tetrahedral inequality \( d(x, y, z) \leq d(x, y, a) + d(x, a, z) + d(a, y, z) \) for all \( x, y, z, a \in X \). \( d(x, y, z) \) is the area of the triangle spanned by \( x, y, z \).

Iseki [11, 12, 13] for the first time developed a fixed point theorem in 2-metric spaces. After this, number of fixed point theorems has been proved by many researchers like H.K. Pathak[9,10] and M.S. Khan for 2-metric spaces by introducing compatible mappings, which are more general than commuting and weakly commuting mappings.

In (1977) S.N. Lal and A.K. Singh give an analogue of Banach's contraction principle for 2-metric spaces. In 1979 M.S. Khan [14, 15] proves some fixed point theorems in 2-metric spaces and also discusses the convergence of sequences of mappings and fixed point in such spaces. Ashok k. Sharma[1,2,3] in (1980) in his note on fixed point in 2-metric space states that theorem given by khan(1979) are wrong he states that the inequalities \( d(f(x),f(y),a) < d(x,y,a) \) or \( d(f(x),g(y),a) < d(x,y,a) \) where \( x, y, a \in X \) and \( x \neq y \) and \( f \) and \( g \) are self mappings defined on 2-metric space (\( X,d \)) appear to be wrong, since \( d(x,y,a) \) vanishes even if the three points \( x, y, a \in X \) are distinct. To observe this fact he gives an example i.e. let \( R^2 \) be the Euclidean space. Let \( d(x,y,z) \) denote the area of the triangle formed by joining the three points \( x,y,z \in R^2 \). Then \( (R^2,d) \) is a 2-metric space and \( d(x,y,z)=0 \) for any three distinct points \( x,y,z \in R^2 \) lying on the same straight line. Theorems of Rhoades [4] (1979) which are mentioned by khan (1979) are also wrong because of the same reason the same reason.

In 1984, B.C. Dhage further generalized 2-metric space to D-metric space. In 1992, Dhage introduced the notion of D-metric space and proved the existence of a unique fixed point of a
self-mapping satisfying a contractive condition. Dhage’s definition uses the symmetry and tetrahedral axioms present in Gahler’s definition, but includes the coincidence axiom that \( d(x, y, z) = 0 \) if and only if \( x = y = z \). In 1994, Adrian Constantin gives a result similar to Hadzic’s common fixed point theorem on metric space for the case of 2-metric space.

In (2000) S. Venkata Ratnam Naidu introduce the notion of compatibility for a pair of self-maps on a 2-metric space and fixed point theorems for pairs as well as quadruples of self-maps on a 2-metric space satisfying certain generalized contraction conditions.

In (2001) Zeqing Liu and Fengrong Zhang give some necessary and sufficient conditions for the existence of common fixed point of a pair of mappings in 2-metric spaces. They also generalize, improve and unify a number of fixed point theorems given by Cho[22, 23], Khan, Fisher, Kubiat, Rhoades and others.

In (2006) R. K. Namdeo, Sweetee Dubey and Kenan Tas obtained a unique coincidence value for a class of self mappings satisfying non-expansive type condition on 2-metric spaces under various conditions and a fixed point theorem is also obtained. In (2008) Agile C. T. and Salunke J. N. also presents some fixed point theorems and common fixed point theorems in 2-metric spaces. In (2008) M.E. Abd EL-Monsef, H.M. Abu-Donia, Kh. Abd-Rabou establish common fixed point theorems for set-valued mappings between 2-metric spaces and generalize some definitions in 2-metric spaces and theorems by Ahmed [Ahmed MA. Common fixed point theorems for weakly compatible mappings], Fisher [B. Fisher Common fixed points of mappings and set-valued mappings on a metric spaces]. Also study common fixed point theorems for four mappings on 2-metric spaces and compact 2-metric spaces.

In 2009 Mantu saha and Debashis dey establish fixed point theorems for a single mapping to the common fixed point and coincidence point results for a certain class of mappings. Also obtained some fixed point results for a class of contractive type mappings in a setting of 2-metric space and deal with the mixed type of contraction mappings. Also proved some fixed point theorems for A-contraction mappings in a 2-metric space.

In (2010) Abdelkrim Aliouche and Carlos Simpson consider bounded 2-metric spaces satisfying an additional axiom, and show that a contractive mapping has either a fixed point or a fixed line.

In (2010) Kieu Phuong Chi* and Hoang Thi Thuy prove a fixed point theorem in 2-metric spaces for a class of maps that satisfy a contractive condition dependent on another function. This is a
proper extension of fixed point theorem by Lal and Singh. In (2010) M. Saha, Debashis Dey and Anamika Ganguly improved and generalized some fixed point results due to B. Fisher, V. Popa and Cho et al.[22, 23] over two different complete 2-metric spaces. In (2010) Rajesh Srivastava, Sabhatkant Dwivedi and ramakant Bhardwaj establish a common fixed point theorem in 2-metric spaces for rational inequality taking three mappings which generalize many known results. In (2011) B. K. Lahiri, Pratulananda Das and Lakshmi Kanta Dey, for the first time, establish Cantor’s intersection theorem and Baire category theorem in 2-metric spaces. And then apply Cantor’s theorem to establish some fixed point theorems in such spaces. Deo Brat Ojha prove some common fixed point theorems in 2-Metric spaces for two pairs of weakly compatible mapping satisfying integral type implicit relation, which improves and extends several known results.

