Chapter 5

**BIVARIATE VARIANCE RESIDUAL LIFE**

5.1. Introduction

It is evident from the earlier discussions that the three basic reliability concepts namely, the survival function, the failure rate and the mean residual life are equivalent, in the sense that knowing any one of them, the other two can be uniquely determined. Another concept which has generated interest in life length studies in recent years is the variance residual life (VRL) which was briefly reviewed in Section 1.2.1.4. A special feature of the VRL is that it does not determine the corresponding life distribution uniquely. This and the fact that monotone behaviour of the VRL generate classes of distribution that are not covered by similar behaviour of the other concepts makes it an interesting proposition to extend the concept from the univariate to the bivariate case. As in the case of bivariate MRL, the bivariate VRL is useful in modelling and analyzing failure data of a two-component system which does not satisfy the assumption of independence among the components.
life times. Apart from looking at several properties of this function, in the present chapter we develop some characterization theorems of the life distribution associated in the previous chapter. Further, we introduce four new classes of life distributions that are determined by the monotone behaviour of the VRL. Also, the monotonic properties of the VRL are characterized in terms of the residual coefficient of variation and is used to study the behaviour of VRL for certain family of bivariate distributions. It is demonstrated that using these different ageing criteria derived from the VRL, one can make comparison among various lifetime models generated in different physical situations.

5.2. Definition and Properties

Let $X = (X_1, X_2)$ be a bivariate random vector admitting absolutely continuous distribution function with respect to Lebesgue measure in the positive octant $R_2^+ = \{(x_1, x_2) | x_1, x_2 > 0\}$ of the two dimensional Euclidean space $R_2$ and having survival function $R(x_1, x_2)$. Assume that $E(X_i^2) < \infty$, $i = 1, 2$. Then the random vector

$$V(x_1, x_2) = (V_1(x_1, x_2), V_2(x_1, x_2))$$

(5.1)
where,

\[ V_i(x_1, x_2) = E[(X_i - x_i)^2 | X_1 \geq x_1, X_2 \geq x_2] - r_i^2(x_1, x_2) \]  

(5.2)

is defined as the bivariate variance residual life and \( V_i(x_1, x_2) \) as its components. The properties of VRL are the following,

1. \[ V_i(x_1, x_2) = 2[R(x_1, x_2)]^{-1} \int_{x_1}^{\infty} \int_{x_2}^{\infty} (t_1 - x_1)^2 R(t_1, t_2) dt_1 dt_2 \]

(6.3)

Proof:

\[ E[(X_i - x_i)^2 | X_1 \geq x_1, X_2 \geq x_2] = [R(x_1, x_2)]^{-1} \int_{x_1}^{\infty} \int_{x_2}^{\infty} \frac{\partial^2 R}{\partial t_1 \partial t_2} dt_1 dt_2 \]

\[ = 2[R(x_1, x_2)]^{-1} \int_{x_1}^{\infty} (t_1 - x_1) R(t_1, x_2) dt_1 \]

\[ = 2[R(x_1, x_2)]^{-1} \int_{x_1}^{\infty} \int_{x_2}^{\infty} R(t_1, x_2) dy_2 dt_1 \]

\[ = 2[R(x_1, x_2)]^{-1} \int_{x_1}^{\infty} r_1(y_1, x_2) R(y_1, x_2) dy_1 \]

(5.4)
Substituting (5.4) on (5.2), we recover (5.3).

On similar lines,

\[ V_2(x_1, x_2) = 2[R(x_1, x_2)]^{-1} \int_{x_2}^{\infty} r_2(x_1, y_2) R(x_1, y_2) dy_2 - r_2^2(x_1, x_2). \]

(5.5)

Formulas (5.3) and (5.5) express the VRL vector in terms of the survival function and the mean residual life of \((X_1, X_2)\).

Formula (5.6) will be used in the sequel to explore the monotone behavior of VRL.