In 2011 Vishal Gupta et al.[19, 20, 21] give coupled fixed point theorem in partially ordered 2-metric spaces. In 2012 Vishal Gupta give some common fixed point theorem and fixed point theorem for a class of contraction on 2-metric spaces.

In (2012) S. N. Mishra, Rajendra Pant and S. Stoffle gives Stability results for a pair of sequences of mappings and their common fixed points in a 2-metric space using certain new notions of convergence are proved. The results obtained extend and unify several known results. The relationship between the convergence of a sequence of self mappings of a Metric space and their fixed points, known as the stability or Continuity of fixed points, has been widely studied in fixed point theory in various settings. The origin of this problem is into a classical result of Bonsall for contraction mappings. Subsequent results by Nadler and others address mainly the problem of replacing the completeness of the space X by the existence of fixed points (which was ensured otherwise by the completeness of X) and various relaxations on the contraction constant: Recently, in attempt to provide the localized versions of certain stability results of Bonsall and Nadler, significant weakening were made to the well-known notions of point wise and uniform convergence by Barbet and Nachi. In addition, these notions have been successfully utilized by them to obtain certain stability results in a metric space. These results have been further generalized by Mishra and Kalinde, Mishra, Singh, Pant and Mishra, Singh and Stoffle. A 2-metric is a function $d(x, y, z)$ symmetric under permutations, satisfying the tetrahedral inequality $d(x, y, z) \leq d(x, y, a)+d(x, a, z)+d(a, y, z)$ for all $x, y, z, a \in X$. $d(x, y, z)$ is the area of the triangle spanned by $x, y, z$. This notion has been considered by several authors, who have
notably generalized Banach's principle to obtain fixed point theorems, for example White, Iseki, Rhoades, Khan, Singh, Tiwari and Gupta, Naidu and Prasad, Naidu and Zhang, Abd El-Monsef, Abu-Donia, Abd-Rabou, Ahmed and others. The contractivity conditions used in these works are usually of the form \( d(F(x), F(y), a) \leq \ldots \) for any \( a \in X \). We may think of this as meaning that \( d(x, y, a) \) is a family of distance-like functions of \( x \) and \( y \), indexed by \( a \in X \). However, Hsiao has shown that these kinds of contractivity conditions don't have a wide range of applications, since they imply colinearity of the sequence of iterates starting with any point. There have also been several different notions of a space together with a function of 3-variables. Dhage's definition uses the symmetry and tetrahedral axioms present in Gahler's definition, but includes the coincidence axiom that \( d(x, y, z) = 0 \) if and only if \( x = y = z \).

**Definition-2.1** A metric space is a set \( X \) together with a function \( d: X \times X \rightarrow \mathbb{R} \) satisfying the properties:

1. \( d(x, y) \geq 0 \) and \( d(x, y) = 0 \iff x = y \),
2. \( d(x, y) = d(y, x) \),
3. \( d(x, y) + d(y, z) \geq d(x, z) \).

**Definition-2.2** A 2-metric space is a set \( X \) with a function \( d: X \times X \times X \rightarrow [0, \infty) \) satisfying the following conditions:

1. For two distinct points \( x, y \in X \), there exists a point \( z \in X \) such that \( d(x, y, z) \neq 0 \),
2. \( D(x, y, z) = 0 \) if at least two of \( x, y, z \) are equal,
3. \( D(x, y, z) = d(x, z, y) = d(y, z, x) \),
4. \( D(x, y, z) \leq d(x, y, u) + d(x, u, z) + d(u, y, z) \) for all \( x, y, z, u \in X \).

**Definition-2.3** A fixed point of a function is a point that is mapped to itself by the function. It is also known as an invariant point. Let \( X \) be a set, \( A \) and \( B \) two nonempty subsets of \( X \) and \( f: A \rightarrow B \) be a map. Then \( x \) called a fixed point of \( f \), When does a point \( x \in A \) such that \( f(x) = x \).

**Definition 2.4** A sequence \( \{ x_n \} \) said to be a Cauchy sequence in 2-metric space \( X \), if for each

\[
\lim_{m,n \to \infty} d(x_m, x_n, a) = 0
\]
Definition 2.5 A sequence \( \{x_n\} \) in 2-metric space \( X \) is convergent to an element \( x \in X \) if for each \( a \in X \), 
\[
\lim_{n \to \infty} d(x_n, x, a) = 0
\]

Definition 2.6 A complete 2-metric space is one in which every Cauchy sequence in \( X \) converges to an element of \( X \).