2. \( \frac{\partial V_i}{\partial x_i} = h_i(x_1, x_2) [V_i(x_1, x_2) - r_i^2(x_1, x_2)] \). (5.6)

3. When \( X_1 \) and \( X_2 \) are independent random variables

Proof:

Differentiating (5.3) with respect to \( x_i \), we have

\[ \frac{\partial V_i}{\partial x_i} = -2r_i(x_1, x_2) \left[1 + \frac{\partial r_i}{\partial x_i}\right] + 2h_i(x_1, x_2) [V_i(x_1, x_2) + r_i^2(x_1, x_2)]. \]

(5.7)

Using the relationship,

\[ h_i(x_1, x_2) = \frac{1 + \frac{\partial r_i}{\partial x_i}}{r_i(x_1, x_2)}, \]
the equation (5.7) becomes

\[
\frac{\partial V_1}{\partial x_1} = -2 r_1^2(x_1, x_2) h_1(x_1, x_2) + h_1(x_1, x_2) [V_1(x_1, x_2) + r_1^2(x_1, x_2)]
\]

\[
= h_1(x_1, x_2) [V_1(x_1, x_2) - r_1^2(x_1, x_2)].
\]

The relationship (5.6) will be used in the sequel to explore the monotone behaviour of VRL.

3. When \( X_1 \) and \( X_2 \) are independent random variables

\[
V(x_1, x_2) = (V_1(x_1, 0), V_2(0, x_2)).
\]

\[
(5.8)
\]

**Proof:**

When \( X_1 \) and \( X_2 \) are independent

\[
r_1(x_1, x_2) = r_1(x_1, 0)
\]

and

\[
R(x_1, x_2) = R(x_1, 0) R(0, x_2).
\]

Thus,

\[
V_1(x_1, x_2) = \frac{2}{R(x_1, 0) R(0, x_2)} \int_{x_1}^{\infty} r_1(y, 0) R(y, 0) R(0, x_2) dy - r_1^2(x_1, 0)
\]

\[
= V_1(x_1, 0).
\]
Similarly
\[ V_2(x_1, x_2) = V_2(o, x_2). \]

It is easy to see that \( V_1(x_1, o) \) and \( V_2(x_1, o) \) are the univariate VRL's of the component variables \( X_1 \) and \( X_2 \).

4. The VRL in general does not determine a distribution uniquely.

Proof:

To prove the assertion, consider the bivariate Pareto model of Lindley and Singpurwalla (1986) with

\[ R(x_1, x_2) = (1+a_1 x_1+a_2 x_2)^{-c}, \quad x_1, x_2 > 0. \] (5.9)

For this distribution, by direct computation we get

\[ \frac{(1+a_1 x_1+a_2 x_2)^2}{a_1^2(c-1)(c-2)} R(x_1, x_2) = \int_{x_1}^{\infty} r_1(y_1, x_2) R(y_1, x_2) dy_1. \]

Substituting in (5.4)

\[ \frac{(1+a_1 x_1+a_2 x_2)^2}{a_1^2(c-1)(c-2)} R(x_1, x_2) = \int_{x_1}^{\infty} r_1(y_1, x_2) R(y_1, x_2) dy_1. \] (5.10)
Differentiating both sides of (5.10) with respect to $x_1$, we have

\[
\frac{(1+a_1 x_1 + a_2 x_2)^2}{a_1^2 (c-1)(c-2)} \frac{\partial R}{\partial x_1} + \frac{2(1+a_1 x_1 + a_2 x_2)}{a_1 (c-1)(c-2)} R(x_1, x_2) = -r_1(x_1, x_2) R(x_1, x_2). \tag{5.11}
\]

Dividing by $R(x_1, x_2)$ and using the definition of bivariate failure, (5.11) becomes

\[
\frac{(1+a_1 x_1 + a_2 x_2)^2}{a_1^2 (c-1)(c-2)} h_1(x_1, x_2) + 2 \frac{(1+a_1 x_1 + a_2 x_2)}{a_1 (c-1)(c-2)} = -r_1(x_1, x_2). \tag{5.13}
\]

Further simplification is achieved by using the relationship between $h_1(x_1, x_2)$ and $r_1(x_1, x_2)$. This leaves the differential equation

\[
\frac{\partial r_1}{\partial x_1} = A(x_1) r_1^2(x_1, x_2) + B(x_1) r_1(x_1, x_2) + C(x_1) \tag{5.12}
\]

where,

\[
A(x_1) = a_1^2 (c-1)(c-2)/(1+a_1 x_1 + a_2 x_2)^2,
\]

\[
B(x_1) = 2a_1/(1+a_1 x_1 + a_2 x_2)
\]

and

\[
C(x_1) = -1. \tag{5.13}
\]
Treating $x_2$ as a constant, (5.12) is Riccati's equation, which can be solved by the method described in Simmons (1974, p. 63).

Since $r_1^2 = \frac{(1 + a_1 x_1 + a_2 x_2)}{a_1(c-1)}$ satisfies (5.12), we choose it as a particular solution to write the general solution of (5.12) as

$$r_1(x_1, x_2) = \frac{1 + a_1 x_1 + a_2 x_2}{a_1(c-1)} + Z(x_1, x_2)$$

where, $Z$ satisfies the equation,

$$\frac{\partial Z}{\partial x_1} - [B(x_1) + 2r_1(x_1, x_2)A(x_1)]Z = A(x_1)Z^2. \quad (5.13)$$

The next step is to solve (5.13). Towards this end, we set $y = \frac{1}{Z}$ in (5.13) to find

$$\frac{\partial y}{\partial x_1} + [B(x_1) + 2r_1(x_1, x_2)A(x_1)]y = -A(x_1) \quad (5.14)$$

which is of the well-known linear form.

Substituting the expressions for $A(x_1)$ and $B(x_1)$ in (5.14), we have

$$\frac{\partial y}{\partial x_1} + \frac{2a_1(c-1)}{1 + a_1 x_1 + a_2 x_2} y = \frac{-a_1^2(c-1)(c-2)}{(1 + a_1 x_1 + a_2 x_2)^2}. \quad (5.15)$$
The solution of the above equation will be

\[ y \cdot e^{\int P(x_1) \, dx_1} = \int Q(x_1) \, e^{\int P(x_1) \, dx_1} \, dx_1 + K(x_2) \]

where,

\[ P(x_1) = \frac{2a_1(c-1)}{1 + a_1 x_1 + a_2 x_2} \]

and

\[ Q(x_1) = \frac{-a_1^2(c-1)(c-2)}{(1 + a_1 x_1 + a_2 x_2)^2} \]

By direct integration,

\[ \int P(x_1) \, dx_1 = (1 + a_1 x_1 + a_2 x_2)^{2(c-1)} \]

and

\[ \int Q(x_1) e^{\int P(x_1) \, dx_1} \, dx_1 = \frac{-(c-1)(c-2)a_1 (1 + a_1 x_1 + a_2 x_2)^{2c-3}}{2c-3} \]

Therefore the solution of the differential equation (5.15) is

\[ y = \frac{K(x_2)(2c-3) - a_1(c-1)(c-2)(1 + a_1 x_1 + a_2 x_2)^{2c-3}}{(2c-3)(1 + a_1 x_1 + a_2 x_2)^{2(c-1)}} \]

where, \( c_1 \) and \( c_2 \) are independent of both \( x_1 \) and \( x_2 \) if and only if \( X=(x_1,x_2) \) is distributed as \( E(a_1,a_2,0) \).
Thus the general solution of (5.12) is prescribed as

\[ r_1(x_1, x_2) = \frac{1+a_1 x_1 + a_2 x_2}{a_1(c-1)} + \frac{1}{y} \cdot \]

A similar result can be obtained in the case of \( r_2(x_1, x_2) \). We have therefore different sequences of truncated moments arising from (5.12) and each such sequence must correspond to a particular distribution and our assertion is completely proved.

5.3. Characterizations

Even though, VRL in general does not determine a distribution uniquely, we can use the relationship among VRL and MRL to characterize certain bivariate life models. Further, in certain cases VRL itself can be employed to characterize some bivariate distribution.

Theorem 5.1.

The VRL vector

\[ V(x_1, x_2) = (c_1, c_2) \quad (5.16) \]

where, \( c_1 \) and \( c_2 \) are independent of both \( x_1 \) and \( x_2 \) if and only if \( X=(X_1, X_2) \) is distributed as \( E(\alpha_1, \alpha_2, 0) \).
Proof:

To prove the assertion we directly from (5.6) that

\[ h_i(x_1, x_2) \left[ c_i - r_i^2(x_1, x_2) \right] = 0, \quad i=1,2. \]

This implies \( r_i(x_1, x_2) = l_i \), a constant independent of both \( x_1 \) and \( x_2 \).

Therefore, from Nair and Nair (1989), \( X_1 \) and \( X_2 \) follows \( E(\alpha_1, \alpha_2, 0) \). Conversely, when \( X = (X_1, X_2) \) follows \( E(\alpha_1, \alpha_2, 0) \), the VRL

\[ V(x_1, x_2) = (\frac{1}{\alpha_1}, \frac{1}{\alpha_2}), \]

is a constant vector and the proof is complete.

Theorem 5.2.

The random vector \( X = (X_1, X_2) \) follows Gumbel's bivariate exponential model \( E(\alpha_1, \alpha_2, \theta) \) if and only if

\[ V(x_1, x_2) = (B_1(x_2), B_2(x_1)). \] \hspace{1cm} (5.17)

Proof:

For the Gumbel's bivariate exponential distribution

\[ V(x_1, x_2) = \left( \frac{1}{(\alpha_1 + \theta x_2)^2}, \frac{1}{(\alpha_2 + \theta x_1)^2} \right) \]
so that the if part is true.

Conversely, from (5.6) and the stipulated form for $V(x_1, x_2)$,

$$h_1(x_1, x_2) [V_1(x_1, x_2) - r_1^2(x_1, x_2)] = 0$$

or

$$V_1(x_1, x_2) = r_1^2(x_1, x_2).$$

Therefore,

$$r_1(x_1, x_2) = A_i(x_j), i, j = 1, 2, i \neq j,$$

which is a characteristic property of the Gumbel's bivariate exponential distribution established in Nair and Nair (1988).

Sometimes it is more convenient to deal with the coefficient of variation of residual life rather than the VRL. We now prove a characterization theorem based on the values assumed by the coefficient of variation

$$C(x_1, x_2) = (C_1(x_1, x_2), C_2(x_1, x_2),$$

where,

$$C_1^2(x_1, x_2) = \frac{V_1(x_1, x_2)}{r_1^2(x_1, x_2)}.$$
Theorem 5.3.

Let \( X = (X_1, X_2) \) be a non-negative random vector admitting absolutely continuous distribution function with respect to Lebesgue measure, such that \( E(X_1^2) < \infty \). Then \( C_1(x_1, x_2) = k \), a constant if and only if \( X \) is

(i) Gumbel's exponential distribution \( E(\alpha_1, \alpha_2, \theta) \) for \( k = 1 \),

(ii) bivariate Pareto Model \( P(a_1, a_2, b, c) \) for \( k > 1 \)

and

(iii) bivariate finite range distribution \( F(p_1, p_2, q, d) \) for \( 0 < k < 1 \).

Proof:

We first prove the necessary part. The given condition can be stated as

\[
V_1(x_1, x_2) = l r_1^2(x_1, x_2),
\]

where,

\[
l = k^2.
\]

Then from (5.6),

\[
h_1(x_1, x_2)(V_1(x_1, x_2) - r_1^2(x_1, x_2)) = l^2 r_1(x_1, x_2) \frac{\partial r_1}{\partial x_1}.
\]
Since,
\[ h_1(x_1, x_2) = r_1(x_1, x_2), \]
we have
\[ \frac{\partial r_1}{\partial x_1} = m, \]
where,
\[ m = \frac{q - 1}{q + 1}. \]

The general solution to the last equation is
\[ r_1(x_1, x_2) = mx_1 + B_1(x_2). \]

Similarly one can obtain,
\[ r_2(x_1, x_2) = mx_2 + B_2(x_1). \]

We conclude that the distribution of X is as stated in Theorem 5.3 from Theorem 3.1.

In seeing the converse is true, note that
for \( E(\alpha_1, \alpha_2, \Theta) \),
\[ C_1^2(x_1, x_2) = 1, \]
for \( P(\alpha_1, \alpha_2, b, c) \),
\[ C_1^2(x_1, x_2) = \frac{c}{c-2} \]
and for \( F(p_1, p_2, q, d) \),
\[ C_1^2(x_1, x_2) = \frac{d}{d+2}. \]
Corollary

When $X_2 \rightarrow 0$ in $V_1(x_1,x_2)$, the univariate property

$$C_1(x_1) = k$$

holds if and only if $X$ is exponential, Pareto and finite range as proved in Mukherjee and Roy (1986).

5.4. Monotone behaviour of VRL

The reliability concepts like failure rate, MRL etc. are used to describe the pattern of functioning of the systems of components. However, in order to have a fuller understanding of the importance of various distributions in reliability theory, various notions of ageing are helpful (see Barlow and Proschan (1975)). Traditionally ageing is conveniently discussed and various life distributions are assessed in terms of the monotonic behaviour of failure rates or MRLFs. The aim of the present section is to introduce new classifications of bivariate distributions based on the monotone behaviour of VRL. The relationship of the classes so defined with other classes existing in literature and the chain of implications among them are also examined. Various classes and
their interpretation in terms of ageing behaviour are detailed below. In defining the various classes we have kept in mind that, the definitions should be based upon conditions imposed upon the joint survival function and not on the constituent random variables, they should be valid for the established definitions in the univariate case and that the arguments that generate bivariate definitions should be natural extensions in some sense of the corresponding univariate definition of VRL.

Definition 5.1.

A bivariate life distribution or random vector is decreasing (increasing) variance residual life–1 (D(I) VRL–1 ) if

\[ V_i(x_1+y_1, x_2+y_2) \leq (\geq) V_i(x_1, x_2), \quad i=1,2 \]  (5.18)

for all \((x_1,y_1)\) and \((x_2,y_2)\) in \(R^2_+\).

The condition (5.18) is appropriate when the components in a system with different ages \(x_i\) are required to survive different times \(y_1, y_2\) and implies that for DVRL–(1) (IVRL–(1)) distribution, the VRL at various ages decreases (increases) as the component ages.
The ages $x_1$ and $x_2$ are chosen to be distinct by anticipating a replacement policy or by considering new components at the same time origin after which time moves at different rates for the two components which is true of accelerate life test situations where the stresses operating on the two components are different. Obviously, the boundary of the two classes satisfy

$$V_i(x_1+y_1,x_2+y_2) = V_i(x_1,x_2), \quad i=1,2 \quad (5.19)$$

in which case $X$ is both DVRL-1 and IVRL-1. The following theorem explains the situation when (5.19) holds good.

**Theorem 5.4.**

The bivariate random variable $X$ is both DVRL-(1) and IVRL (1) if and only if $X_1$ and $X_2$ are independent and exponentially distributed.

**Proof.**

Condition (5.19) is equivalent to

$$V_i(x_1,x_2) = c_1$$

a constant independent of $x_1$ and $x_2$. The result now follows from Theorem 5.1.
Definition 5.2.

A bivariate distribution or random variable is said to be DVRL-(2) (IVRL-(2)) if

\[ V_1(x_1+y,x_2) \leq (\geq) V_1(x_1,x_2) \]

and

\[ V_2(x_1,x_2+y) \leq (\geq) V_2(x_1,x_2) \]  \hspace{1cm} (5.20)

for all \((x_1,x_2)\) in \(R_2^+\) and \(y > 0\).

The physical situation when condition (5.20) is of interest occurs when the individual lives of the components when the other has survived a specific lifetime are subject to study. In fact (5.20) means that given a two-component system with ages \(x_1\) and \(x_2\), the VRL of the \(i\)th component can be decreased (increased) on replacing it by a similar component of larger age.

Using differential calculus \((X_1,X_2)\) is DVRL (2) (IVRL(2)) according as

\[ \frac{\partial V_1}{\partial x_1} \leq (\geq) 0 \]  \hspace{1cm} (5.21)

and

\[ \frac{\partial V_2}{\partial x_2} \leq (\geq) 0 \]  \hspace{1cm} (5.22)

The boundary of the class is obviously the one satisfying the equality in (5.20). More precisely we have,
Theorem 5.5.

A bivariate distribution will be DVRL-(2) and IVRL-(2) if and only if it is the Gumbel's bivariate exponential model.

Proof:

From (5.21), the distribution satisfies the conditions of the theorem if and only if

\[ \frac{\partial V_1}{\partial x_1} = 0 \]

and

\[ \frac{\partial V_2}{\partial x_2} = 0 \]

or when

\[ V(x_1, x_2) = (B_1(x_2), B_2(x_1)) \]

which is the characteristic property of Gumbel's exponential model by Theorem 5.2.

There exists an implication between the DVRL-(2) (IVRL-(2)) class and the DMRL-(2) (IMRL-(2)) class, where the latter is defined by the condition

\[ r_1(x_1 + y, x_2) \leq (\geq) r_1(x_1, x_2) \]

and

\[ r_2(x_1, x_2 + y) \leq (\geq) r_2(x_1, x_2) \]

(5.22)

for all \((x_1, x_2)\) in \(\mathbb{R}_2^+\) and \(y > 0\).
Theorem 5.6.

Let $E(X_1^2)$ be finite and the life distribution is DMRL (2) (IMRL (2)). Then $X = (X_1, X_2)$ is DVRL (2) (IVRL (2)) at all points for which $R(x_1, x_2) > 0$.

Proof:
We prove the result only in the DMRL case.

The proof for the dual case will follow by reversing the inequality signs. Since the random vector has DMRL (2) distribution, one can write from (5.22) that

$$r_1(t_1, x_2) \leq r_1(x_1, x_2)$$
and

$$r_2(x_1, t_2) \leq r_2(x_1, x_2)$$

for all $t_1 \geq x_1 > 0$ and $t_2 \geq x_2 > 0$. Since $R(x_1, x_2)$ is positive for all $x_1, x_2 > 0$, it is true from (5.23) that

$$2[R(x_1, x_2)]^{-1} \int_{x_1}^{\infty} R(t_1, x_2) [r_1(t_1, x_2) - r_1(x_1, x_2)] dt_1 \leq 0.$$  

(5.24)

The first term on the left of the equation (5.24) is

$$E[(X_1 - x_1)^2 | X_1 > x_1, X_2 > x_2]$$

The result follows from the identity,
from equation (5.4) and using the relationship
\[ R(x_1, x_2) r_1(x_1, x_2) = \int_{x_1}^{\infty} R(t_1, x_2) dt_1, \]
the second term simplifies to \(-2r_1^2(x_1, x_2)\). Thus (5.24) reduces to
\[ \nu_1(x_1, x_2) - r_1^2(x_1, x_2) \leq 0. \]

Hence from (5.6), \( \frac{\partial \nu_1}{\partial x_1} \leq 0 \). Similarly there holds
the inequality \( \frac{\partial \nu_2}{\partial x_2} \leq 0 \) and thus the distribution is DVRL (2).

There exists a characterization of the DVRL-(2) (IVRL-(2)) models in terms of the coefficient of variation of residual life, which is presented in the following theorem.

Theorem 5.7.

The distribution of \( X=(X_1, X_2) \) has DVRL-(2) (IVRL-(2)) if and only if \( C_1(x_1, x_2) \leq (\geq) 1 \), where \( C_1(x_1, x_2) \) is defined in section 5.3.

Proof:

The result follows from the identity,
Definition 5.3.

A bivariate random variable $X$ on its distribution is said to have DVRL-(3) (IVRL-(3)) if

$$ V_i(x_1+y, x_2+y) \leq (\geq) V_i(x_1, x_2) $$

(5.25)

for all $(x_1, x_2)$ in $\mathbb{R}_2^+$ and $y > 0$.

This condition is of interest when the components have different ages $x_1$ and $x_2$ and our concern is to a common time horizon $y$. The difference in ages contemplated here can arise out of a replacement policy and the common time $y$ is of particular significance when we look at a series system. The condition (5.25), on the other hand, can be interpreted in the following way. As the system ages, the VRL of all components decrease (increase). Again the distribution separating the DVRL-(3) class and IVRL-(3) class should satisfy the property

$$ V_i(x_1+y, x_2+y) = V_i(x_1, x_2) $$

(5.26)

for all $(x_1, x_2)$ in $\mathbb{R}_2^+$ and $y > 0$. 

Theorem 5.8:

A bivariate random variable $X$ in the support of $\mathbb{R}_2^+$ with VRL-(3) and IVRL-(3) as it is distributed in the Marshall-Olkin bivariate exponential form.

\[
\frac{\partial V_i}{\partial x_1} = \frac{h_i(x_1, x_2)}{r_1^2(x_1, x_2)} [C_i(x_1, x_2) - 1]
\]

derived from (5.6).
Theorem 5.8.

A bivariate random vector $X$ in the support of $R_2^+$ with $E(X_1^2) < \infty$, is both DVRL-(3) and IVRL-(3) if it is distributed in the Marshall-Olkin bivariate exponential form.

Proof:

When $X$ follows Marshall and Olkin exponential model, it satisfies the bivariate lack of memory property

$$R(x_1+y, x_2+y) = R(x_1,x_2) R(y,y) \quad (5.27)$$

and from Zahedi (1985),

$$r_i(x_1+y, x_2+y) = r_i(x_1,x_2), \quad i=1,2. \quad (5.28)$$

Using (5.27) and (5.28) in the equation (5.3), we have

$$V_i(x_1+y, x_2+y) = V_i(x_1,x_2).$$

Thus $X$ has both DVRL-3 and IVRL-3

Definition 5.4.

The random vector $X$ or its distribution is said to possess the DVRL-(4) (IVRL-(4)) property if

$$V_i(x+y, x+y) \leq (\geq) V_i(x,y), \quad i=1,2 \quad (5.29)$$

for all $x,y > 0$. 
The physical situation contemplated here is that a two-component system starts working at the same time and our interest lies in observing its performance after a time \( y \). It follows that the condition (5.29) has the interpretation that the VRL of each component decreases (increases) as the components age.

The following relationship hold among the various classes of DVRL (IVRL) distributions

\[
\text{DVRL-(1)} \rightarrow \text{DVRL-(3)} \rightarrow \text{DVRL-(4)}
\]

\[
\text{DVRL-(2)}
\]

and

\[
\text{IVRL-(1)} \rightarrow \text{IVRL-(3)} \rightarrow \text{IVRL-(4)}
\]

\[
\text{IVRL-(2)}
\]

These implications follow directly from the respective definitions of the various classes. At the same time, the question of reverse implications in each case is also of considerable importance. Some results in this connection are presented in the following theorems.

Theorem 5.9.

IVRL-(2) does not imply IVRL-(3) and IVRL-(2) does not imply IVRL-(4).
Proof:

Consider the Gumbel's exponential distribution with

\[ R(x_1, x_2) = e^{-\theta x_1 - \theta x_2 - \alpha_1 x_1 - \alpha_2 x_2}; \quad x_1, x_2 > 0. \]

From direct calculations

\[ V_1(x_1, x_2) = \frac{1}{(\alpha_1 + \theta x_2)^2} \]

and

\[ V_2(x_1, x_2) = \frac{1}{(\alpha_2 + \theta x_1)^2}. \]

Clearly,

\[ V_1(x_1 + y, x_2) = V_1(x_1, x_2) \]

and

\[ V_2(x_1, x_2 + y) = V_2(x_1, x_2). \]

Thus \( X \) has IVRL-(2) property.

Since,

\[ V_1(x_1 + y, x_2 + y) \not\leq V_1(x_1, x_2) \]

and

\[ V_2(x_1 + y, x_2 + y) \not\leq V_2(x_1, x_2), \]

\( X \) does not satisfy the property IVRL-(3). Similarly, \( X \) does not have the property IVRL-(4) and the proof is complete.
Theorem 5.10.

IVRL-(1) does not imply DVRL-(2) and 
IVRL-(1) does not imply DVRL-(4).

Proof:

Taking 
\[ R(x_1, x_2) = (1+a_1x_1+a_2x_2)^{-c}, \quad x_1, x_2 > 0, \]
the VRL vector is 
\[ V(x_1, x_2) = \left( \frac{c(1+a_1x_1+a_2x_2)^2}{a_1^2(c-1)(c-2)}, \quad \frac{c(1+a_1x_1+a_2x_2)^2}{a_2^2(c-1)(c-2)} \right). \]

Obviously, 
\[ V_i(x_1+y_1, x_2+y_2) > V_i(x_1, x_2), \quad i=1,2, \]
but 
\[ V_1(x_1+y, x_2) \leq V_1(x_1, x_2) \]
and 
\[ V_2(x, x_2+y) \leq V_2(x_1, x_2). \]

Thus IVRL-(1) does not imply DVRL-(2).

Similarly,
\[ V_i(x+y, x+y) \leq V_i(x, x). \]

Hence, IVRL-(1) does not imply DVRL-(4).
Theorem 5.11.

DVRL-(1) does not imply IVRL-(2) and also
DVRL-(1) does not imply IVRL-(4).

Proof:

From,
\[ R(x_1, x_2) = (1-p_1 x_1 - p_2 x_2)^d, \quad 0 < x_i < \frac{1}{p_i}, \]
the VRL,
\[ V_i(x_1, x_2) = \frac{d(1-p_1 x_1 - p_2 x_2)^2}{p_i^2(d+1)(d+2)}, \quad i=1,2. \]

It is evident that
\[ V_i(x_1+y_1, x_2+y_2) \leq V_i(x_1, x_2), \quad i=1,2. \]

Since,
\[ V_1(x_1+y, x_2) \notin V_1(x_1, x_2) \]
and
\[ V_2(x_1, x_2+y) \notin V_2(x_1, x_2) \]

DVRL-(1) does not imply IVRL-(2).

On similar lines, we have
\[ V_i(x+y, x+y) \notin V_i(x, x), \quad i=1,2. \]

Thus DVRL-(1) does not imply IVRL-(4).

The results in sections 5.2 and 5.3 of the present chapter form part of Sankaran and Nair (1992 c).