Definition 2.7 Let \( A \) and \( S \) be mappings from a metric space \( (X, d) \) in to itself, \( A \) and \( S \) are said to be weakly compatible if they commute at their coincidence point. i.e. \( Ax = Sx \) for some \( x \in X \) \( \Rightarrow ASx = SAx \)

Definition 2.8 Two self maps \( f \) and \( g \) of a metric space \( (X, d) \) are called compatible if
\[
\lim_{n \to \infty} d(fgx_n, gfx_n) = 0 \quad \text{whenever} \quad \{x_n\} \quad \text{is a sequence in} \quad X \quad \text{such that} \quad \lim_{n \to \infty} fx_n = \lim_{n \to \infty} gx_n = t \quad \text{for some} \quad t \quad \text{in} \quad X.
\]

Definition 2.9 Two self maps \( f \) and \( g \) of a metric space \( (X, d) \) are called non compatible if \( \exists \) at least one sequence \( \{x_n\} \) such that
\[
\lim_{n \to \infty} fx_n = \lim_{n \to \infty} gx_n = t \quad \text{for some} \quad t \quad \text{in} \quad X \quad \text{but}
\]
\[
\lim_{n \to \infty} d(fgx_n, gfx_n) \quad \text{is either} \quad \text{non zero or non existent.}
\]

Definition 2.10 Maps \( f \) and \( g \) are said to be commuting if \( fgx = gf \) for all \( x \in X \).

Definition 2.11 A mapping \( f \) from a 2-metric space \( (X, d) \) into itself is said to be continuous at \( x \in X \) if for every sequence \( \{x_n\} \) in \( X \) such that \( \lim_{n \to \infty} d(x_n, x, a) = 0 \) for all \( a \in X \), \( \lim_{n \to \infty} d(fx_n, fx, a) = 0 \). \( f \) is called continuous on \( X \) if it is so at all points of \( X \).

Definition 2.12 A self-map \( T \) on a metric space \( X \) is said to be \( A \)-contraction, if it satisfies the condition
\[
d(Tx, Ty) \leq \alpha [d(x, y), d(x, Tx), d(y, Ty)] \quad \text{for all} \quad x, y \in X \quad \text{and some} \quad \alpha \in A.
\]

Definition 2.13 A 2-metric space \( (X, d) \) is said to be bounded if there is a constant \( K \) such that
\[
d(a, s, b, c) \leq K \quad \text{for all} \quad a, b, c \in X.
\]
Definition-2.14 A function $\phi : [0, \infty) \rightarrow [0, \infty)$ is said to be contractive modulus if $\phi : [0, \infty) \rightarrow [0, \infty)$ and $\phi(t) < t$ for $t > 0$.

Definition-2.15 A real valued function $\phi$ defined on $X \subseteq R$ is said to be upper semi continuous if $\lim \sup_{x \to t_n} \phi(t_n) \leq \phi(t)$, for every sequence $(t_n) \in X$ with $t_n \to t$ as $n \to \infty$.

Definition-2.16 Let $M$ be a set and the distance function $d$ is a mapping from $M \times M$ into $R^+$, set of non-negative real numbers, satisfying the following conditions:

1. $d(x, y) \geq 0$ for $x \neq y$, (non-negativity)
2. $d(x, y) = 0$ iff $x = y$. (identity of indiscernibles)
3. $d(x, z) \leq d(x, y) + d(y, z)$, (subadditivity / triangular inequality)

Then the pair $(M, d)$ is called Quasi-metric space.

Definition-2.17 Let $M$ be a set and the distance function $d$ is a mapping from $M \times M$ into $R^+$, set of non-negative real numbers, satisfying the following conditions:

1. $d(x, y) \geq 0$ for $x \neq y$, (non-negativity)
2. $d(x, y) = 0$ iff $x = y$. (identity of indiscernibles)
3. $d(x, y) = d(y, x)$, (symmetry),

Then the pair $(M, d)$ is called Semi-metric space.

Definition-2.18 Let $M$ be a set and the distance function $d$ is a mapping from $M \times M$ into $R^+$, set of non-negative real numbers, satisfying the following conditions:

1. $d(x, y) \geq 0$ for $x \neq y$, (non-negativity)
2. $d(x, y) = d(y, x)$, (symmetry),
3. $d(x, z) \leq d(x, y) + d(y, z)$, (subadditivity / triangular inequality),
4. $d(x, x) = 0$, (reflexivity), Then, the pair $(M, d)$ is called Pseudometric space.

Definition-2.19 A mapping $F: R^+ \rightarrow R^+$ is called a distribution function if it is non-decreasing and left-continuous with $\inf_{t \in R} F(t) = 0$, and $\sup_{t \in R} F(t) = 1$. We denote $\Delta_n$ to be the class of all distribution functions $F$ such that $F(0) = 0$. A distance distribution function is a mapping.
F: [0, \infty) \rightarrow [0, 1] which is non-decreasing, left continuous on (0, \infty) and F(0) = 0. Also, \(D_+\) is the subset of \(\Delta_+\) containing all functions \(F\) with the condition \(\lim_{t \to \infty} F(t) = 1\).

Definition-2.20 A metric space which is continuous in all its three argument is called continuous.

2. Purposed work:
The aim of this Dissertation is to discuss the concept of 2-metric spaces and to prove some common fixed point theorems for different type of mapping in 2-metric spaces.
3. References:


